

# Lyapunov exponents, dual Lyapunov exponents, and multifractal analysis

Aihua Fan<sup>a)</sup>

Faculté de Mathématiques et d'Informatique, Université de Picardie, 33, Rue Saint Leu, 80039 Amiens  
and Analyse Harmonique, Université de Paris Sud, 91405, Orsay, France

Yunping Jiang<sup>b)</sup>

Department of Mathematics, Queens College of CUNY, Flushing, New York 11367  
and Department of Mathematics, Graduate School of CUNY, New York, New York 10036

(Received 19 March 1999; accepted for publication 13 June 1999)

It is shown that the multifractal property is shared by both Lyapunov exponents and dual Lyapunov exponents related to scaling functions of one-dimensional expanding folding maps. This reveals in a quantitative way the complexity of the dynamics determined by such maps. © 1999 American Institute of Physics. [S1054-1500(99)00204-9]

**One of the topics in dynamical systems is to study the geometric structure of some invariant set associated with a given dynamical system. The scaling function has been used to describe the geometric structure of the maximal invariant set of a certain dynamical system studied in this article. In many cases the maximal invariant set is a fractal. More interestingly, the graph of the scaling function in many cases also looks like a fractal. Therefore to analyze a certain fractal property of the maximal invariant set and the dual symbolic space where the scaling function lives becomes important. The multifractal analysis provides us with a tool to conduct this study. In this article, we use the Lyapunov exponent and the dual Lyapunov exponent for such a given dynamical system to give some multifractal decompositions for the maximal invariant set and the dual symbolic space. We then use Ruelle–Perron–Frobenius operators in the thermodynamical formalism to calculate the Hausdorff dimensions of the subsets in the multifractal decompositions. Our analysis says that the Lyapunov exponent and the dual Lyapunov exponent for a dynamical system studied in this article are indeed multifractal functions.**

## I. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

The graph of a scaling function for a folding map (see Ref. 1, p. 115) looks like a fractal. However, the direct study of the fractal property of the graph is rather difficult. In this article, we study the dual Lyapunov exponent which we define as the ergodic mean value of the scaling function. We prove that the dual Lyapunov exponent is a multifractal function (Theorem 2). We also study the Lyapunov exponent of the folding map and show that it is also a multifractal function (Theorem 1).

Scaling functions for expanding  $C^{1+\alpha}$  folding maps are introduced in mathematics by Sullivan in Ref. 2 (see Ref. 1,

pp. 8–9 for a more comprehensive definition). It has a root in physics, in particular, in the study of period doubling bifurcations in chaos (see Refs. 3,4,5). Scaling functions are also defined for Markov maps of intervals in Ref. 1, p. 76. Although the results in this article can be proved for any Markov map of an interval, we will only state them for folding maps of an interval (in order to have notational simplicity).

Let  $I$  be the interval  $[-1, 1]$ . Suppose  $f: I \rightarrow \mathbf{R}$  is a map. Let  $f^n$  denote the  $n$ -time composition of  $f$ . Suppose,

- (i)  $f(-1) = f(1) = -1$  and  $f(0) > 1$ ,
- (ii)  $f|_{[-1,0]}$  is increasing and  $f|_{[0,1]}$  is decreasing,
- (iii)  $f$  is continuously differentiable on  $f^{-1}(I)$  with non-zero derivatives (at an endpoint of an interval we only consider one-side derivatives).

We call such a map a  $C^1$  folding map. The maximal invariant set of  $f$  is

$$\Lambda = \bigcap_{n=0}^{\infty} f^{-n}(I).$$

Our main concern is the dynamical system  $(\Lambda, f)$ . We say a folding map  $f$  is *expanding* if, furthermore

- (iv) There are constants  $C > 0$  and  $\lambda > 1$  such that

$$|(f^n)'(x)| \geq C\lambda^n, \quad \left( \forall x \in \bigcup_{j=0}^{n-1} f^{-j}(I) \right).$$

By a theorem due to Mather (see Ref. 6 for a proof), for an expanding map  $f$ , there exists a  $C^1$  change of coordinate such that under the new coordinate we have  $|f'(x)| \geq \lambda > 1$  ( $\forall x \in f^{-1}(I)$ ). In the following, “expanding” will mean this last condition.

We will denote by  $\omega_{f'}$  the modulus of continuity of  $f'$  on  $f^{-1}(I)$  which is defined in the usual way,

$$\omega_{f'}(r) = \sup_{|x-y| \leq r} |f'(x) - f'(y)|,$$

where  $x$  and  $y$  stay in a same component of  $f^{-1}(I)$ . We say that  $f'$  satisfies the *Dini condition* if

$$\int_0^1 \omega_{f'}(r) \frac{dr}{r} < \infty.$$

We say that  $f'$  satisfies the *strong Dini condition* if

<sup>a)</sup>Electronic mail: fan@math.pst.u-cergy.fr

<sup>b)</sup>Electronic mail: yunqc@yunping.math.qc.edu; Current address: Institute for Mathematical Research (FIM), ETHZ, 8092 Zurich, Switzerland. Electronic mail: jiang@math.ethz.ch

$$\int_0^1 \omega_{f'}(r) \frac{|\log r|}{r} dr < \infty.$$

The *Lyapunov exponent* of  $(\Lambda, f)$  at  $x \in \Lambda$  is defined as

$$L(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{df^n}{dx}(x) \right|$$

(if the limit exists). It follows from the ergodic theorem that  $L(x)$  is almost everywhere constant with respect to an  $f$ -invariant ergodic measure and the constant is the integral of  $\log|f'(x)|$  with respect to this measure (called the Lyapunov exponent of the measure). Since there are a lot of invariant ergodic measures, we should get different constants. In this paper, we first describe more quantitatively how the Lyapunov exponents are different by discussing the sizes via Hausdorff dimensions of the following sets:

$$E_\alpha = \{x \in \Lambda : L(x) = \alpha\}.$$

Our study of the Hausdorff dimension  $\dim E_\alpha$  is based on transfer operators. For  $\beta \in \mathbf{R}$ , consider the transfer operator,

$$T_\beta \psi(x) = \sum_{y \in f^{-1}(x)} |f'(y)|^\beta \psi(y), \quad (x \in I).$$

The operator  $T_\beta$  acts on the space  $C(I)$  of continuous functions. The logarithm of its spectral radius is denoted as  $P_{\log|f'|}(\beta)$  or simply  $P(\beta)$  which is called the pressure function.<sup>7,8</sup>

**Theorem 1:** *Suppose  $f$  is an expanding folding map whose derivative satisfies the strong Dini condition. Then  $P(\beta)$  is everywhere differentiable and if  $\alpha = P'(\beta)$  for some  $\beta$ , we have*

$$\dim E_\alpha = \frac{P(\beta) - \alpha\beta}{\alpha}.$$

Now let us look at the scaling function  $s$  of  $(\Lambda, f)$ . It is defined on the dual Cantor set  $\Sigma^*$  (see Ref. 1, p. 9) which is composed of sequences of the form  $\omega^* = \dots i_n \dots i_1 i_0$  and is equipped with the usual ultrametric. Let  $\theta^*$  be the shift on  $\Sigma^*$  defined as  $\theta^*: \dots i_n \dots i_1 i_0 \mapsto \dots i_n \dots i_1$ . Instead of directly studying  $f$ , we shall study its inverses. Let  $g_0$  and  $g_1$  be the left and right inverse branches restricted on  $I$  of  $f$ . By the above hypothesis made on  $f$ , the maps  $g_0: I \rightarrow g_0(I)$  and  $g_1: I \rightarrow g_1(I)$  are  $C^1$  diffeomorphisms. For a finite sequence  $\omega_n = j_0 i_1 \dots j_{n-1}$  (read from left to right) of symbols  $\{0, 1\}$ , let

$$g_{\omega_n} = g_{j_0} \circ g_{j_1} \circ \dots \circ g_{j_{n-1}}, \quad I_{\omega_n} = g_{\omega_n}(I).$$

Let  $\omega_n^* = i_{n-1} \dots i_0$  denote the same  $\omega_n$  but read from right to left. The *scaling function*  $s$  of  $(\Lambda, f)$  is defined as

$$s(\omega^*) = \lim_{n \rightarrow \infty} s(\omega_n^*), \quad (\omega^* = \dots i_{n-1} \dots i_1 i_0 \in \Sigma^*)$$

(if the limit exists for every  $\omega^*$ ), where

$$s(\omega_n^*) = \frac{|I_{i_{n-1} \dots i_1 i_0}|}{|I_{i_{n-1} \dots i_1}|}$$

( $|J|$  denoting the length of an interval  $J$ ). It will be seen (Theorem 3) that the scaling function is well defined when  $f'$

satisfies the Dini condition. Note that  $s$  is a  $C^1$ -invariant (refer to Ref. 1). For a scaling function  $s$  defined on  $\Sigma^*$ , we define

$$\text{var}_n(s) = \sup\{|s(\omega^*) - s(\omega^{*'})| : \omega_n^* = \omega_n^{*'}\}$$

as the  $n$ -variation of  $s$ . We say a scaling function is of *summable variation* if

$$\sum_{n=1}^{\infty} \text{var}_n(s) < \infty.$$

Furthermore, we prove that  $s$  is of *summable variation* if  $f'$  satisfies the strong Dini condition (Corollary 1).

We define the *dual Lyapunov exponent* of  $(\Lambda, f)$  at  $\omega^* \in \Sigma^*$  as

$$L^*(\omega^*) = - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log s(\theta^{*k} \omega^*)$$

(if the limit exists). Our purpose is to study the sizes of the following sets:

$$E_\alpha^* = \{\omega^* \in \Sigma^* : L^*(\omega^*) = \alpha\}.$$

We prove the following theorem:

**Theorem 2:** *Suppose  $f$  is an expanding folding map whose derivative satisfies the strong Dini condition. If  $\alpha = P'(\beta)$  for some  $\beta$ , we have*

$$\dim E_\alpha^* = \frac{P(\beta) - \alpha\beta}{\log 2}.$$

We call  $\cup_\alpha E_\alpha^*$  (resp.  $\cup_\alpha E_\alpha$ ) the multifractal decomposition of  $\Sigma^*$  (resp.  $\Lambda$ ) with respect to dual Lyapunov exponents (resp. Lyapunov exponents). They are  $C^1$  invariants. Theorems 1 and 2 say that Hausdorff dimensions of  $E_\alpha^*$  and  $E_\alpha$  are well calculated, that is, the dual Lyapunov exponent and the Lyapunov exponent for an expanding  $C^1$  folding map whose derivative satisfies the strong Dini condition are multifractal functions.

## II. DINI CONDITION, STRONG DINI CONDITION, AND DISTORTION

For  $n \geq 0$ , denote

$$\mathcal{I}_n = \{I_{\omega_n} : \omega_n = j_0 \dots j_{n-1} \in \{0, 1\}^n\}, \quad \delta_n = \max_{I \in \mathcal{I}_n} |I|.$$

Consider the space  $\Sigma$  of infinite sequences  $\omega = j_0 j_1 \dots j_{n-1} \dots$  of 0's and 1's. For  $\omega = j_0 j_1 \dots \in \Sigma$  and an integer  $n \geq 1$ , we write  $\omega_n = j_0 \dots j_{n-1}$ . Let  $\theta: \Sigma \rightarrow \Sigma$  be the shift map defined by  $(\theta\omega)_n = j_{n+1}$ . Let  $u_j: \Sigma \rightarrow \Sigma$  ( $j=0, 1$ ) be the inverses of  $\theta$ , which can be defined by  $u_j \omega = j\omega$ .

*Lemma 1:* *Suppose  $f$  is a folding map and suppose  $\lim_{n \rightarrow \infty} \delta_n = 0$ . Then, the limit*

$$\pi(\omega) = \lim_{n \rightarrow \infty} g_{\omega_n}(x), \quad (\forall x \in I)$$

*exists and is independent of  $x$ . The function  $\pi: \Sigma \rightarrow \Lambda$  is continuous and*

$$\pi \circ \theta = f \circ \pi, \quad \pi \circ u_j = g_j \circ \pi \quad (j=0, 1).$$

*Proof:* Note that  $\{I_{\omega_n}\}$  is a sequence of embedded intervals whose length tends to zero. So,  $\bigcap_{n=0}^{\infty} I_{\omega_n} = \{\bar{x}\}$  for some  $\bar{x}$ . Since  $g_{\omega_n}(x) \in I_{\omega_n}$  (for all  $x \in I$ ), we get the existence of the limit and  $\pi(\omega) = \bar{x}$ . Recall that  $d(\omega, \omega') = 2^{-n}$ , where  $n = \sup\{m: \omega_m = \omega'_m\}$ . If  $d(\omega, \omega') = 2^{-n}$ , we have

$$\pi(\omega) = g_{\omega_n}(z), \quad \pi(\omega') = g_{\omega_n}(z')$$

for some  $z, z' \in I$ . So,  $|\pi(\omega) - \pi(\omega')| \leq |I_{\omega_n}| \leq \delta_n$ . To get the last equalities in the lemma, it suffices to check them by using the definition of  $\pi$  and the fact that  $f^{\circ} g_j$  is the identity of the interval  $I$  for  $j=0$  or  $1$ .  $\square$

It follows from the last lemma that we have the following relations:

$$\Lambda = \pi(\Sigma), \quad \Lambda = g_0(\Lambda) \cup g_1(\Lambda).$$

The first relation says that  $\Lambda$  is the image of  $\Sigma$  under  $\pi$ . The second relation means that  $\Lambda$  is a self-similar set (see Ref. 9).

*Lemma 2:* Suppose  $f$  is a  $C^1$  folding map. For any finite sequence  $\omega_n$  of 0's and 1's of length  $n$ ,  $g_{\omega_n} \circ f^n$  is the identity on  $I_{\omega}$ ,  $f^n \circ g_{\omega_n}$  is the identity on  $I$ , and we have

$$\frac{df^n}{dx}(g_{\omega_n}(y)) = \frac{1}{g'_{\omega_n}(y)} \quad (\forall y \in I).$$

*Proof:* Immediate.  $\square$

If  $f'$  satisfies the Dini condition, we can then define

$$\Omega_{f'}(t) = \int_0^t \frac{\omega_{f'}(r)}{r} dr.$$

*Lemma 3 (naive distortion):* Suppose  $f$  is an expanding  $C^1$  folding map whose derivative satisfies the Dini condition and has expanding constant  $\lambda > 1$ . Then for any  $x, y \in I$  and any  $\omega_n = j_0 j_1 \dots j_{n-1}$ , we have

$$\left| \frac{g'_{\omega_n}(x)}{g'_{\omega_n}(y)} \right| \leq \exp[C \Omega_{f'}(|x-y|)],$$

where  $C = (1/\lambda \log \lambda) > 0$  (which is a constant independent of  $x, y$ , and  $\omega_n$ ).

*Proof:* Let  $x_n = g_{\omega_n}(x)$  and  $y_n = g_{\omega_n}(y)$  and let  $x_{n-i} = f^i(x_n)$  and  $y_{n-i} = f^i(y_n)$  for  $0 \leq i \leq n$ . Note that  $x_0 = x = f^n(x_n), y_0 = y = f^n(y_n)$ . By Lemma 2, we have

$$\left| \frac{g'_{\omega_n}(x)}{g'_{\omega_n}(y)} \right| = \left| \frac{(f^n)'(y_n)}{(f^n)'(x_n)} \right|.$$

It follows that

$$\begin{aligned} \log \left| \frac{g'_{\omega_n}(x)}{g'_{\omega_n}(y)} \right| &\leq \left| \sum_{i=0}^{n-1} \log |f'(y_{n-i})| - \log |f'(x_{n-i})| \right| \\ &\leq \frac{1}{\lambda} \sum_{i=0}^{n-1} |f'(y_{n-i}) - f'(x_{n-i})| \\ &\leq \frac{1}{\lambda} \sum_{i=0}^{n-1} \omega_{f'}(y_{n-i} - x_{n-i}). \end{aligned}$$

Note that  $|y_{n-i} - x_{n-i}| \leq \lambda^{-(n-i)} |x - y|$ . The last sum is bounded by

$$\begin{aligned} \sum_{i=0}^{n-1} \omega_{f'}(\lambda^{-(n-i)} |x-y|) &\leq \int_0^{\infty} \omega_{f'}(\lambda^{-t} |x-y|) dt \\ &= \frac{1}{\log \lambda} \int_0^{|x-y|} \frac{\omega_{f'}(t)}{t} dt. \end{aligned}$$

$\square$

*Theorem 3:* Suppose  $f$  is an expanding  $C^1$  folding map whose derivative satisfies the Dini condition. Then the scaling function  $s$  exists and satisfies

$$\text{var}_n(s) \leq C \Omega_{f'}(\delta_n),$$

where  $C > 0$  is a constant.

*Proof:* For  $\omega^* = \dots i_n i_{n-1} \dots i_0 \in \Sigma^*$ , let  $\omega_n^* = i_{n-1} \dots i_0$ . If  $m > n$ , we have

$$\begin{aligned} |s(\omega_m^*) - s(\omega_n^*)| &= \left| \frac{|I_{i_{m-1} \dots i_1 i_0}|}{|I_{i_{m-1} \dots i_1}|} - \frac{|I_{i_{n-1} \dots i_1 i_0}|}{|I_{i_{n-1} \dots i_1}|} \right| \\ &= \left| \frac{|g_{i_{m-1} \dots i_n}(I_{i_{n-1} \dots i_1 i_0})|}{|g_{i_{m-1} \dots i_n}(I_{i_{n-1} \dots i_1})|} - \frac{|I_{i_{n-1} \dots i_1 i_0}|}{|I_{i_{n-1} \dots i_1}|} \right| \\ &= \left| \frac{|g'_{i_{m-1} \dots i_n}(\xi_{m,n})|}{|g'_{i_{m-1} \dots i_n}(\eta_{m,n})|} - 1 \right| \cdot \frac{|I_{i_{n-1} \dots i_1 i_0}|}{|I_{i_{n-1} \dots i_1}|}, \end{aligned}$$

where  $\xi_{m,n}$  and  $\eta_{m,n}$  are some two points in  $I_{i_{m-1} \dots i_1}$ . By the above naive distortion lemma, we have

$$\left| \frac{|g'_{i_{m-1} \dots i_n}(\xi_{m,n})|}{|g'_{i_{m-1} \dots i_n}(\eta_{m,n})|} - 1 \right| \leq C \Omega_{f'}(\xi - \eta) \leq C \Omega_{f'}(\delta_n),$$

where  $\xi = f^{m-n}(\xi_{m,n})$  and  $\eta = f^{m-n}(\eta_{m,n})$  are some two points in  $I_{i_{n-1} \dots i_1}$ . Thus we have proved that

$$|s(\omega_m^*) - s(\omega_n^*)| \leq C \Omega(\delta_n), \quad (m > n).$$

Since  $\delta_n \rightarrow 0$ ,  $\{s(\omega_n^*)\}$  is a Cauchy sequence. The existence of the scaling function  $s(\omega)$  is proved. Letting  $m \rightarrow \infty$  in the above inequality, we get

$$|s(\omega^*) - s(\omega_n^*)| \leq C \Omega(\delta_n).$$

Now it is easy to get the claimed result by replacing  $C$  by  $2C$ .  $\square$

Moreover, we have that

*Corollary 1:* Suppose  $f$  is an expanding  $C^1$  folding map whose derivative satisfies the strong Dini condition. Then the scaling function  $s$  is of summable variation.

*Proof:* According to Theorem 3,

$$\int_0^1 \Omega_{f'}(r) \frac{dr}{r} < \infty$$

implies that  $s$  is of summable variation. Now let us see how to control  $\int_0^1 \Omega_{f'}(r) (dr/r)$  from the strong Dini condition. From

$$\int_0^1 \Omega_{f'}(r) \frac{dr}{r} = \lim_{\epsilon \rightarrow 0} \Omega_{f'}(\epsilon) \log \epsilon + \int_0^1 \omega_{f'}(r) \frac{|\log r|}{r} dr$$

and

$$\Omega_{f'}(\epsilon) \log |\epsilon| \leq \int_0^\epsilon \omega_{f'}(r) \frac{|\log r|}{r} dr,$$

we see that  $\int_0^1 \Omega_{f'}(r) (dr/r) < \infty$  if

$$\int_0^1 \omega_{f'}(r) \frac{|\log r|}{r} dr < \infty.$$

This completes the corollary. □

We have seen that  $\Lambda$  is related to  $\Sigma$  via  $\pi$ . Define the following transfer operator on  $C(\Sigma)$ :

$$S_\beta \varphi(\omega) = q_\beta(u_0(\omega)) \varphi(u_0(\omega)) + q_\beta(u_1(\omega)) \varphi(u_1(\omega)),$$

where

$$q_\beta(\omega) = |g'_i(\pi\omega)|^{-\beta} \quad \text{if } \omega \in u_j(\Sigma).$$

Now we see that  $S_\beta$  is associated to  $T_\beta$  in the following way. For simplicity, we write simply  $T = T_\beta$  and  $S = S_\beta$  in the following theorem.

**Theorem 4:** *Suppose  $f$  is an expanding  $C^1$  folding map with expanding constant  $\lambda > 1$ , i.e.,  $|f'(x)| \geq \lambda$  and whose derivative satisfies the Dini condition. Then we have*

- (1) *The operators  $S$ ,  $T|_{C(I)}$  and  $T|_{C(\Lambda)}$  have the same spectral radius  $\rho$ .*
- (2) *There are unique probability measure  $\nu$  and  $\mu$  respectively supported by  $\Sigma$  and  $\Lambda$  such that  $S^* \nu = \rho \nu$ ,  $T^* \mu = \rho \mu$ , where  $S^*$  and  $T^*$  are respectively the adjoint operator of  $S$  and  $T$ .*
- (3)  *$\mu$  is the image of  $\nu$  under  $\pi$ .*
- (4)  *$\mu$  has the following Gibbs property: there is a constant  $C > 0$  such that*

$$C^{-1} \leq \frac{\mu(I_{\omega_n})}{\rho^{-n} G_{\omega_n}(x)} \leq C$$

holds for any sequence  $\omega_n = j_0 \cdots j_{n-1}$ , any  $x \in I$ , where

$$G_{\omega_n}(x) = \prod_{k=0}^{n-1} |g'_{j_k}(g_{j_{k+1}} \cdots j_n(x))|^{-\beta}.$$

*Proof:* The proof is based on Lemma 1, the naive distortion lemma and the Ruelle transfer operator theorem on symbolic space (see also Refs. 7,8,10). Its details are given in Ref. 11.

### III. MULTIFRACTAL ANALYSIS I: THE PROOF OF THEOREM 1

Let  $x = \pi\omega$  with  $\omega = j_0 j_1 \cdots$ . Let  $\omega_n = j_0 \cdots j_{n-1}$ . By Lemma 2, we have

$$(f^n)'(x) = \frac{1}{g'_{\omega_n}(f^n(x))} = \frac{1}{\prod_{k=0}^{n-1} g'_{j_k}(g_{j_{k+1}} \cdots j_n(\xi))},$$

where  $\xi = f^n(x)$ . By the naive distortion lemma, we have

$$L(x) = - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |g'_{j_k}(g_{j_{k+1}} \cdots j_n(\xi))|$$

$$= - \lim_{n \rightarrow \infty} \frac{\log |I_{\omega_n}|}{n}.$$

The limit does not depend upon  $\xi$ . So take  $\xi = \pi(\theta^n \omega)$ . We have

$$g'_{j_k}(g_{j_{k+1}} \cdots j_n(\xi)) = g'_{j_k}(\pi \theta^k \sigma) = g'_{j_k}(\pi(j_k j_{k+1} \cdots)).$$

Now let us define a function  $q$  on  $\Sigma$  as

$$q(\omega) = \log |g'_{j_0}(\pi\omega)|$$

for  $\omega = j_0 j_1 \cdots j_n \cdots$ . Then we have that

$$L(x) = - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} q(\theta^k \omega) = - \lim_{n \rightarrow \infty} \frac{\log |I_{\omega_n}|}{n}.$$

Note that the middle term in the last equation is in the form of ergodic means of the Birkhoff ergodic theorem. Therefore, for any  $\theta$  (as well as  $f$ ) invariant ergodic measure  $\nu$ ,  $L(\pi\omega)$  exists and equals a constant for almost every  $\nu$  point. The constant equals  $-\int_\Sigma q(\omega) d\nu$ .

Let  $\mu_\beta$  be the Gibbs measure of the operator  $T_\beta$ . By the Gibbs property of  $\mu_\beta$ , we have

$$\lim_{n \rightarrow \infty} \frac{\log \mu_\beta(I_n(x))}{\log |I_n(x)|} = -\beta - P(\beta) \lim_{n \rightarrow \infty} \frac{n}{\log |I_n(x)|},$$

where  $I_n(x) = I_{\omega_n}$  means the interval in  $\mathcal{I}_n$  containing  $x$ . For  $\mu_\beta$ -almost everywhere we also have

$$\lim_{n \rightarrow \infty} \frac{\log |I_{\omega_n}(x)|}{n} = \int_\Sigma q(\tau) d\mu_\beta(\tau) = -P'(\beta)$$

(if  $P(\beta)$  is differentiable at  $\beta$ ). So, if  $\alpha = P'(\beta)$ , we have  $\mu_\beta(E_\alpha) = 1$ . On the other hand, if  $x \in E_\alpha$ ,

$$\lim_{n \rightarrow \infty} \frac{\log \mu_\beta(I_n(x))}{\log |I_n(x)|} = -\beta + P(\beta)/\alpha = \frac{P(\beta) - \alpha\beta}{\alpha}.$$

Now the dimension formula follows from the Billingsley theorem.<sup>12</sup> We postpone the proof of differentiability of  $P(\beta)$  at the end of the proof of Theorem 2.

### IV. MULTIFRACTAL ANALYSIS II: THE PROOF OF THEOREM 2

The following result is well-known in symbolic dynamical systems (see Ref. 13). Let  $\Phi$  be a real valued continuous function defined on  $\Sigma^*$ . Let  $P_\Phi(\beta)$  be the pressure of the following transfer operator:

$$T_\Phi \varphi = e^{\beta\Phi(u_0^* \omega^*)} \varphi(u_0^* \omega^*) + e^{\beta\Phi(u_1^* \omega^*)} \varphi(u_1^* \omega^*),$$

where  $u_i^*$  for  $i=0$  and  $1$  are the inverses of the dual shift  $\theta^*$  and can be defined as  $u_i^*(\omega^*) = \omega^* i$ . If  $P_\Phi$  is differentiable at some point  $\beta$ , by noting that  $\alpha = P'_\Phi(\beta)$ , we have

$$\begin{aligned} \dim \left\{ \omega^* \in \Sigma^* : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Phi(\theta^{*k} \omega^*) = \alpha \right\} \\ = - \frac{1}{\log 2} [\alpha \beta - P_\Phi(\beta)]. \end{aligned}$$

It suffices to apply this to  $\Phi(\omega^*) = -\log s(\omega^*)$ . But we have to show that  $P_{-\log s}(\beta) = P(\beta)$ , the pressure function associated with the operators  $T_\beta$ , to show that  $P(\beta)$  is everywhere differentiable. Note that the spectral radius of  $T_\beta$  can be expressed as

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{x \in \text{Fix}(f^n)} \prod_{k=0}^{n-1} |f'(f^k(x))|^\beta \right) \\ = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \text{Fix}(f^n)} |(f^n)'(x)|^\beta \\ = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega^* \in \text{Fix}(\theta^{*n})} \prod_{k=0}^{n-1} |s(\theta^{*k} \omega^*)|^{-\beta}. \end{aligned}$$

(See Ref. 1 pp. 276 and 76 for the above two equalities). The last quantity is just the spectral radius of the transfer operator associated with the potential function  $-\beta \log s$ . Thus we have proved the equality  $P(\beta) = P_{-\log s}(\beta)$ .

From Corollary 1, the scaling function under the assumption of Theorem 2 is of summable variation. It is well known from Thermodynamical Formalism for symbolic dynamical systems that for a potential being of summable variation, there is a unique equilibrium state corresponding to it.<sup>7,10,14</sup> It is also known that the uniqueness in equilibrium states for a given potential function implies the differentiability of the pressure function (see Ref. 15). Therefore,  $P(\beta)$  and  $P_{-\log s}(\beta)$  are both differentiable at  $\beta$ . Thus we proved Theorem 2 and completed the last argument of the proof of Theorem 1.

**V. SOME REMARKS**

(1) Theorem 1 and Theorem 2 provide us the relation,

$$\dim E_\alpha = \frac{\log 2}{\alpha} \dim E_\alpha^*.$$

Note that metrics are different for the two dimensions. The reason for the factor  $\log 2/\alpha$  would be that the shift  $\theta$  has a constant ‘Lyapunov exponent’  $\log 2$  and that the map  $f$  restricted on  $E_\alpha$  has a constant Lyapunov exponent  $\alpha$ .

- (2) If  $q(\omega) = |g'_{j_0}(\pi\omega)|$  is not constant on  $\Sigma$ ,  $\log q(\omega)$  can not be written as  $u(\theta\omega) - u(\omega) + C$  for any continuous function  $u$  and any constant  $C$ .<sup>16</sup> Then the dimension function in Theorem 2 is strictly concave on some interval.<sup>8</sup>
- (3) From a result in Ref. 17, we know that if  $f$  is just continuously differentiable, the set  $L_f$  of all possible

Lyapunov exponents is just the set of integrals  $\int \log |f'| d\mu$ , where  $\mu$  varies in the set of  $f$ -invariant measures.<sup>18</sup>

- (4) The method in this paper also works for any  $C^1$  expanding Markov maps satisfying the Dini or Strong Dini condition. See Ref. 1 for Markov maps and their scaling functions.

**ACKNOWLEDGMENTS**

The final version of this paper was completed when the second author (Y.J.) visited the Faculté de Mathématiques et Informatique at Université de Picardie Jules Verne in France. During this research the first author (A.F.) visited the Institute of Mathematical Science (IMS) at the Chinese University of Hong Kong and the second author (Y.J.) visited the Institute for Mathematical Research (FIM) at ETH-Zurich in Switzerland. The authors would like to thank these institutes for their hospitality and support. The second author (Y.J.) is supported in part by NSF grants and PSC-CUNY awards.

- <sup>1</sup>Y. Jiang, *Renormalization and Geometry in One-Dimensional and Complex Dynamics, Advanced Series in Nonlinear Dynamics* (World Scientific, Singapore, 1996), Vol. 10.
- <sup>2</sup>D. Sullivan, ‘‘Differentiable structure on fractal like sets determined by intrinsic scaling functions on dual Cantor sets,’’ in *Proceedings of the Symposia in Pure Mathematics*, Vol. 48, 1988 (unpublished).
- <sup>3</sup>P. Coullet and C. Tresser, ‘‘Itération d’endomorphismes et groupes de renormalisation,’’ *C. R. Acad. Sci. Paris Ser. A-B* **287**, A577–A580 (1978); *J. Phys. Colloq.* **39**, C5 (1978), 25–28; supplément au 39:8 (1975).
- <sup>4</sup>M. Feigenbaum, *Scaling Function Dynamics*, Lecture notes prepared by Zoltan Kovács, NATO Adv. Sci. Inst. Ser. B Phys., Lake Como, 1990 (Plenum, New York, 1991), Vol. 280, pp. 1–23.
- <sup>5</sup>M. Feigenbaum, ‘‘Quantitative universality for a class of nonlinear transformations,’’ *J. Stat. Phys.* **19**, 25–52 (1978).
- <sup>6</sup>H. Hirsch and C. Pugh, ‘‘Stable manifolds and hyperbolic sets,’’ *Proc. Symp. Pure Math* **14**, 133–163 (1970).
- <sup>7</sup>R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. LNM 470* (Springer, Berlin, 1975).
- <sup>8</sup>D. Ruelle, ‘‘Thermodynamic formalism: The mathematical structures of classical equilibrium statistical mechanics,’’ *Ency. Math. Appl.* (Addison–Wesley, Reading, 1978), Vol. 5.
- <sup>9</sup>K. Falconer, *Fractal Geometry* (Wiley, New York, 1990).
- <sup>10</sup>A. H. Fan, ‘‘A proof of the Ruelle operator theorem,’’ *Rev. Math. Phys.* **7**, 1241–1247 (1995).
- <sup>11</sup>A. H. Fan and K. S. Lau, ‘‘Iterated function systems and Ruelle operator,’’ *J. Math. Anal. Appl.* (to be published).
- <sup>12</sup>P. Billingsley, *Ergodic Theory and Information* (Wiley, New York, 1975).
- <sup>13</sup>A. H. Fan, ‘‘Sur les dimensions de mesures,’’ *Stud. Math.* **111**, 1–17 (1994).
- <sup>14</sup>P. Walters, ‘‘Invariant measures and equilibrium states for some mappings which expand distances,’’ *Trans. Am. Math. Soc.* **236**, 121–153 (1978).
- <sup>15</sup>F. Takens and E. Verbitski, ‘‘Multifractal analysis of local entropies for expansive homeomorphisms with specification’’ (preprint, 1998).
- <sup>16</sup>A. H. Fan and K. S. Lau, ‘‘Asymptotic behavior of multiperiodic functions  $G(x) = \prod_{n=1}^\infty g(x/2^n)$ ,’’ *J. Fourier Anal. Appl.* **4**, 129–150 (1998).
- <sup>17</sup>A. H. Fan and D. J. Feng, ‘‘Ergodic means, dimensions, and entropies’’ (in preparation).
- <sup>18</sup>Y. Jiang, ‘‘Asymptotic differentiable structure on Cantor set,’’ *Commun. Math. Phys.* **155**, 503–509 (1993).