Scaling Functions And Gibbs Measures And Teichmüller Spaces Of Circle Endomorphisms

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Abstract. We study the scaling function of a $C^{1+h}$ expanding circle endomorphism. We find necessary and sufficient conditions for a Hölder continuous function on the dual symbolic space to be realized as the scaling function of a $C^{1+h}$ expanding circle endomorphism. We further represent the Teichmüller space of $C^{1+h}$ expanding circle endomorphisms by the space of Hölder continuous functions on the dual symbolic space satisfying our necessary and sufficient conditions and study the completion of this Teichmüller space in the universal Teichmüller space.

1. Introduction
In the study of critical phenomena and universality in physics (see, for example, [6,10]) scaling functions have been used to describe the finer geometric structure of a dynamical system or a family of dynamical systems. The concept was introduced to the study of hyperbolic Cantor sets on the real line in [25] by Sullivan (see [14, §1.3] for a more comprehensive description). Scaling functions are also defined for Markov maps and are used in the study of the geometric structure of geometrically finite maps (see [14, Chapter 3] for a summary of this work). For a Markov map, one can construct a semiconjugacy between the map and a symbolic dynamical system of finite type. However, the scaling function (if it exists) is defined on the symbolic space dual to this shift of finite type (see §2 for the definition). The scaling function for a Markov map satisfies a summation condition (see §2). A natural question is which functions defined on a given dual symbolic space are the scaling functions of a Markov map?

A circle endomorphism is a Markov map with a standard Markov partition. Using this Markov partition, one can get a semiconjugacy between the one-sided full shift dynamical system on $n$ symbols and the circle endomorphism, where $n$ is the degree of the circle endomorphism. This allows us to define the scaling function of a circle endomorphism on the associated dual symbolic space. If this circle endomorphism is $C^{1+h}$ expanding, then we know that its scaling function exists and is a Hölder continuous function on the dual symbolic space (see Theorem 1). In this paper, we study the above question for $C^{1+h}$ expanding circle endomorphisms. Although our ideas work for $C^{1+h}$ circle endomorphisms of any degree, for notational simplicity we will only formulate our results for circle endomorphisms of degree two. The most well-known result in the study of circle endomorphisms is that a $C^{1+h}$-expanding circle endomorphism has a unique absolutely continuous invariant measure. This

* Supported in part by grant from NNSF of China.
** Supported in part by grants from NSF and PSC-CUNY.
result is closely related to Gibbs measure theory in symbolic dynamical systems (see [27] for more details). Somewhat surprisingly, without the Hölder condition on the derivative, a $C^1$ expanding circle endomorphism can have more than one absolutely continuous invariant measure (see [23] for more details). We will also study the geometric structure of the space of $C^{1+h}$ circle expanding maps as a whole, that is, we will study Teichmüller theory for circle expanding endomorphisms.

The reader may refer to [5,18,23,27] for a description of the requisite Gibbs measure theory, to [14] for the scaling function theory, and to [1,2,4,7,8,11,12,13,14,19,26] for the quasiconformal mapping theory and Teichmüller theory and its application to dynamical systems.

The paper is arranged as follows. In §2, we give the definition of the scaling function of an expanding degree two orientation-preserving circle endomorphism and present some properties of scaling functions. In §3, we study two conditions satisfied by the scaling function of a $C^{1+h}$ expanding degree two orientation-preserving circle endomorphism. One is called the summation condition and the other is the compatibility condition. This second condition allows us to lift objects from the symbolic space to the circle. The main result in this section says that a Hölder continuous function on a dual symbolic space is the scaling function of a $C^{1+h}$ expanding orientation-preserving circle endomorphism if and only if it satisfies these two conditions. In §4, we study the geometric structure of the space of all $C^{1+h}$ expanding degree two orientation-preserving circle endomorphisms, that is, the Teichmüller space of this space. We give a representation of the Teichmüller space by the space of scaling functions and study the completion of the Teichmüller space in the universal Teichmüller space. This part partly overlaps with the paper [13] (Theorem 6) but uses different methods.

Acknowledgment. This work was started when Yunping Jiang visited the Nonlinear Centre at Cambridge University. He would like to thank Robert MacKay and the Nonlinear Center for hospitality and support. He also thanks Dennis Sullivan and Fred Gardiner for helpful conversations.

2. Circle expanding maps, scaling functions and Gibbs measures

Let $S^1$ be the unit circle equipped with normalized Lebesgue measure. Suppose $f$ is a degree two orientation-preserving covering map from $S^1$ onto $S^1$. The map $f$ is said to be expanding if there are constants $C > 0$ and $\lambda > 1$ such that $|f^n(I)| \geq C\lambda^n|I|$ for any sub-interval $I$ of $S^1$ and any integer $n > 0$ so that $|f^n(I)| < 1$. The constant $\lambda$ in this inequality is called an expanding constant of $f$. We will simply call a degree two expanding orientation-preserving covering map from $S^1$ onto $S^1$ a circle map. A circle map $f$ has a unique fixed point, say $\alpha$. The inverse image of $\alpha$ under $f$ has two points, one is $\alpha$ and the other we denote as $\beta$. Using this two points, we get a standard Markov partition for $f$ as follows: Let $I_0$ and $I_1$ be the closures of the two intervals in $S^1 \setminus \{\alpha, \beta\}$. Then we have

1. $S^1 = I_0 \cup I_1$,
2. $f[I_0]$ and $f[I_1]$ are injective,
3. $f(I_0) = f(I_1) = S^1$.

A partition of $S^1$ satisfying these three conditions is called a Markov partition. We call the partition $\Sigma_0 = \{I_0, I_1\}$ the standard partition for the circle map $f$.

Let $g_0$ and $g_1$ be the inverses of $f[I_0]$ and $f[I_1]$. Thus $g_0$ maps $S^1 \setminus \{\alpha\}$ to $I_0$ and $g_1$ maps $S^1 \setminus \{\alpha\}$ to $I_1$. For any string $w_n = i_0 \ldots i_n$ of 0’s and 1’s, let $g_{w_n} = g_{i_0} \circ \cdots \circ g_{i_n}$ and let $I_{w_n}$ be the closure of $g_{w_n}(S^1 \setminus \{\alpha\})$. Let

$$
\Sigma_n^+ = \{w_n = i_0 i_1 \ldots i_{n-1}\} \quad \text{and} \quad \Sigma_n^- = \{w_n^* = i_n \ldots i_2 i_1\}
$$
be the collection of strings of 0’s and 1’s of length $n$ but read differently and define
\[ \sigma_{+,n}(i_0 i_1 \ldots i_{n-1}) = i_1 \ldots i_{n-1} : \Sigma^+_n \to \Sigma^+_{n-1} \]
and
\[ \sigma_{n,-}(i_n \ldots i_2 i_1) = i_n \ldots i_2 : \Sigma^-_n \to \Sigma^-_{n-1}. \]
By considering the inverse limits of
\[ (\Sigma^+_n, \sigma_{+,n})_{n=1}^\infty \quad \text{and} \quad (\Sigma^-_n, \sigma_{-,n})_{n=2}^\infty, \]
we have two symbolic dynamical systems
\[ \Sigma^+ = (\{ w = i_0 i_1 \ldots \}, \sigma_+) \quad \text{and} \quad \Sigma^- = (\{ w^* = i_2 i_1 \}, \sigma_-). \]
We will also make use of a third system, which is a common extension of $\Sigma^+$ and $\Sigma^-$, namely $\Sigma$ defined by
\[ \Sigma = (\{ \ldots i_{-2} i_{-1} \cdot i_0 i_1 i_2 \ldots \}, \sigma), \]
where $\sigma$ is the left shift operation: $\sigma(x)_n = x_{n+1}$. There are projections $\pi^+$ and $\pi^-$ from $\Sigma$ to $\Sigma^+$ and $\Sigma^-$ respectively defined by
\[ \pi^+(\ldots i_{-2} i_{-1} \cdot i_0 i_1 i_2 \ldots) = i_0 i_1 i_2 \ldots \]
\[ \pi^-(\ldots i_{-2} i_{-1} \cdot i_0 i_1 i_2 \ldots) = \ldots i_{-2} i_{-1}. \]
The importance of these spaces arises as there is a natural semiconjugacy from $\Sigma^+$ to $S^1$:
\[ p(w) = \bigcap_{n=0}^\infty I_{w_n}. \]
This map has the property that
\[ f \circ p = p \circ \sigma_+ \]
on $\Sigma^+$, where $w = i_0 i_1 i_2 \ldots$ and $p$ is a continuous onto map from $\Sigma^+$ to $S^1$ (refer to [14, Chapter 3]). Further $p$ is one–one off a countable set.

The second space, $\Sigma^-$ is called the dual symbolic space of $f$ (see [14, Chapter 3]) and we define
\[ s(w^*_n) = \frac{|I_{w^*_n}|}{|I_{\sigma_{-,n}(w^*_n)}|}. \]
The quantities $s(w^*_n)$ are called the pre-scalings of $f$. Thus we have the following definition (see [14, Chapter 3]).

**Definition 1.** If for every $w^*_n = \ldots w^*_1 \in \Sigma^-$, $\lim_{n \to \infty} s(w^*_n)$ exists, then we define a function $s(w^*) = \lim_{n \to \infty} s(w^*_n)$ on $\Sigma^-$. This function is called the scaling function of $f$.

We say a circle map $f$ is $C^{1+h}$ if it is $C^1$ and its derivative is Hölder continuous and write $C^{1+h}$ for the set of $C^{1+h}$ circle maps. A function $s$ on $\Sigma^-$ is called Hölder continuous if there are constants $C > 0$ and $0 < \tau < 1$ such that
\[ |s(w^*) - s(v^*)| \leq C \tau^n \]
as long as the first $n$ digits of $w^*$ and $v^*$ from the right are the same. The following theorem is known (see [14, Chapter 3]) and is important in this paper. We give a full proof.
Theorem 1. The scaling function of a map in \( C^{1+h} \) exists and is a Hölder continuous function on \( \Sigma^- \).

The proof of this theorem uses the naive distortion lemma.

**Lemma 1 (Naive Distortion Lemma).** Suppose \( f \in C^{1+h} \). Then there are constants \( C > 0 \) and \( 0 < \gamma \leq 1 \) such that for any integer \( k > 0 \), any open interval \( I \neq S^i \) such that \( f^k|I \) is injective, and any \( \xi \) and \( \eta \) in \( I \),

\[
\left| \log \left( \frac{|(f^k)'(\xi)|}{|(f^k)'(\eta)|} \right) \right| \leq C|f^k(\xi) - f^k(\eta)|^\gamma.
\]

**Proof of Theorem 1.** Let \( w^* \) be a point in \( \Sigma^- \). Suppose \( w^* = \cdots w_m^* \) and \( w_m^* = i_{m-1} \cdots i_0 \). Then \( f^k(I_{w^*_m}) = I_{w^*_n} \) for \( n = m - k \geq 0 \). Thus we have

\[
|s(w_m^*) - s(w_n^*)| = \left| 1 - \frac{|(f^k)'(\xi)|}{|(f^k)'(\eta)|} \right| s(w_n^*) = \left| 1 - \frac{|(f^k)'(\xi)|}{|(f^k)'(\eta)|} \right| \frac{|(f^n)'(\xi)|}{|(f^n)'(\eta)|} s(w_n^*),
\]

where \( \xi \) and \( \eta \) are in \( I_{\sigma, -n(w_m^*)} \) and \( \xi' \) and \( \eta' \) are in \( I_{\sigma, -n(w_n^*)} \). Applying the naive distortion lemma, we have that

\[
|s(w_m^*) - s(w_n^*)| \leq C_0|f^k(\xi) - f^k(\eta)|^\gamma \leq C_0|I_{w^*_n}|^\gamma \leq C_1 \lambda^{-n\gamma},
\]

where \( C_0 \) and \( C_1 \) are constants and \( \lambda > 1 \) is an expanding constant of \( f \). This means that \( \{s(w_n^*)\}_{n=0}^\infty \) is a Cauchy sequence. So

\[
s(w) = \lim_{n \to \infty} s(w_n^*)
\]

exists. Similarly, if \( w^* = \cdots w_n^* \) and \( v^* = \cdots v_n^* \) are two points in \( \Sigma^- \) we also have

\[
|s(w^*) - s(v^*)| \leq C_0|I_{w^*_n}|^\gamma \leq C_1 \lambda^{-n\gamma}.
\]

This implies that \( s \) is Hölder continuous. \( \square \)

The pre-scalings \( \{s(w_n^*)\} \) of \( f \) satisfy

\[
s(w_0) + s(w_1) = 1.
\]

Taking the limit, it follows that

\[
s(w^*0) + s(w^*1) = 1 \quad (*)
\]

for each \( w^* \in \Sigma^- \). If a function \( s \) satisfies (*) , we say that it satisfies the summation condition.

We mention one more property of scaling functions. Suppose \( f \) is a \( C^{1+h} \) circle map. A point \( a \) is called a periodic point of period \( n \) if \( f^n(a) = a \) but \( f^i(a) \neq a \) for all \( 0 < i < n \). The eigenvalue \( e_a \) of \( f \) at a periodic point \( a \) of period \( n \) is by definition \( e_a = (f^n)'(a) \). In [14, Chapter 3], there is a relation between the scaling function of \( f \) and all eigenvalues of \( f \) as follows. Suppose \( f^j(a) \in I_{i_1 \cdots i_{n-1}} \). Let \( w_n = i_0 \cdots i_{n-1}, \ w_n^* = i_0 \cdots i_{n-1} \). Then \( w = w_n^\infty \) is a point in \( \Sigma^+ \) corresponding to \( a \). Let \( w^*(a) = (w_n^*)^\infty \).
**Proposition 1.** For any periodic point $a$ of $f$ of period $n$,

$$
\frac{1}{\varepsilon_a} = \prod_{j=0}^{n-1} s(\sigma^j(w^*(a))).
$$

Since a major issue in what follows is whether a given function can arise as the scaling function of a circle map, it is natural that one needs to have a notion of which continuous functions on a shift space can be lifted to give continuous functions on the circle. We call a continuous function $h: \Sigma^+ \to \mathbb{R}$ circle-continuous if

- $h(11\ldots) = h(00\ldots)$,
- $h(011\ldots) = h(100\ldots)$ and
- $h(w011\ldots) = h(w100\ldots)$ for any finite string $w$.

A function $h$ is circle continuous if and only if it can be written as $g \circ p$ where $p$ is the natural semiconjugacy introduced earlier.

In §3, we will make extensive use of results about Gibbs measures. We outline here the necessary background.

Sets of the form $\{x \in \Sigma^+: x_0 = i_0, \ldots, x_n = i_n\}$ will be called cylinder sets in $\Sigma^+$ and will be denoted $[i_0\ldots i_n]$. Similarly, cylinder sets in $\Sigma^−$ will be denoted by $[i_n \ldots i_1]$.

A $\sigma_+$-invariant measure $\nu^+$ on $\Sigma^+$ will be called a Gibbs measure if there is a continuous function $g: \Sigma^+ \to (0,1)$ such that

$$
\lim_{n \to \infty} \frac{\nu^+([i_0\ldots i_n])}{\nu^+([i_1\ldots i_n])} = g(i_0i_1\ldots)
$$

for all sequences $i_0i_1\ldots \in \Sigma^+$. Such a function $g$ necessarily satisfies $g(0x)+g(1x) = 1$, a condition analogous to the summability condition above. The function $g$ is called the conditional probability function of $\nu^+$. Measures satisfying this property have in the past (see [Ke]) been called $g$-measures. Gibbs measures often refer to measures on $\Sigma$. There is however a very close connection between the kinds of measures as mentioned below.

Similarly, a $\sigma_-$-invariant measure $\nu^−$ on $\Sigma^−$ will be called a dual Gibbs measure if there is a function $s: \Sigma^- \to (0,1)$ such that

$$
\lim_{n \to \infty} \frac{\nu^−([i_n\ldots i_1])}{\nu^−([i_n\ldots i_2])} = s(\ldots i_2i_1).
$$

The function $s$ is called the dual conditional probability function of $\nu^−$.

In all that follows, all conditional probability functions and dual conditional probability functions will be Hölder continuous. In this case, it may be shown (see [Bo]) that if $g$ is a conditional probability function satisfying a summability condition ($g(0x) + g(1x) = 1$ for all $x$), then there is a unique measure $\nu^+$ with this function as its conditional probability distribution function. Further, the measure $\nu^+$ is the unique invariant measure satisfying the inequality

$$
C_1 \leq \frac{\nu^+([x_n\ldots x_0])}{\exp(\log g(x_000\ldots) + \log g(x_1x_000\ldots) + \ldots + \log g(x_n\ldots x_000\ldots))} \leq C_2
$$

for appropriate constants $C_1$ and $C_2$. Note that the denominator is written as the exponential of a sum of logarithms rather than as a product in order to make explicit
the connection with the definition in [Bo]. There is a similar characterization of \( \nu^- \) in terms of the exponential of sums of terms involving the logarithm of \( s \). This characterization of Gibbs measures is used as the definition in [Bo].

It is well-known that there is a bijection between the set of \( \sigma^- \)-invariant measures on \( \Sigma \) and the set of \( \sigma^+ \)-invariant measures on \( \Sigma^+ \), namely \( \nu \mapsto \nu \circ \pi^+ \). Similarly, there is a bijection between the \( \sigma^- \)-invariant measures on \( \Sigma \) and the set of \( \sigma^- \)-invariant measures on \( \Sigma^- \). If \( \nu^+ \) and \( \nu^- \) are the measures corresponding on \( \Sigma^+ \) and \( \Sigma^- \) corresponding to \( \nu \), then we have \( \nu^+([i_0 \ldots i_{n-1}]) = \nu^-([i_0 \ldots i_{n-1}]) \), where the subset \([i_0 \ldots i_{n-1}]\) of \( \Sigma^- \) consists of those \( w^+ = \ldots x_n \ldots x_1 \) such that \( x_n = i_0, x_{n-1} = i_1, \ldots, x_1 = i_{n-1} \).

Suppose now that \( f \in C^{1+h} \) with unique absolutely continuous invariant measure \( \mu \). Let \( \rho \) denote the density of \( \mu \) with respect to Lebesgue measure. Since \( p \) is one--one off a countable set, there is a unique invariant measure \( \nu^+ \) on \( \Sigma^+ \) such that \( \mu = \nu^+ \circ p^- \). The measure \( \nu^+ \) will be called the symbolic measure of \( f \). It is a Gibbs measure with conditional probability function

\[
g(x) = \frac{\rho(p(x))}{f'(p(x))\rho(p(x))}.
\]

Since \( \rho \) is known to be Hölder continuous, we see that \( g \) is Hölder continuous and circle-continuous. Applying the above-mentioned bijections, there are measures \( \nu \) and \( \nu^- \) on \( \Sigma \) and \( \Sigma^- \) corresponding to \( \nu^+ \). The measure \( \nu^- \) is called the dual symbolic measure of \( f \).

The key observation which makes Gibbs measures of relevance to questions involving scaling functions is the following.

**Proposition 2.** Suppose \( f \in C^{1+h} \), \( \nu^- \) is the dual symbolic measure of \( f \) and \( s: \Sigma^- \to (0,1) \) is the scaling function of \( f \). Then the dual conditional probability function of \( \nu^- \) is \( s \).

**Proof.** This follows from the observation that \( \nu^-[i_n \ldots i_1] = \nu^+[i_n \ldots i_1] = \mu(I_{i_n \ldots i_1}) \). It follows that

\[
\frac{\nu^-[i_n \ldots i_1]}{\nu^-[i_n \ldots i_2]} = \frac{\mu(I_{i_n \ldots i_1})}{\mu(I_{i_n \ldots i_2})}.
\]

Since the density \( \rho \) is a continuous function, \( I_{i_n \ldots i_1} \subset I_{i_n \ldots i_2} \) and \( |I_{i_n \ldots i_2}| \to 0 \) as \( n \to \infty \), we see that the dual conditional probability function is given by

\[
s'(i_n \ldots i_1) = \lim_{n \to \infty} \frac{|I_{i_n \ldots i_1}|}{|I_{i_n \ldots i_2}|}.
\]

This is equal to \( s(i_n \ldots i_1) \). \( \square \)

We will make use at one stage of Ruelle-Perron-Frobenius operators, and offer the following introduction. For proofs and more details, the reader is referred to [Bo, W]. Given a continuous function \( \phi \) on \( \Sigma^+ \), the Ruelle-Perron-Frobenius operator with potential \( \phi \) is defined by

\[
\mathcal{L}_\phi: C(\Sigma^+) \to C(\Sigma^+); \quad \mathcal{L}_\phi[g](x) = \sum_{i \in \{0,1\}} e^{\phi(ix)} g(ix).
\]

A potential \( \psi \in C(\Sigma^+) \) is called normalized if \( \mathcal{L}_\psi[1] = 1 \). This is equivalent to saying that \( \exp(\psi) \) satisfies the summability condition. For any Hölder continuous \( \phi \) on \( \Sigma \), there always exists a constant \( P(\phi) \) (known as the pressure of \( \phi \)) and a normalized Hölder continuous function \( \psi \) on \( \Sigma^+ \) such that \( \psi \circ \pi^+ = \phi - P(\phi) - h + h \circ \sigma_+ \) for
a Hölder continuous function $h$. This says that any Hölder continuous function on $\Sigma$ is cohomologous to a normalized Hölder continuous function up to an additive constant. A version of this also holds for Hölder continuous functions on $\Sigma^+$: Given any Hölder continuous function $\phi$ on $\Sigma^+$, there is a normalized Hölder continuous function $\psi$ on $\Sigma^+$ and a Hölder $h$ such that $\psi = \phi - P(\phi) - h + h \circ \sigma_+$. Both $\psi$ and $P(\phi)$ are uniquely determined by $\phi$ and $h$ is determined up to an additive constant.

The potential $\psi$ is called the normalization of $\phi$. If $\psi$ is normalized, then $P(\psi) = 0$. If $\psi$ is normalized and Hölder continuous, then it may be shown that $L^\infty_n[g]$ converges uniformly to a constant for all continuous functions $g$. This constant turns out to be $\int g \, d\nu^+$ where $\nu^+$ is the unique Gibbs measure with conditional probability function $e^\psi$. If $\phi' = \phi - H + H \circ \sigma_+$, then one can check that $L^n[\phi'] = e^H L^n[e^{-H}g].$

3. Characterization of the scaling functions of circle maps

If $f$ is in $C^{1+h}$, then Theorem 1 says that its scaling function $s$ exists and is a Hölder continuous function on $\Sigma^-$. As we have seen the scaling function satisfies the summation condition,

$$s(w^*0) + s(w^*1) = 1$$

for all $w^* \in \Sigma^-$. We will show in the next theorem that there is another condition for $s$ to be a scaling function of a map in $C^{1+h}$. Suppose $s$ is a Hölder continuous function on $\Sigma^-$. We define a sequence of functions on $\Sigma^-$:

$$C_{s,N}(w^*) = \prod_{n=0}^N \frac{s(w^*10\ldots0_n)}{s(w^*01\ldots1_n)}.$$ 

Provided that $s(\ldots000) = s(\ldots111)$, the sequence of functions $C_{s,N}(w^*)$ is necessarily uniformly geometrically convergent to a limit function $C_s(w^*)$. We say that $s$ satisfies the compatibility condition if

$$s(\ldots000) = s(\ldots111);$$

and

$$C_s(w^*)$$

is independent of $w^*$.

(In other words, $s$ satisfies the compatibility condition if $s(\ldots000) = s(\ldots111)$ and

$$\prod_{n=0}^\infty \frac{s(w^*10\ldots0_n)}{s(w^*01\ldots1_n)}$$

is independent of $w^*$).

**Theorem 2.** Let $s$ be a Hölder continuous function on $\Sigma^-$. Then $s$ is the scaling function of a map in $C^{1+h}$ if and only if $s$ satisfies the summation and compatibility conditions.

Before proving Theorem 2, we will need a number of lemmas. These address the relationship between a Gibbs measure and its dual Gibbs measure, showing that a Gibbs measure has a circle-continuous conditional probability function if and only if the dual Gibbs measure has a compatible dual conditional probability function. It will then follow that if $f \in C^{1+h}$, the symbolic measure is a Gibbs measure with a Hölder and circle-continuous conditional probability function so the dual symbolic measure has a compatible dual continuous conditional probability function. Conversely if $s$ satisfies the conditions, we construct a dual Gibbs measure, get the corresponding Gibbs measure, which then has to have a circle-continuous conditional probability function and finally lift this to a map of the circle with the required scaling function.
Lemma 2. Let \( \nu \) be an invariant measure on \( \Sigma \) and let \( \nu^+ = \nu \circ \pi^+ \) and \( \nu^- = \nu \circ \pi^- \). Then \( \nu^+ \) is a Gibbs measure on \( \Sigma^+ \) with Hölder continuous conditional probabilities if and only if \( \nu^- \) is a dual Gibbs measure on \( \Sigma^- \) with Hölder continuous conditional probabilities. If either of these properties holds, then \( \log g \circ \pi^+ \) is Hölder continuously cohomologous to \( \log s \circ \pi^- \) where \( g \) is the conditional probability function of \( \nu^+ \) and \( s \) is the dual conditional probability function of \( \nu^- \).

If either of the properties in the lemma holds, then we call \( \nu \) a two-sided Hölder Gibbs measure. It should be noted that no theorem of this type is true if the Hölder condition is removed (see [Ka] for an example where this fails).

Proof. By symmetry, it is clearly sufficient to demonstrate that if \( \nu^- \) is a dual Gibbs measure on \( \Sigma^- \) with Hölder continuous conditional probabilities, then \( \nu^+ \) is a Gibbs measure on \( \Sigma^+ \) with Hölder continuous conditional probabilities.

Let \( s \) denote the dual conditional probability function of \( \nu^- \). Then \( s \) is a Hölder continuous function on \( \Sigma^- \). It follows that \( s \circ \pi^- \) is a Hölder continuous function on \( \Sigma \). Since \( s \) is a conditional probability function, we have \( P(\log s \circ \pi^-) = 0 \). It follows (see [3]) that \( \log s \circ \pi^- \) is Hölder continuously cohomologous to a Hölder continuous function \( \phi \) which is normalized and depends only on the positive coordinates. Write \( \phi = \log g \circ \pi^+ \). We now show that \( \nu^+ \) is a Gibbs measure on \( \Sigma^+ \) with conditional probability function \( g \).

To see this, note that \( \log s \circ \pi^- = \log g \circ \pi^+ - k \circ \sigma + k \) for a Hölder continuous function \( k \). Since \( \nu^- \) is a dual Gibbs measure with conditional probability function \( s \), there are constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \leq \frac{\nu^-[x_n \ldots x_0]}{\exp(\log s(00\ldots0) + \log s(00\ldots0\ldots x_n x_{n-1}) + \ldots + \log s(00\ldots0\ldots x_n x_0\ldots x_0))} \leq C_2.
\]

Since \( \log s \circ \pi^- \) is cohomologous to \( \log g \circ \pi^+ \), we have

\[
\log s(00\ldots0) + \log s(00\ldots0\ldots x_n x_{n-1}) + \ldots + \log s(00\ldots0\ldots x_n x_0) = \sum_{i=1}^{n+1} \log s \circ \pi^-(\sigma^i(00\ldots0\ldots x_n x_0)) = \sum_{i=1}^{n+1} \log g \circ \pi^+(\sigma^i(00\ldots0\ldots x_n x_0)) - k \circ \sigma^{n+2}(00\ldots0\ldots x_n x_0) + k \circ \sigma(00\ldots0\ldots x_n x_0) + \log g(x_n x_{n-1} \ldots x_0 0) + \log g(x_{n-2} \ldots x_0 0) + \ldots + \log g(00\ldots0) - k(00\ldots0\ldots x_0 0) + k(00\ldots0\ldots x_n x_{n-1} x_0 \ldots x_0).
\]

Since \( \nu^+[x_n \ldots x_0] = \nu^-[x_n \ldots x_0] \), and \( \exp k \) and \( g \) are bounded away from 0 and \( \infty \), it follows that there are constants \( C_3 \) and \( C_4 \) such that

\[
C_3 \leq \frac{\nu^+[x_n \ldots x_0]}{\exp(\log g(x_0 0) + \log g(x_1 x_0 0) + \ldots + \log g(x_n 0 0))} \leq C_4.
\]

This shows that \( \nu^+ \) is a Gibbs measure with Hölder continuous conditional probability function \( g \) as required. \( \square \)

We will now give necessary and sufficient conditions under which a Hölder continuous function \( \phi \) is continuously cohomologous to a circle-continuous function.
Before doing this, we will need to introduce the following quantities which measure the deviation of a function on $\Sigma^+$ from a circle-continuous function. If $\phi$ is a continuous function on $\Sigma^+$, then its deviations from circle-continuity are given by

$$
\delta_0(\phi) = \phi(00\ldots) - \phi(11\ldots)
$$

$$
\delta_1(\phi) = \phi(100\ldots) - \phi(011\ldots)
$$

$$
\Delta_{x_nx_{n-1}\ldots x_1}(\phi) = \phi(x_nx_{n-1}\ldots x_1100\ldots) - \phi(x_nx_{n-1}\ldots x_1011\ldots).
$$

A function is circle-continuous if and only if its deviations from circle-continuity vanish.

**Lemma 3.** Let $\phi$ be a Hölder continuous function defined on $\Sigma^+$. Then $\phi$ is continuously cohomologous to a circle-continuous function if and only if

1. $\phi(00\ldots) = \phi(11\ldots)$; and
2. there exists a constant $C(\phi)$ such that $\sum_{n=1}^\infty \Delta_{x_n\ldots x_1}(\phi) = C(\phi)$ for all $w^* = \ldots x_n \ldots x_1 \in \Sigma^-$. 

**Proof.** Suppose first that $\phi$ is continuously cohomologous to a circle-continuous function $\psi$. Then $\psi(x) = \phi(x) - \chi(x) + \chi(|x_1|)$ for a continuous function $\chi$. Then we have $\phi(00\ldots) = \psi(00\ldots) = \psi(11\ldots) = \phi(11\ldots)$, so (1) holds. Since $\psi$ is circle-continuous, we have $\Delta_{x}(\psi) = 0$ for all finite strings $a$. From the cohomology equation, we now see

$$
\Delta_{x_nx_{n-1}\ldots x_1}(\chi) = \Delta_{x_nx_{n-1}\ldots x_1}(\phi) + \Delta_{x_{n-1}\ldots x_1}(\chi)
$$

and

$$
\Delta_{x_1}(\chi) = \Delta_{x_1}(\phi) + \delta_1(\chi).
$$

for any non-empty finite string $x_nx_{n-1}\ldots x_1$. We also have $\delta_1(\chi) = \delta_1(\phi) + \delta_0(\chi)$. Combining the above equations, we see that

$$
\Delta_{x_nx_{n-1}\ldots x_1}(\chi) = \Delta_{x_nx_{n-1}\ldots x_1}(\phi) + \Delta_{x_{n-1}\ldots x_1}(\phi) + \ldots + \Delta_{x_1}(\phi) + \delta_1(\phi) + \delta_0(\chi).
$$

for each finite string $x_n \ldots x_1$.

Since $\chi$ is required to be continuous, we have $\Delta_{x_n\ldots x_1}(\chi)$ converges uniformly to 0 as $n \to \infty$. This implies that $\sum_{n=1}^\infty \Delta_{x_n\ldots x_1}(\phi) = -\delta_1(\phi) - \delta_0(\chi)$ is independent of $w^* = \ldots x_2x_1$ which is condition (2).

Conversely, suppose $\phi(00\ldots) = \phi(11\ldots)$ and

$$
\Delta_{x_1}(\phi) + \Delta_{x_2x_1}(\phi) + \ldots + \Delta_{x_nx_{n-1}\ldots x_1}(\phi) + \ldots = C(\phi)
$$

for each $x = \ldots x_nx_{n-1} \ldots x_1 \in \Sigma^-$. Then set

$$
\delta_0 = -\delta_0(\phi) - C(\phi);
$$

$$
\delta_1 = -C(\phi); \quad \text{and}
$$

$$
\Delta_{x_n\ldots x_1} = \Delta_{x_n\ldots x_1}(\phi) + \Delta_{x_{n-1}\ldots x_1}(\phi) + \ldots + \Delta_{x_1}(\phi) - C(\phi).
$$

Then by assumption, $\Delta_{x_n\ldots x_1} \to 0$ as $n \to \infty$. Moreover, since $\phi$ is Hölder continuous, there are constants $K > 0$ and $\lambda > 1$ such that $\Delta_{x_n\ldots x_1}(\phi) \leq K/\lambda^n$ for all strings $x_n \ldots x_1$. We then have $\Delta_{x_n\ldots x_1} \leq K/(\lambda - 1) \cdot 1/\lambda^n$. We now construct a continuous function $\chi$ on $\Sigma^+$ such that $\delta_0(\chi) = \delta_0$, $\delta_1(\chi) = \delta_1$, and $\Delta_{x_n\ldots x_1}(\chi) = \Delta_{x_n\ldots x_1}$ for each string $x_n \ldots x_1$. Then $\phi - \chi + \chi \circ \sigma_+ \to 0$, will be a continuous function and will have vanishing deviations. To construct $\chi$, define $h : \Sigma^+ \to \mathbb{R}$ by

$$
h(x) = \frac{x_0}{2} - \sum_{n=1}^\infty \frac{x_n}{2^{n+1}}.
$$
and $k: \Sigma^+ \to \mathbb{R}$ by
\[
k(x) = 1 - \sum_{n=0}^{\infty} \frac{x_n}{2^n+1}
\]
where $x = x_0x_1 \ldots$ Then $h$ is a continuous function on $\Sigma^+$ with $\|h\|_\infty = 1/2$, $\delta_1(h) = 1$, $\Delta_0(h) = 0$ for all strings $a$ and $\delta_0(h) = 0$. Similarly, $\delta_0(k) = 1$, $\delta_1(k) = 0$ and $\Delta_a(k) = 0$ for all finite strings $a$. Now for each finite string $a$, let
\[
h_a(x) = 1_{|a|}(x) h(\sigma_+^{[a]}(x)) \Delta_a
\]
where $|a|$ means the length of the string $a$ and $1_S$ denotes the characteristic function of the set $S$. Then $\Delta_a(h_a) = \Delta_a$ and all other deviations vanish. Now forming $H_n = \sum_{|a|=n} h_a$ gives a function $H_n$ with
\[
\|H_n\|_\infty = \sup_{|a|=n} \|\delta_a\| \leq \frac{K}{(\lambda - 1)\lambda^n}.
\]
Now let $H = (\delta_0)k + (\delta_1)h + \sum_{n=1}^{\infty} H_n$. Then $H$ is the uniform limit of continuous functions, and hence is continuous. Further, $H$ has the property that $\Delta_a(H) = \Delta_a$ for all strings $a$, $\delta_1(H) = \delta_1$ and $\delta_0(H) = \delta_0$. Now set $\psi = \phi - H + H \circ \sigma_+$. Then $\psi$ is a continuous function. Its deviations are given by
\[
\delta_0(\psi) = \delta_0(\phi) = 0;
\delta_1(\psi) = \delta_1(\phi) - \delta_1 + \delta_0 = \delta_1(\phi) - (\phi) + (-\delta_1(-\phi) - (\phi)) = 0;
\Delta_{x_1}(\psi) = \Delta_{x_1}(\phi) - \Delta_{x_1} + \Delta_1 = \Delta_{x_1}(\phi) - \Delta_{x_1}(\phi) - (\phi) - (\phi) = 0; \text{ and }
\Delta_{x_0...x_1}(\psi) = \Delta_{x_0...x_1}(\phi) - \Delta_{x_0...x_1} + \Delta_{x_0...x_1} = 0.
\]
So $\psi$ has vanishing deviations and is therefore circle-continuous. We now know that $\phi$ is cohomologous to a circle-continuous function $\psi$. $\square$

We now show that if $\phi$ is Hölder continuous and continuously cohomologous to a circle continuous function $\psi$, then $\phi$ is in fact Hölder continuously cohomologous to a circle continuous function $\psi'$. We note that in fact the ‘transfer function’ $H$ produced above can be shown to be Hölder continuous by performing a sequence of elementary estimates. We choose however to present an alternative argument, which might have other applications. The argument essentially shows that if a Hölder continuous function $\phi$ is continuously cohomologous to a function possessing some symmetry (in this case, circle-continuity), then the normalized Hölder continuous function corresponding to $\phi$ also possesses that symmetry.

**Lemma 4.** If a Hölder continuous function on $\Sigma^+$ is continuously cohomologous to a circle-continuous function, then its normalization is a circle-continuous function.

**Proof.** Let $\phi$ be a Hölder continuous function which is continuously cohomologous to a circle continuous function $\psi$. From this we deduce that $\phi$ is Hölder cohomologous to a circle-continuous function as follows: We know that $\phi = C + \phi' - K + K \circ \sigma_+$ for a constant $C$ and a Hölder continuous function $K$ where $\phi'$ is a normalized potential. It follows that $\psi - C = \phi' - G + G \circ \sigma_+$ for a continuous function $G$. Since $\psi - C$ is circle-continuous, it follows that $L_{\psi - C}$ sends the space of circle-continuous functions to itself. But $L_{\psi - C} \|1(x) = \exp(G(x))L_{\phi' - G}(\exp(-G))(x)$, which (by continuity of $G$) converges uniformly to a constant (non-zero) multiple of $\exp(G)$. However, since the circle-continuous functions form a closed subspace of $C(X)$, it follows that $\exp(G)$ is circle-continuous. It then follows that $\phi'$ is circle-continuous. Since $\phi$ was Hölder cohomologous to $\phi'$ (up to a constant), it follows that the normalization of $\phi$ is circle-continuous as required. $\square$
Lemma 5. If $\nu$ is a two-sided Hölder Gibbs measure such that $\nu^−$ has conditional probability function $s$ defined on $\Sigma^−$ and $\nu^+$ has conditional probability function $g$ defined on $\Sigma^+$, then $\log g$ is Hölder cohomologous to $\Phi: \Sigma^+ \to \mathbb{R}$ where

$$\Phi(x_0x_1 \ldots) = \sum_{n=1}^{\infty} \left( \log s(\ldots 0x_0 \ldots x_n) - \log s(\ldots 0x_1 \ldots x_n) \right) + \log s(\ldots 0x_0)$$

Proof. Define $\delta_n: \Sigma \to \mathbb{R}$ by

$$\delta_n(x) = \log s(\ldots 0x_0 \ldots x_n) - \log s(\ldots 0x_0 \ldots x_{n-1})$$

for $n > 1$ and $\delta_1(x) = \log s(\ldots 0x_0)$. Note that by Hölder continuity of $s$, we have $\|\delta_n\|$ converges geometrically to 0 and $\sum_{n=1}^{\infty} \delta_n = \log s \circ \pi^-$. We also have $\delta_n(\sigma^n(x)) = \log s(\ldots 0x_0 \ldots x_{n-1}) - \log s(\ldots 0x_1 \ldots x_{n-1})$ for $n > 1$, while for $n = 1$, $\delta_1(\sigma(x)) = \log s(\ldots 0x_0)$. Summing, we get

$$\sum_{n=1}^{k} \delta_n \circ \sigma^n(x) = \sum_{n=2}^{k} \left( \log s(\ldots 0x_0 \ldots x_{n-1}) - \log s(\ldots 0x_1 \ldots x_{n-1}) \right) + \log s(\ldots 0x_0).$$

Taking the limit, we have $\sum_{n=1}^{\infty} \delta_n \circ \sigma^n(x) = \Phi \circ \pi^+(x)$. That this limit exists and is continuous follows from the fact that $\|\delta_n\|_\infty$ converges to 0 exponentially. The function $\Phi$ may also be seen to be Hölder continuous.

It remains to verify that $\Phi \circ \pi^+$ is Hölder continuously cohomologous to $\log s \circ \pi^-$. To see this, note that

$$\Phi \circ \pi^+ - \log s \circ \pi^- = \sum_{n=1}^{\infty} (\delta_n \circ \sigma^n - \delta_n).$$

This may be written as $H \circ \sigma - H$ where $H = \sum_{n=1}^{\infty} h_n$ and $h_n = \sum_{i=0}^{n-1} \delta_n \circ \sigma^i$. Again, it is possible to verify directly that $H$ is Hölder continuous, but in this case, the result follows more simply from Livšic’s theorem (see [20,21]). We check that $\|h_n\|_\infty$ converges to 0 geometrically. It then follows that $H$ is continuous. Livšic’s theorem then applies as $\Phi \circ \pi^+$ and $\log s \circ \pi^-$ are two Hölder continuous functions on $\Sigma$ which are cohomologous by an $L^\infty$ function, allowing us to deduce that in fact $\Phi \circ \pi^+$ and $\log s \circ \pi^-$ are Hölder continuously cohomologous.

Since we have already seen (Lemma 2) that $\log s \circ \pi^-$ and $\log g \circ \pi^+$ are Hölder continuously cohomologous, it now follows that $\log g \circ \pi^+$ and $\Phi \circ \pi^+$ are Hölder continuously cohomologous.

A further application of Livšic’s theorem allows us to deduce that $\log g$ and $\Phi$ are Hölder continuously cohomologous (as functions on $\Sigma^+$). Let $x = wwww \ldots$ be any periodic point in $\Sigma^+$, where $w$ is any finite string. Let $y$ be the corresponding periodic point in $\Sigma \ldots wwww \ldots$. Since $\log g \circ \pi^+$ and $\Phi \circ \pi^+$ are cohomologous, it follows that

$$\sum_{i=0}^{\text{per}(w)-1} \log g \circ \pi^+ (\sigma^i(y)) = \sum_{i=0}^{\text{per}(w)-1} \Phi \circ \pi^+ (\sigma^i(y)).$$

This implies

$$\sum_{i=0}^{\text{per}(w)-1} \log g(\sigma^i(x)) = \sum_{i=0}^{\text{per}(w)-1} \Phi(\sigma^i(x)),$$

for any finite word $w$ which is known by Livšic’s theorem to be a necessary and sufficient condition for $\Phi$ and $\log g$ to be Hölder continuously cohomologous. \qed
Lemma 6. If \( \nu \) is a H"older Gibbs measure, \( \nu^+ \) has conditional probability function \( g \) and \( \nu^- \) has conditional probability function \( s \), then \( g \) is circle-continuous if and only if \( s \) is compatible.

Proof. Let \( \Phi \) be as in the statement of Lemma 5. We know that \( \Phi \) is H"older continuously cohomologous to \( \log g \). It follows that \( \log g \) is the normalization of \( \Phi \). From Lemmas 3 and 4, we see that \( \log g \) is circle-continuous if and only if \( \Phi \) satisfies the conditions of Lemma 3.

From the definition of \( \Phi \), we see that \( \Phi(00\ldots) = \log s(\ldots00) \) and \( \Phi(11\ldots) = \log s(\ldots11) \). It follows that \( s \) satisfies the first part of the compatibility condition if and only if \( \Phi \) satisfies the first condition from Lemma 3.

To finish the proof, we show that under the assumption that \( s \) satisfies the first compatibility condition, \( s \) satisfies the second compatibility condition if and only if \( \Phi \) satisfies the second condition from Lemma 3.

We now calculate the quantities involved in condition (2) of Lemma 3. Set \( C_n(x) = \Delta_{x_n\ldots x_1}(\Phi) + \ldots + \Delta_{x_1}(\Phi) \) and \( C(x) = \lim_{n \to \infty} C_n(x) \). The condition is then that \( C(x) \) is independent of \( x \). We have

\[
\Delta_{x_n\ldots x_1}(\Phi) = \Phi(x_n \ldots x_100\ldots) - \Phi(x_n \ldots x_1011\ldots)
\]

\[
= \sum_{k=0}^{\infty} \left[ \log s(x_{n-k}00x_{n-k-1}\ldots010\ldots0) - \log s(x_{n-k}00x_{n-k-1}\ldots010\ldots0) \right]
\]

Summing, we see

\[
C_n(x) = \sum_{k=0}^{\infty} \left[ \log s(x_{n-k}00x_{n-k-1}\ldots010\ldots0) - \log s(x_{n-k}00x_{n-k-1}\ldots010\ldots0) \right]
\]

Since \( \log s \) is H"older and \( \log s(\ldots00) = \log s(\ldots11) \), the sum can be split as:

\[
C_n(x) = \left[ \sum_{k=0}^{\infty} \log s(x_{n-k}00x_{n-k-1}\ldots010\ldots0) - \log s(x_{n-k}00x_{n-k-1}\ldots010\ldots0) \right]
\]

The second summation is clearly independent of \( x \), so we see that \( C(x) \) is independent of \( x \) if and only if

\[
\sum_{k=0}^{\infty} \log s(x_{n-k}00x_{n-k-1}\ldots010\ldots0) - \log s(x_{n-k}00x_{n-k-1}\ldots010\ldots0)
\]

is independent of \( x \), but this is just the second part of the compatibility condition on \( s \).

It follows that \( \Phi \) is continuously cohomologous to a circle-continuous function if and only if \( s \) satisfies the compatibility condition and hence \( g \) is circle-continuous if and only if \( s \) satisfies the compatibility condition. \( \square \)
**Lemma 7.** If \( \nu^+ \) is a Gibbs measure with a Hölder continuous conditional probability function \( g \), then there is a circle map \( f \in C^{1+h} \) whose symbolic measure is \( \nu^+ \).

For a fuller version of this proof, the reader is referred to [23]. Here is a summary of the argument.

**Proof.** Define a map \( f_0 : S^1 \to S^1 \) by \( f_0(x) = 2x \pmod{1} \). Let \( p : \Sigma^+ \to S^1 \) be defined by \( p(x) = \sum_{n=0}^{\infty} x_n/2^{n+1} \). This is the natural semiconjugacy between \( \sigma_+ \) and \( f_0 \). The map \( p \) is also a measure-theoretic isomorphism between \( (\Sigma^+, \nu^+, \sigma_+) \) and \( (S^1, \nu^+ \circ p^{-1}, f_0) \). To create the map \( f \), we conjugate \( f_0 \). Let \( \theta : S^1 \to S^1 \) be defined by \( \theta(x) = \nu^+(p^{-1}([0,x])) \). The map \( \theta \) is a homeomorphism from the circle to itself as Gibbs measures are automatically non-atomic and fully supported (see [W]). Then set \( f(x) = \theta(f_0(\theta^{-1}(x))) \) preserving the measure \( \nu^+ \circ p^{-1} \circ \theta^{-1} \). Calculating, we see that \( \nu^+ \circ p^{-1}(\theta^{-1}([0,x])) = \nu^+ \circ p^{-1}([0,\theta^{-1}(x)]) = x \). It follows that \( \nu^+ \circ p^{-1} \circ \theta^{-1} \) is Lebesgue measure. So \( f \) preserves Lebesgue measure. It may be checked that \( f \in C^{1+h} \). Let \( |\cdot| \) denote Lebesgue measure. The triples \( (\Sigma, \nu^+, \sigma_+) \) and \( (S^1, |\cdot|, f) \) are measure-theoretically isomorphic by the map \( \theta \circ p \). The map \( \theta \circ p \) may also be seen to be the natural semiconjugacy so that \( \nu^+ \) is indeed the symbolic measure corresponding to the map \( f \) as required. \( \square \)

We now combine these Lemmas to complete the proof of Theorem 2.

**Proof of Theorem 2.** Suppose \( f \in C^{1+h} \). Then let \( \mu \) be the unique absolutely continuous invariant measure for \( T \) with density \( \rho \). Let \( \nu^+ \) be the corresponding symbolic measure and \( p \) be the natural semiconjugacy from \( \Sigma^+ \) to \( S^1 \). Then \( \nu^+ \) is a Gibbs measure with conditional probability function given by

\[
g(x) = \frac{\rho(p(x))}{f'(p(x))\rho(f(p(x)))}.
\]

This is clearly circle-continuous and Hölder continuous. By Lemmas 2 and 6, it follows that the scaling function \( s \) (which is also the conditional probability function of \( \nu^- \)) is compatible and Hölder continuous. As pointed out in §2, scaling functions automatically satisfy the summability condition.

Conversely, suppose that \( s \) is a Hölder continuous function satisfying the summability condition and the compatibility condition. By results in [W], there is a unique Gibbs measure \( \nu^- \) on \( \Sigma^- \) which has \( s \) as its dual conditional probability function. This may be extended to give a measure \( \nu \) on \( \Sigma \) and by Lemma 2, \( \nu^+ \) is a Gibbs measure with Hölder continuous conditional probability function, \( g \) say. Since \( s \) is compatible, it follows that \( g \) is circle-continuous. Now by Lemma 7, there is a map \( f \in C^{1+h} \) which has \( \nu^+ \) as its symbolic measure. This map necessarily has \( \nu^- \) as its dual symbolic measure and so its scaling function is just the conditional probability function of \( \nu^- \) which is \( s \). \( \square \)

4. **Teichmüller space of circle endomorphisms**

In this section, we study the geometric structure of \( C^{1+h} \) and represent its Teichmüller space by the space of Hölder continuous functions on \( \Sigma^- \) satisfying the summation and compatibility conditions. We also study the completion of the Teichmüller space.

Consider a fixed element \( q(z) = z^2 \) in \( C^{1+h} \). For every element \( f \) in \( C^{1+h} \), there is a unique homeomorphism \( h \) of \( S^1 \) such that

\[
hf = qh.
\]
Here \( h \) is called the conjugacy from \( f \) to \( q \). The conjugacy here is not only a homeomorphism but also satisfies a very important geometric property called quasisymmetry defined as follows. Suppose \( h : I \to J \) is a homeomorphism. Define

\[
\rho(h, x, t) = \max \left\{ \frac{|h(x + t) - h(x)|}{|h(x) - h(x - t)|}, \frac{|h(x) - h(x - t)|}{|h(x + t) - h(x)|} \right\}
\]

for \( x \in I \) and \( t > 0 \) with \( x \pm t \in I \),

\[
\rho(h, t) = \sup_{x \in I} \rho(h, x, t),
\]

and

\[
\rho(h) = \sup_{t \to 0, t > 0} \rho(h, t).
\]

The map \( h \) is said to be \( \rho \)-quasisymmetric if \( \rho(h) \leq \rho \) and to be symmetric if \( \rho(h, t) \to 1 \) as \( t \to 0 \). A \( C^1 \)-diffeomorphism is an example of a symmetric homeomorphism but a symmetric homeomorphism may be very singular; it may map a measure zero set into a positive measure set. The proof of the following theorem can be found, for example in [14, §3.5].

**Theorem 3.** Suppose \( f \) is in \( C^{1+\h} \) and \( h \) is the conjugacy from \( f \) to \( q \). Then \( h \) is quasisymmetric.

Now we consider the space of pairs \([f, h]\) where \( f \) is in \( C^{1+\h} \) and \( h \) is the conjugacy from \( f \) to \( q \). For any two pairs \([f_1, h_1]\) and \([f_2, h_2]\), \( h_2^{-1} h_1 \) is the conjugacy from \( f_1 \) to \( f_2 \) and is again quasisymmetric. Two pairs \([f_1, h_1]\) and \([f_2, h_2]\) are equivalent (denoted by \([f_1, h_1] \sim [f_2, h_2]\)) if \( h_2^{-1} h_1 \) is a \( C^1 \)-diffeomorphism. It is easy to verify that \( \sim \) is an equivalence relation. Let \( \kappa = \{[f, h]\} \) mean an equivalence class. Then the Teichmüller space, denoted as \( TC^{1+\h} \), of \( C^{1+\h} \) is the collection of all equivalence classes \( \kappa = \{[f, h]\} \) for \( f \) in \( C^{1+\h} \).

The following theorem is a consequence of Shub-Sullivan’s theorem in [24] and Proposition 1. A more general version for geometrically finite maps is proved in [15,16].

**Theorem 4.** Suppose \( f_1 \) and \( f_2 \) are in \( C^{1+\h} \). Let \( h \) be the conjugacy from \( f_1 \) to \( f_2 \), i.e., \( h f_1 = f_2 h \). Then \( h \) is a \( C^{1+\h} \) diffeomorphism if and only if the scaling functions of \( f_1 \) and \( f_2 \) are the same.

This theorem says that any two elements in an equivalence class in the Teichmüller space \( TC^{1+\h} \) have the same scaling function. This combined with Theorem 2 gives us the following important corollary. Let \( S^h \) be the space of all Hölder continuous functions on \( \Sigma^- \) satisfying the summation and compatibility conditions. For any \( \kappa = \{[f, h]\} \) in \( TC^{1+\h} \), let \( \iota(\kappa) \) mean the scaling function of \( f \).

**Corollary 1.** The map \( \iota \) from \( TC^{1+\h} \) to \( S^h \) is bijective.

We are going to embed \( TC^{1+\h} \) as well as \( S^h \) into the universal Teichmüller space and study its completion in the universal Teichmüller space.

Let \( QS \) be the set of all quasisymmetric homeomorphisms of the circle modulo the space of all Möbius transformations of the circle (\( QS \) may be identified with the set of all quasisymmetric homeomorphisms of the circle fixing three points). Let \( S \subset QS \) be the subset of all symmetric homeomorphisms of the circle. For any \( h \in QS \), let \( \mathcal{E}_h \) be the set of all quasiconformal extensions \( H \) of \( h \) into the unit disk. Let \( K_H \) be the quasiconformal dilatation of \( H \in \mathcal{E}_h \) (see [19]). Using quasiconformal dilatation, one defines a distance in \( QS \) by

\[
d(h_1, h_2) = \frac{1}{2} \inf \{ \log K_{H^{-1}_2 H_1}, H_1 \in \mathcal{E}_{h_1}, H_2 \in \mathcal{E}_{h_2} \}.
\]
It is known that the space \((QS, d)\) is a complete space under the metric \(d\). This space is called the universal Teichmüller space (see [19]). The topology coming from the metric \(d\) on \(QS\) induces a topology on \(QS/S\). Defining
\[
\overline{d}(Sh_1, Sh_2) = \inf_{k_1, k_2 \in S} d(k_1 h_1, k_2 h_2),
\]
gives a metric on \(QS/S\) (see [12]). The topology on \((QS/S, \overline{d})\) is the finest topology which makes the projection \(\Pi : QS \rightarrow QS/S\) continuous. An equivalent topology can be defined as follows. For any \(h \in QS\), let \(H_{loc}\) be a quasiconformal extension of \(h\) to a small neighborhood \(U\) of \(S^1\) in the complex plane. Let
\[
\mu = \mu_{H_{loc}} = \frac{(H_{loc})^\pi}{(H_{loc})_z},
\]
Let
\[
k_{H_{loc}} = \|\mu(z)\|_\infty
\]
and
\[
C_{H_{loc}} = \frac{1 + k_{H_{loc}}}{1 - k_{H_{loc}}},
\]
Then
\[
C_h = \inf C_{H_{loc}}
\]
where the infimum is taken over all quasiconformal extensions \(H_{loc}\) of \(h\) to a neighborhood of the circle. The number \(C_h\) is called the boundary dilatation of \(h\). It is known that \(h\) is symmetric if and only if \(C_h = 1\). Define
\[
\overline{d}(h_1, h_2) = \frac{1}{2} \log C_{h_2^{-1} h_1}.
\]
The topology of \((QS/S, \overline{d})\) is equivalent to the topology of \((QS/S, \overline{d})\). For any \(\kappa = \{[f, h]\} \in TC^{1+h}\), define \(\Phi(\kappa) = hS\), where \(hS\) means the equivalence class of \(h\) in \(QS/S\). Then \(\Phi\) embeds \(TC^{1+h}\) into \(QS/S\) and induces a topology on the Teichmüller space \(TC^{1+h}\). By Corollary 1, we also get an embedding of \(S^h\) into \(QS/S\). Now we are going to discuss the completion of \(TC^{1+h}\) in \(QS/S\).

Let \(R\) be the real line. Then \(E(x) = e^{\pi i x} : R \rightarrow S^1\) is a covering map. For any circle map, let \(F : R \rightarrow R\) be a lift of \(f\), i.e., \(F\) is an orientation-preserving homeomorphism such that \(E \circ F = f \circ E\). Then \(F(x+1) = F(x)+2\). If we impose the additional condition \(F(0) = 0\), then \(F\) is uniquely defined. A circle map \(f\), is said to be uniformly symmetric (refer to [26]) if
\[
\rho_f(t) = \sup \{\rho((F^n)^{-1}, t)\} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0.
\]
It is equivalent to say that all inverse branches of \(f^n|I\) for all \(n > 0\) and all open intervals \(I \neq S^1\) are uniformly symmetric. An example of a uniformly symmetric circle map is a \(C^{1+h}\) circle map. This fact follows from the naive distortion lemma directly. Let \(SC\) be the space of all uniformly symmetric circle maps. Then \(C^{1+h} \subset SC\). The following lemma can be found in [19]. A normalized homeomorphism \(h\) of the real line is a homeomorphism such that \(h(0) = 0\), \(h(1) = 1\), and \(h(\infty) = \infty\).

**Lemma 8.** Let \([a, b]\) be a closed interval on the real line, and \(\epsilon > 0\). Then there is a \(\delta > 0\) such that for any normalized quasisymmetric function \(h\),
\[
|h(x) - x| < \epsilon, \quad x \in [a, b],
\]
whenever \(h\) is \((1 + \delta)\)-quasisymmetric.

By using this lemma and referring to the proof of Theorem 1, we get the following theorem.
Theorem 5. Suppose \( f \) is a uniformly symmetric circle map. Then its scaling function exists. Moreover, this scaling function is continuous.

By applying Lemma 8 again, we see that the scaling function is left invariant if the map is conjugated by a symmetric homeomorphism.

Lemma 9. Suppose \( f_1 \) and \( f_2 \) are two uniformly symmetric circle maps. Let \( h \) be the conjugacy between \( f_1 \) and \( f_2 \). If \( h \) is symmetric, then the scaling functions of \( f_1 \) and \( f_2 \) are the same.

Similarly we define the Teichmüller space of \( SC \) as follows. Consider a fixed element \( q(z) = z^2 \) in \( SC \). For every element \( f \) in \( SC \), there is a homeomorphism \( h \) of \( S^1 \) such that

\[
hf = qh.
\]

The conjugacy \( h \) here is again quasisymmetric. The proof of this fact is similar to the proof of Theorem 3 by applying bounded geometry (refer to [14, §3.5]). Now we consider the space of pairs \([f, h]\) where \( f \) is in \( SC \) and \( h \) is the conjugacy from \( f \) to \( q \). Two pairs \([f_1, h_1]\) and \([f_2, h_2]\) are equivalent (denote as \([f_1, h_1] \sim [f_2, h_2]\) again) if \( h_1^{-1} h_2 \) is symmetric. It is easy to verify that \( \sim \) is an equivalence relation. Let \( \kappa = \{[f, h]\} \) denote an equivalence class. Then the Teichmüller space, denote as \( TSC \), of \( SC \) is the collection of all equivalence classes \( \kappa = \{[f, h]\} \) for \( f \) in \( TSC \).

For any \( \kappa = \{[f, h]\} \) in \( SC \), \( \Phi(\kappa) = hS \), where \( hS \) means the equivalence class of \( h \) in \( QS/S \). \( \Phi \) defines an embedding of \( TSC \) into \( QS/S \) and induces a topology on the Teichmüller space \( TSC \). Following Lemma 9 and Theorem 4, if both of \( f_1 \) and \( f_2 \) are in \( C^{1+h} \) and \([f_1, h_1]\) and \([f_2, h_2]\) are equivalent in \( SC \), then they are also equivalent in \( C^{1+h} \). The following lemma also appeared in [13], but we offer a different proof.

Lemma 10. The space \( TSC \) is a complete space.

Proof. Take any Cauchy sequence \( \{\kappa_n\}_{n=1}^{\infty} = \{[f_n, h_n]\}_{n=1}^{\infty} \) in \( TSC \). Then

\[
\bar{d}(Sh_n, Sh_m) \to 0 \quad \text{as} \quad m, n \to \infty.
\]

We may assume by working modulo \( S \) that \( h_n^{-1} h_m \) tends to the identity map as \( m \) and \( n \) go to infinity. Therefore, \( \{h_n\}_{n=1}^{\infty} \) is a Cauchy sequence in the universal Teichmüller space and \( h_n \) tends to a quasisymmetric map \( h \) as \( n \) goes to infinity. Let \( l_n = h_n^{-1} h \) and \( f = h^{-1} qh \). Define maps \( f_n \) by \( f_n = l_n f l_n^{-1} \) for all \( n > 0 \). Let \( \rho(l_n) \) be the quasisymmetric constant of \( l_n \). Then \( \rho(l_n) \to 1 \) as \( n \to \infty \). Let \( g_{n,w_k} \) be the inverse of \( f_k^\alpha : I_{n,w_k} \to S^1 \setminus \{\alpha_n\} \), where \( \alpha_n \) is the fixed point of \( f \). Let \( g_{w_k} \) be the inverse of \( f^\alpha : I_{w_k} \to S^1 \setminus \{\alpha\} \) where \( \alpha \) is the fixed point of \( f \). Let \( \rho(g_{n,w_k}, t) \) be the symmetric distortion of \( g_{n,w_k} \). Then there is a function \( \epsilon_n(t) \to 1 \) as \( t \to 0 \) such that \( \rho(g_{n,w_k}, t) \leq \epsilon_n(t) \) for all \( w_k \). Let \( \rho(g_{w_k}, t) \) be the symmetric distortion of \( g_{w_k} \). Since \( g_{w_k} = l_n f_n w_k l_n^{-1} \),

\[
\rho(g_{w_k}, t) \leq (\rho(l_n))^2 \epsilon_n(t)
\]

for all \( w_k \) and all \( n > 0 \). Therefore,

\[
\rho_f(t) \leq \inf_{n>0} \{(\rho(l_n))^2 \epsilon_n(t)\}.
\]

So \( \rho_f(t) \to 1 \) as \( t \to 0 \), this means that \( f \) is uniformly symmetric and \([f, h] \in TSC \). So \( TSC \) is complete. \( \square \)
Lemma 11. For any \([f, h] \in TSC\) and any \(\epsilon > 0\), there is an analytic circle map \(f_\epsilon\) in \(C^{1+h}\) such that \([f_\epsilon, h_\epsilon]\) is in the \(\epsilon\)-neighborhood of \([f, h]\) in \(TSC\).

Proof. For any \([f, h] \in TSC\), we construct an analytic circle map approximating \(f\) in the quasiconformal metric. The technique in the construction was developed in complex dynamics (see \([9]\)).

Consider a quasiconformal extension \(H\) of \(h\) to the complex plane. Then \(T = HqH^{-1}\) is a quasiregular map of the complex plane. Let

\[
\mu_T^n(z) = \frac{(T^n)_z}{(T^n)_z}
\]

be the Beltrami coefficient of \(T^n\). Since \(f\) is uniformly symmetric, we can pick an extension \(T\) (equivalently, pick an extension \(H\) of the conjugacy \(h\)) such that there is a function \(\gamma(t) \to 0\) as \(t \to 0\) and such that \(|\mu_T^n(z)| \leq \gamma(|z|^{2n} - 1)\) for all \(n > 0\) and a.e. \(z\). (This can be taken as an equivalent definition of a uniformly symmetric circle map (refer to \([12]\)).) Let

\[
\mu_H(z) = \frac{H_z}{H_z}
\]

be the Beltrami coefficient of \(H\). By calculus,

\[
\mu_{T^n}(z) = \frac{\mu_H(q^n(z)) - \mu_H(z)}{1 + \mu_H(q^n(z))\mu_H(z)} \Theta(z), \quad \text{where } |\Theta(z)| = 1.
\]

This implies that

\[
|\mu_H(q^n(z)) - \mu_H(z)| \leq C \gamma(|z|^{2n} - 1)
\]

for all \(n > 0\) and a.e. \(z\) where \(C > 0\) is a constant. For any \(\epsilon > 0\), we have a \(\delta > 0\) such that \(\gamma(t) < \epsilon/C\) for all \(0 \leq t < \delta\). Let

\[
A_0 = \{z \in \mathbb{C} \mid 1 - \delta < |z| < (1 - \delta)^{1/2}\} \cup \{z \in \mathbb{C} \mid (1 + \delta)^{1/2} < |z| < 1 + \delta\}
\]

and set \(A_n = q^{-n}(A_0)\). Define \(\mu(z) = \mu_H(z)\) for \(z \in \mathbb{C} \setminus \cup_{n=1}^{\infty} A_n\) and \(\mu = \mu_H(q^n(z))\) for \(z \in A_n\) and \(n > 0\). Then \(\mu\) is a Beltrami coefficient defined on the complex plane. Let \(R\) be a quasiconformal homeomorphism solving the Beltrami equation \(R_\epsilon = \mu(z)R_z\) and let \(T_\epsilon = RqR^{-1}\). By calculus,

\[
\mu_{T_\epsilon}(z) = \frac{\mu(q(z)) - \mu(z)}{1 + \mu(q(z))\mu(z)} \Theta(z), \quad \text{where } |\Theta(z)| = 1.
\]

So \((T_\epsilon)_z = 0\) for \((1 - \delta)^{1/2} < |z| < (1 + \delta)^{1/2}\), that is, \(f_\epsilon = T_\epsilon|S^1\) is analytic. Because \(|\mu(z) - \mu_{T_\epsilon}(z)| < \epsilon\) for all \(z \in \mathbb{C}\), \(T_\epsilon\) is \(\epsilon\)-approximate to \(F\) in the quasiconformal metric.

Since \(R|S^1\) is quasisymmetric, \(f_\epsilon\) has bounded geometry meaning that all pre-scalings \(s_{\epsilon}(w_n)\) of \(f_\epsilon\) are greater than a constant. A real analytic degree two circle endomorphism having bounded geometry is expanding (refer to \([14, \text{Chapter 3}]\)). This completes the proof. \(\square\)

Theorem 6. The completion of \(TC^{1+h}\) is \(TSC\).

Proof. It follows from Lemmas 10 and 11.

The study of \(TSC\) and its representation by the space of scaling functions is still an important problem.
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Received October 1997; final version received March 1999.