

# THE JULIA SET OF FEIGENBAUM QUADRATIC POLYNOMIAL

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(In memory of Professor Liao Shantao)

## Abstract

In this paper, we give a proof of Sullivan's complex bounds for the Feigenbaum quadratic polynomial and show that the Julia set of the Feigenbaum quadratic polynomial is connected and locally connected.

## 1. Introduction

Quadratic polynomials  $Q_c = z^2 + c$ , where  $c$  is a complex parameter, define a rich family of dynamical systems on the Riemann sphere  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . For any parameter  $c \in \mathbb{C}$ , the infinity  $\infty$  is a super attracting fixed point in the sense that the derivative of  $Q_c$  at  $\infty$  is zero. The *attracting basin*  $F_\infty(Q_c)$  of  $Q_c$  at  $\infty$  is defined to be the set of all points converging to the infinity under the iterates of  $Q_c$ . And the boundary of  $F_\infty(Q_c)$  is called the *Julia set* of  $Q_c$ . It is easy to see that the Julia set  $J(Q_c)$  is connected if and only if the forward orbit of the critical point 0 under  $Q_c$  is bounded. The so-called *Mandelbrot set*  $M$  is defined to be the set of the parameter  $c$  for which the Julia set  $J(Q_c)$  is connected. Douady and Hubbard proved that  $M$  is connected [DH1]. The study of further regularities of  $M$  becomes very important since the local connectivity of  $M$  implies the open density of the hyperbolic ones in this family. Here by a *hyperbolic* polynomial we mean that all critical points are in the basins of attracting periodic cycles. For a survey, see [Mc1]. An apriori problem is to study the local connectivity of the Julia set  $J(Q_c)$  when  $c \in M$ . From a theorem in [DH2], the Julia set  $J(Q_c)$  is connected and locally connected if the critical point of  $Q_c$  is either contained in a periodic attracting basin or a parabolic attracting basin or lands on a repelling periodic orbit. This implies

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that  $J(Q_c)$  is locally connected if the parameter  $c$  is in the interior of  $M$ . When  $c$  is on the boundary  $\partial M$  of  $M$ , the local connectivity of  $J(Q_c)$  is uncertain. There are a lot of  $c$ 's on  $\partial M$  for which  $J(Q_c)$  is locally connected, and also  $c \in \partial M$  for which  $J(Q_c)$  is not locally connected. More precisely, Yoccoz showed that  $J(Q_c)$  is locally connected if  $c \in \partial M$ ,  $Q_c$  has neither Cremer periodic points nor Siegel disks, and  $Q_c$  is not *infinitely renormalizable* (for the references on this matter, see [Mi2] or [Hub] or [Ji4]). Sullivan and Douady proved that  $J(Q_c)$  is never locally connected if it contains a Cremer periodic point (see [Su1] or [Do]), and Peterson gave the examples of  $Q_c$  which has a Siegel disk and a locally connected Julia set [Pe]. And Douady and Hubbard gave an example of an infinitely renormalizable  $Q_c$  whose Julia set is connected but not locally connected (this unpublished example was described by Milnor in [Mi2]). A natural question arises: Is there  $c \in \partial M$  such that  $Q_c$  is *infinitely renormalizable* (see section 2 for the definition) and  $J(Q_c)$  is connected and locally connected? In 1992, one (YJ) of us outlined an idea in [Ji2] to answer this question by assuming some presumptions such as complex bounds and unbranch condition. This result is published in [Ji3] (see also [Ji4]). In 1993, we found the first positive example (see [JH] and refer to [Ji4]) in the course of studying Sullivan's result announced at that time, i. e., Sullivan's complex bounds for the renormalizations of the Feigenbaum quadratic polynomial [Su2]. The purpose of this note is to give a historic remark, a proof of Sullivan's complex bounds without using Sullivan's sector lemma, and a complete proof of the result of [JH], that is,

**Theorem 1.1** *The Feigenbaum quadratic polynomial  $Q_F$  has a connected and locally connected Julia set.*

A precise definition of the Feigenbaum quadratic polynomial  $Q_F$  and its infinitely renormalizability are given in section 2. We first give, in section 3, the statement of so-called Sullivan's complex bounds for the renormalizations of  $Q_F$ , and then in section 5 we provide a direct proof of this geometric property by using some ideas in [LY, LS] instead of applying Sullivan's sector lemma. A key step in the proof of Theorem 1.1 is to show that for each renormalization domain  $U_n$ , there exists a central puzzle piece  $W_n$  (the one containing the critical point) such that  $W_n \subset U_n$ . Sullivan's complex bounds of the renormalizations of  $Q_F$  implies that the renormalization domains  $U_n$  shrink to a point as  $n \rightarrow \infty$ . Therefore the puzzle pieces  $W_n$  shrink to a point, and then  $J(Q_F)$  is locally connected at the critical point 0. Similar arguments show the local connectivity of  $J(Q_F)$  at the points which converge to the critical orbit under the iterates of  $Q_F$ , and the proof of the local connectivity of  $J(Q_F)$  at the other points is simple. All of these are arranged in sections 4. We finish this note by mentioning a few open questions about  $J(Q_F)$  in the last section.

**Remark 1.1** One can find a proof of Sullivan’s complex bounds by using Sullivan’s sector lemma as well as a proof of Sullivan’s sector lemma in [Far, Ji4]. For the initial idea of Sullivan’s complex bounds and an outlined proof, one can find in [Su2, MS].

**Remark 1.2** It is worth to mention that the method in our proof contains an important improvement of the method used by Yoccoz in his work (refer to [Ji3, Ji4]), that is, Yoccoz puzzle pieces are pullbacks of some fixed initial puzzle pieces selected at the beginning, but the initial puzzle pieces in our method will change as the renormalization level changes. Similar ideas were used in [Ji1]. More general puzzles for infinitely renormalizable quadratic polynomials were introduced and studied initially in [Ji3], which lead to a more general result about the local connectivity of infinitely renormalizable quadratic polynomials. For a survey on these more general puzzles, see [Ji5]. For an application of these general puzzles to the study of the local connectivity of the Mandelbrot set, see [Ji6]. After the preprints [Ji3, JH] were circulated and the second author talked to people privately, they realized that it is possible to show that  $J(Q_c)$  is locally connected for any  $c \in \partial M \cap \mathbb{R}$  by just showing the complex bounds of renormalizations for all types of infinitely renormalizable quadratic real polynomials. This has been carried out in [LS, GS, LY].

**Remark 1.3** The Julia set of the Feigenbaum quadratic polynomial also supplies an example of the fractal sets, produced by dynamics, which can be understood in certain ways. Some further investigation of  $J(Q_F)$  can be found in [Hu].

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## 2. Real Dynamics of the Feigenbaum quadratic polynomial

Consider the family of real quadratic polynomials

$$Q_r(x) = x^2 + r,$$

where  $r$  is a real parameter. Let  $r_n$  denote the parameter  $r$  such that 0 is a periodic point of period  $2^n$  under  $Q_r$ , where  $n \in \{0\} \cup \mathbb{N}$ . The bounded monotone decreasing sequence  $\{r_n\}_{n=0}^{\infty}$  converges to a limit  $r_{\infty}$ , and numerical data indicate  $r_{\infty} = -1.401155 \dots$ . We call the quadratic polynomial  $Q_{r_{\infty}}$  the *Feigenbaum* quadratic polynomial because of his remarkable discovery of the asymptotic geometric convergence of  $r_n$  to  $r_{\infty}$  in 1978 [Fe1, Fe2]. We simply denote it by  $Q_F$ .

The map  $Q_F$  has two fixed points. Let  $p$  denote the fixed point where  $Q_F$  has a positive derivative. Clearly  $Q_F$  maps the interval  $[-p, p]$  into itself, and hence the orbit of the critical point 0 under  $Q_F$  lie in the interval  $[-p, p]$ , and hence the Julia set  $J(Q_F)$  is connected ([Ju] [Fat]). By Sullivan's nonwandering domain theorem [Su3], the Julia set  $J(Q_F)$  is equal to the complement of  $F_{\infty}(Q_F)$  in  $\hat{\mathbb{C}}$ , and hence  $J(Q_F) \cap \mathbb{R} = [-p, p]$ .

In the following, we summarize some known properties of the dynamics of  $Q_F$  on the real line  $\mathbb{R}$ , see [Gu1, Gu2], [Mis], or [Su2] for other references.

(1) The period  $n$  of a real periodic point  $x$  under  $Q_F$  is called a *real period* of  $Q_F$ . Then the set  $RP(Q_F)$  of all real periods of  $Q_F$  equals to the set

$$RP(Q_F) = \{2^n : n = 0, 1, 2, \dots, m, \dots\}.$$

Furthermore, the Feigenbaum quadratic polynomial has exactly one real periodic orbit of period  $2^n$  for each  $n \in \mathbb{N}$ , and every real periodic point of  $Q_F$  is a repelling periodic point of period  $2^n$  for some  $n \in \mathbb{N} \cup \{0\}$ . They are called real period-doubling periodic points.

(2) Given any integer  $n \in \mathbb{N}$ , let  $p_n$  be the real periodic point of period  $2^{n-1}$  under  $Q_F$  closest to the critical point 0 and let  $I_0^n$  denote the interval  $[-|p_n|, |p_n|]$ . Then the  $2^n$ -th iterate  $Q^{2^n}$  of  $Q_F$  maps the interval  $I_0^n$  into itself with one critical point at 0, and hence the restriction  $Q_F^{2^n}|_{I_0^n} : I_0^n \rightarrow I_0^n$  is a quadratic-like map and is called a *renormalization* of  $Q_F$ . And  $Q_F$  is said to be *infinitely renormalizable* in the sense that there exists an infinite sequence  $\{I_0^n\}_{n=1}^{\infty}$  of nested intervals centered at the critical point 0 satisfying that  $|I_0^n| \rightarrow 0$  as  $n \rightarrow \infty$ , and

$$Q_F^{2^n}|_{I_0^n} : I_0^n \rightarrow I_0^n$$

is a quadratic-like map for any  $n \in \mathbb{N}$ .

(3) Let  $I_i^n = Q_F^i(I_0^n)$ ,  $1 \leq i \leq 2^n - 1$ . For each  $n \in \mathbb{N}$ , the intervals  $I_i^n$ ,  $0 \leq i \leq 2^n - 1$ , have no pairwise intersection except at their ends. In fact,  $\bigcap_{n=1}^{\infty} \bigcup_{i=0}^{2^n-1} I_i^n$  is the closure  $K$  of the forward orbit of the critical point 0, i.e.,

$$\bigcap_{n=1}^{\infty} \bigcup_{i=0}^{2^n-1} I_i^n = \overline{\{Q_F^n(0) : n = 0, 1, 2, \dots, n, \dots\}}.$$

And hence  $K$  is a Cantor set, called the *critical orbital Cantor set attractor* of  $Q_F$  since almost every point of the interval  $[-p, p]$ , with respect to the Lebesgue

measure, converges to the Cantor set  $K$  under the iterates of  $Q_F$ . Furthermore, each point of  $K$  has a dense forward orbit in  $K$  under the iterates of  $Q_F$ .

(4) **Asymptotic constant rescaling ratios:** The ratios of the lengths of two adjacent intervals  $I_0^n$ 's converge to a constant  $> 0$  and  $< 1$ . Numerical data show

$$\lim_{n \rightarrow \infty} \frac{|I_0^{n+1}|}{|I_0^n|} = 0.3995 \dots$$

It implies that the lengths of the renormalization intervals  $I_0^n$  converge to zero geometrically fast as  $n \rightarrow \infty$ . Besides Feigenbaum, this phenomenon was also independently discovered by Coulet and Tresser around same time in Europe (see [CT]).

(5) The real preimages of the critical point 0, i.e., the set

$$\{x \in \mathbb{R} : Q_F^n(x) = 0 \text{ for some } n \in \mathbb{N} \cap \{0\}\}$$

is dense in the interval  $[-p, p]$ .

### 3. Sullivan's complex bounds of renormalizations

From now on we simply denote by  $Q$  the Feigenbaum quadratic polynomial  $Q_F$ . Clearly the renormalizations  $F_n = Q^{2^n}|_{I_0^n} : I_0^n \rightarrow I_0^n$ ,  $n \in \mathbb{N}$ , can analytically extend to open domains  $U_n$ , containing  $I_0^n$ , in the complex plane  $\mathbb{C}$ .

Let  $U$  and  $V$  be two simply connected open domains in  $\mathbb{C}$  with  $\bar{U} \subset V$ . There exists a conformal mapping between the annulus  $A = V \setminus \bar{U}$  and a standard annulus  $\{z : 1 < |z| < r\}$  for some  $r > 1$ , where  $r$  is uniquely determined by  $A$  [Ah1]. The *modulus* of  $A$  is defined as  $\frac{1}{\pi} \log r$ .

A *polynomial-like mapping* of degree  $d$  is a triple  $(U, V, F)$ , where  $U$  and  $V$  are two simply connected open domains with  $\bar{U} \subset V$ , and  $F : U \rightarrow V$  is an analytic proper mapping of degree  $d$ . And a polynomial-like mapping of degree 2 is called a *quadratic-like mapping* [DH2].

**Theorem 3.1 (Sullivan's complex bounds [Su2])** *There exists  $\epsilon > 0$  and  $M > 0$  such that for any  $n \in \mathbb{N}$ , the  $n^{\text{th}}$  renormalization*

$$F_n = Q^{2^n}|_{I_0^n} : I_0^n \rightarrow I_0^n$$

*can be analytically extended to a quadratic-like mapping  $F_n : U_n \rightarrow V_n$  satisfying:*

- (1)  $I_0^n \subset U_n$  and the closure  $\bar{U}_n \subset V_n$ ,
- (2) the modulus of  $V_n \setminus \bar{U}_n$  is greater than  $\epsilon$ , and
- (3) the ratio  $|U_n \cap \mathbb{R}|/|I_0^n|$  is bounded from above by  $M$ .

We will first apply this property to prove our main theorem in the next section. A direct proof of Sullivan's complex bounds will be given in section 5 by using some ideas of [LY, LS] instead of applying the sector lemma in [Su2].

## 4. The local connectivity of $J(Q_F)$

Suppose  $F : U \rightarrow V$  is a quadratic-like map with connected Julia set  $J_F$ . Suppose  $m \geq 2$  is an integer. We say  $F : U \rightarrow V$  is  $m$  renormalizable if there are two simply connected domains isomorphic to an open unit disc such that  $F' = F^m : U' \rightarrow V'$  is again a quadratic-like map with connected Julia set  $J_{F'}$ . The quadratic-like map  $F' : U' \rightarrow V'$  is called an  $m$ -renormalization of  $F : U \rightarrow V$ . From the definition, the Julia set  $J_{F'}$  depends on the choice of  $U'$ , but the following lemma proved in [Ji3] assuring that for a given  $m$ ,  $J_{F'}$  is unique if  $F : U \rightarrow V$  is  $m$  renormalizable. This result plays a key role in the study of the local connectivity of the Julia set of an infinitely renormalizable quadratic polynomial.

**Lemma 4.1** *Suppose  $F : U \rightarrow V$  is an  $m$  renormalizable quadratic-like map. Let  $F' : U \rightarrow V'$  and  $F'' : U'' \rightarrow V''$  are two  $m$ -renormalizations of  $F : U \rightarrow V$ . Let  $J_{F'}$  and  $J_{F''}$  be the corresponding Julia sets. Then  $J_{F'} = J_{F''}$ .*

A special version of Lemma 4.1 when  $m = 2^n$  will be raised and shown in the proof of Lemma 4.4, and it is all needed for this note. For interesting readers, please refer to [Ji3] for a proof of Lemma 4.1.

**Lemma 4.2** [Koebe Distortion Lemma [Ah2]] *Let  $F$  be an univalent function defined on the unit disk  $\{z : |z| < 1\}$ . Then for any  $0 < r < 1$ , there exists a constant  $M(r) > 0$  such that  $\frac{1}{M(r)} \leq \frac{|F'(x)|}{|F'(y)|} \leq M(r)$  for any  $x, y \in \{z : |z| \leq r\}$ .*

By applying Riemann mapping theorem [Ah2], it is easy to have

**Corollary 4.1** *For any constant  $\epsilon > 0$ , there exists a constant  $M(r) > 0$  such that if  $f$  is univalent on a simply connected open domain  $V$  and  $U$  is a simply connected open subdomain of  $V$  with the modulus of  $V \setminus \bar{U}$  greater than  $\epsilon$ , then  $\frac{1}{M(r)} \leq \frac{|F'(x)|}{|F'(y)|} \leq M(r)$  for any  $x, y \in \bar{U}$ .*

From now on, we denote  $q(n) = 2^n$ . Let  $F_n : U_n \rightarrow V_n$  be the same as in Sullivan's complex bounds. Clearly  $F_n$  is called a  $q(n)$ -renormalization in Lemma 4.1.

**Lemma 4.3** *Let  $\text{diam}(U_n)$  denote the diameter of  $U_n$ . Then*

$$\text{diam}(U_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Proof.* Let  $D = \{z \in \mathbb{C} : |z| < 1\}$  and  $G_n : D \rightarrow V_n$  be the Riemann mapping. Clearly the modulus of  $D \setminus G_n^{-1}(\bar{U}_n)$  is equal to the modulus of  $V_n \setminus \bar{U}_n$ , which is greater than a constant  $\epsilon > 0$ . Hence there exists  $0 < r < 1$

such that  $G_n^{-1}(U_n) \subset D_r = \{z : |z| < r\}$ . By Koebe distortion theorem, the nonlinearity of  $G_n$  is bounded from below and from above on  $D_r$ , and hence on  $G_n^{-1}(U_n)$ . Therefore  $\text{diam}(U_n)$  is commensurate to  $|U_n \cap \mathbb{R}|$ . Since  $|U_n \cap \mathbb{R}|$  is commensurate to  $|I_0^n|$ ,  $\text{diam}(U_n)$  is commensurate to  $|I_0^n|$ . We know that  $|I_0^n| \rightarrow 0$  as  $n \rightarrow \infty$ , so  $\text{diam}(U_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

We define the *filled-in Julia set* of  $F_n$  as  $K(F_n) = \bigcap_{k=1}^{\infty} F_n^{-k}(U_n)$ . And the boundary of  $K(F_n)$  is called the *Julia set*  $J(F_n)$  of  $F_n$ . In literature, they are called *small (filled-in) Julia sets* of  $Q$ . In our consideration,  $K(F_n) = J(F_n)$ , in another word, the small filled-in Julia set has no interior point. This is easily concluded from Sullivan's nonwandering domain theorem [Su2]. From Lemma 4.3, we have

**Proposition 4.1** *As  $n \rightarrow \infty$ ,  $\text{diam}(J(F_n)) \rightarrow 0$ .*

**Remark 4.1** An alternative proof of this proposition is given in [Ji3] by applying so-called modulus inequality in renormalization proved in [Ji3].

Before we start to prove the local connectivity of  $J(Q_F)$  at the critical point, we need to introduce a few more concepts.

A renormalization  $F_n : U_n \rightarrow V_n$  satisfies *unbranched condition* if

$$\text{Orb}(Q, 0) \cap V_n = \text{Orb}(Q^{q(n)}, 0),$$

where  $\text{Orb}(f, x)$  denotes the orbit of the point  $x$  under the map  $f$ .

In the following, we explain why the renormalizations  $F_n : U_n \rightarrow V_n$  can satisfies unbranched condition and have complex bounds simultaneously.

In the study of the geometric properties of the critical orbital Cantor-set attractor

$$K = \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{2^n-1} I_i^n,$$

one found the following interesting result: if  $I_i^n$  has a space (an interval)  $J$  (which is commensurate with the length of  $I_i^n$ ) in the complement of  $\bigcup_{i=0}^{q(n)-1} I_i^n$  on one side of  $I_i^n$ , then in the next level of renormalization there is a space (an interval)  $J'$  (which is commensurate with the length of  $I_j^{n+1}$ ) in the complement of  $\bigcup_{i=0}^{q(n+1)-1} I_i^{n+1}$  at the other side of  $I_j^{n+1}$  whenever  $I_j^{n+1} \subset I_i^n$ . Therefore we have the following proposition.

**Proposition 4.2** ([Mc1]) *There exists  $\delta > 0$  such that for each  $n \in \mathbb{N}$  there are intervals  $J_n$  and  $J'_n$  on  $\mathbb{R}$  in the complement of  $K$  in  $\mathbb{R}$  connected to the end points of  $I_0^n$  respectively and satisfying*

$$\frac{|J_n|}{|I_0^n|} > \delta \quad \text{and} \quad \frac{|J'_n|}{|I_0^n|} > \delta.$$

**Proposition 4.3** *The renormalizations  $F_n : U_n \rightarrow V_n$  of the Feigenbaum quadratic polynomial can satisfy unbranched condition and have complex bounds simultaneously.*

*Proof.* By Sullivan's complex bounds and Proposition 4.2, there exists  $m > 0$ , independent of  $n$ , such that  $F_n^{-m}(V_n) \cap \mathbb{R}$  is contained in  $J_n \cup I_0^n \cup J'_n$ . Clearly the restriction of  $F_n$  on  $\overline{F_n^{-(m+1)}(V_n)}$  is a polynomial-like mapping with the modulus of  $F_n^{-m}(V_n) \setminus F_n^{-(m+1)}(V_n)$  greater than  $\frac{\epsilon}{2^m}$ . Since the orbit  $\text{Orb}(Q, 0)$  of the critical point is contained in  $K$ , it is easy to check that the unbranched condition is satisfied.

It is easy to see that at the infinity  $\infty$ , the Feigenbaum quadratic polynomial  $Q$  is locally analytically conjugate to the square map  $Q_0(z) = z^2$ , and the conjugation can be extended to the whole attracting basin  $F_\infty(Q)$ . Therefore there exists a holomorphic homeomorphism  $\Phi$  from  $F_\infty(Q)$  onto the unit open disk  $D = \{z : |z| < 1\}$  such that  $\Phi(\infty) = 0$ ,  $\Phi'(\infty) = 1$ , and  $\Phi$  conjugates  $Q|_{F_\infty(Q)} : F_\infty(Q) \rightarrow F_\infty(Q)$  to the square map  $S : D \rightarrow D : z \mapsto z^2$ . The map  $\Phi$  is called the *local normalization* of  $Q$  at the infinity. Given any real number  $0 \leq \theta < 1$ , the preimage under  $\Phi$  of the ray  $\{z : z = \rho e^{2\pi i \theta}, 0 \leq \rho < 1\}$  is called an *external ray* of the Julia set  $J(Q)$ ,  $\theta$  is called the *angle* of the external ray, and it is denoted by  $\theta$ . For any real number  $0 < \rho < 1$ , the preimage under  $\Phi$  of the circle  $\{z : |z| = \rho\}$  is called an *equipotential curve* for the Julia set  $J(Q)$ .

Now we start to prove the local connectivity of  $J(Q)$  at the critical point 0.

It was known in [Su1] that there exist finitely many external rays landing on each repelling periodic point. Let  $p_n$  denote the end point of the renormalization interval  $I_0^n$  which is a repelling periodic point of period  $2^{n-1}$ . Let  $\gamma_i^n, i = 1, 2, \dots, k(n)$ , denote the external rays which land on the orbit of  $p_n$ . We use these external rays and an equipotential curve  $\Gamma$  surrounding the Julia set  $J(Q)$  to have a partition  $\{D_{0,i}^n\}_{i=1}^{j(n,1)}$  of the domain containing  $J(Q)$  and bounded by  $\Gamma$ . By pulling back the external rays and the equipotential curve, we have a deeper partition of the domain bounded by  $Q^{-1}(\Gamma)$ . Go on pulling back, we get deeper and deeper partitions  $\{D_{a,i}^n\}_{i=1}^{j(n,a)}, a = 1, 2, 3, \dots$ . Let  $D_{a,i(0)}^n$  denote the puzzle piece in the  $a^{\text{th}}$  partition containing the critical point 0. Let  $\hat{J}_n = \bigcap_{a=1}^{\infty} \overline{D_{a,i(0)}^n}$ .

**Lemma 4.4**  *$\hat{J}_n \subset J(F_n)$ , where  $F_n$  is the  $n^{\text{th}}$  renormalization of  $Q$  which also satisfies unbranched condition.*

*Proof.* Let  $V'_n = D_{q(n-1),i(0)}^n$  and  $U'_n = D_{q(n-1)+q(n),i(0)}^n$  denote the puzzle pieces containing 0 when  $a = q(n-1)$  or  $q(n-1) + q(n)$  respectively. Because of renormalizability of  $Q$ , the  $q(n)^{\text{th}}$  iterate of  $Q$  maps  $\overline{U'_n}$  into  $\overline{V'_n}$ , which

satisfies unbranched condition. In fact  $F'_n = Q^{q(n)} : \text{int}(U'_n) \rightarrow \text{int}(V'_n)$  is a proper mapping, but it is not a polynomial-like mapping since  $\overline{U'_n}$  and  $\overline{V'_n}$  have some common boundaries. The good thing is that one can thicken the puzzle pieces  $V'_n$  and  $U'_n$  to have two open domains  $V''_n$  and  $U''_n$  containing  $\overline{U'_n}$  and  $\overline{V'_n}$  respectively such that  $F''_n = Q^{q(n)} : U''_n \rightarrow V''_n$  is polynomial-like mapping (see [Mi2] or [Hub]) and satisfies unbranched condition.

Let  $F_n : U_n \rightarrow V_n$  be the  $n^{\text{th}}$  renormalization of  $Q$  which satisfies the unbranched condition and has the complex bounds. By the property of polynomial-like mappings, one has

$$J(F_n) = \overline{\cup_{i \in \mathbb{Z}} (F_n)^i(0)} \quad \text{and} \quad J(F''_n) = \overline{\cup_{i \in \mathbb{Z}} (F''_n)^i(0)}.$$

By the unbranched condition and our particular constructions of  $F''_n$ , one has

$$\text{Orb}(F_n, 0) = I_0^n \cap \text{Orb}(Q, 0) = \text{Orb}(F''_n, 0).$$

Clearly 0 has two preimages under  $F_n$  in  $U_n$  and has two preimages under  $F''_n$  in  $U''_n$ . And  $F_n$  and  $F''_n$  follow the same renormalization cycle, their inverses are compositions of  $Q^{-1}$ . Among this composition, except the last pull back has two preimages, all of them are monotone pullbacks. Therefore  $(F_n)^{-1}(0) = (F''_n)^{-1}(0)$ . Inductively  $(F_n)^{-i}(0) = (F''_n)^{-i}(0)$  for all  $i \in \mathbb{N}$ . Hence  $\cup_{i \in \mathbb{Z}} (F_n)^i(0) = \cup_{i \in \mathbb{Z}} (F''_n)^i(0)$ , and hence  $J(F_n) = J(F''_n)$ . Clearly

$$J(F''_n) = \cap_{i \in \mathbb{N}} (F''_n)^{-i}(V''_n) \supset \cap_{i \in \mathbb{N}} (F'_n)^{-i}(\overline{V'_n}) \supset \hat{J}_n.$$

Therefore  $\hat{J}_n \subset J(F_n)$ .

**Lemma 4.5** *The Julia set  $J(Q)$  is locally connected at the critical point 0.*

*Proof.* Following the proof of the above lemma, for each  $n$ , one can find a puzzle piece  $D_{a(n), i(0)}^n$  in a deep enough partition such that  $0 \in D_{a(n), i(0)}^n \subset U_n$ , where  $U_n$  is the same as in Lemma 4.4. We know that the diameter of  $U_n$ , and hence the diameter of  $D_{a(n), i(0)}^n$ , goes down to zero as  $n \rightarrow \infty$ . Since  $D_{a(n), i(0)}^n \cap J(Q)$  is connected,  $J(Q)$  is locally connected at 0.

An immediate corollary of the pullback is the following.

**Corollary 4.2** *The Julia set  $J(P)$  is locally connected at any point  $z$  in the grand orbit  $\text{Orb}_G(Q, 0) = \cup_{i \in \mathbb{Z}} Q^i(0)$  of the critical point 0.*

By moving around the initial renormalization domain  $I_0^n$ , the same methods of the proofs of Lemma 4.4 and Lemma 4.5 can show

**Corollary 4.3** *The Julia set  $J(Q)$  is locally connected at any point  $z$  in  $K$ , and hence locally connected at any point in the grand orbit of  $z$ .*

In the rest of this section, we will finish a proof of our Theorem 1.1 by separating the remaining points  $z$  of  $J(Q)$  in two cases, that is,  $0 \in \omega(z)$  or  $0 \notin \omega(z)$ , where  $\omega(z)$  is the  $\omega$ -limit set of  $z$  under  $Q$ .

**Lemma 4.6** *If  $z \in J(Q)$  and  $0 \in \omega(z)$ , then  $J(Q)$  is locally connected at  $z$ .*

*Proof.* We assume that the point  $z$  is not equal to any point which has been considered in Corollary 4.3. From Lemma 4.5, there exists an infinite sequence  $\{W_n\}_{n=1}^\infty$  of open neighborhoods of 0 such that  $W_n \cap J(Q)$  is connected for any  $n \in \mathbb{N}$  and  $\text{diam}(W_n) \rightarrow 0$  as  $n \rightarrow \infty$ . In fact we take  $W_n$  to be the interior of  $D_{a(n),i(0)}^n$  in Lemma 4.5. So  $W_n \subset U_n \subset V_n$ , where  $U_n$  and  $V_n$  are the same as in Lemma 4.4. It is easy to see that there are  $q(n)$  polynomial-like mappings along the orbit of  $U_n$  under  $Q$ , which are  $Q^{q(n)} : Q^i(F_n^{-1}(U_n)) \rightarrow Q^i(U_n)$  for  $i = 1, 2, \dots, q(n)$ . All of them satisfy the unbranched condition. We denote these polynomial-like mappings by  $F_n^i = Q^{q(n)} : U_n^i = Q^i(F_n^{-1}(U_n)) \rightarrow V_n^i = Q^i(U_n)$  for  $i = 1, 2, \dots, q(n)$ . We also denote  $W_n^i = Q^i(F_n^{-1}(W_n))$  for  $i = 1, 2, \dots, q(n)$ .

Since  $0 \in \omega(z)$  and 0 is dense in  $K$ , we have  $K \subset \omega(z)$ . We also know  $K \subset \cup_{i=1}^{q(n)} W_n^i$  for each  $n \in \mathbb{N}$ . Also notice that  $W_n^i, i = 1, 2, \dots, q(n)$ , are pairwise disjoint and  $\cup_{i=1}^{q(n)} W_n^i$  does not cover the Julia set  $J(Q)$ . We will use this property later.

Suppose that the first return time for  $z$  entering  $\cup_{i=1}^{q(n)} W_n^i$  is  $l(n)$ . We denote by  $W_n^{i(z)}$  the puzzle piece containing  $Q^{l(n)}(z)$ . There exists a  $l(z)^{\text{th}}$  monotone pullback of  $W_n^{i(z)}$  under  $Q$  containing  $z$ , denoted by  $D^n(z)$ . By unbranched condition, there exists a  $l(z)^{\text{th}}$  monotone pullback of  $V_n^{i(z)}$  containing  $D^n(z)$ . So

$$D^n(z) \subset Q^{-l(z)}(U_n^{i(z)}) \subset Q^{-l(z)}(V_n^{i(z)}).$$

Claim:  $\cap_{n=1}^\infty D^n(z) = \{z\}$ .

Suppose that  $\cap_{n=1}^\infty D^n(z)$  contains at least two points  $z$  and  $y$ . Since  $Q^{-l(z)} : V_n^{i(z)} \rightarrow Q^{-l(z)}(V_n) \supset D^n(z)$  is monotone and the modulus of  $V_n^{i(z)} \setminus \overline{U_n^{i(z)}}$  is greater than a constant  $\epsilon > 0$  when  $n$  is large enough. By the corollary of Koebe distortion theorem, the monotone pullback branches  $Q^{-l(z)} : V_n^{i(z)} \rightarrow Q^{-l(n)}(V_n) \supset D^n(z)$ ,  $n \in \mathbb{N}$ , have uniformly bounded nonlinearity on  $U_n^{i(z)}$ . Since  $z, y \in D^n(z)$  and  $|z - y| \neq 0$ ,  $D^n(z)$  contains an open domain  $W$  for any  $n \in \mathbb{N}$ . Then  $z \in W \subset \cap_{n=1}^\infty D^n(z)$ . Since  $z$  is a point in  $J(Q)$ , there exists  $m \in \mathbb{N}$  such that  $Q^m(W) \supset J(Q)$  (see [Mi1]). But  $Q^{l(n)}(W) \subset W_n^{i(n)}$  and  $\{W_n^i\}_{i=1}^{q(n)}$  does not cover  $J(Q)$ , a contradiction. Therefore the claim  $\cap_{n=1}^\infty D^n(z) = \{z\}$  holds, and hence  $J(Q)$  is locally connected at  $z$ .

**Lemma 4.7** *If  $z \in J(Q)$  and  $0 \notin \omega(z)$ , then  $J(Q)$  is locally connected at  $z$ .*

*Proof.* Since the orbit of each point in  $K$  is dense in  $K$ ,  $0 \notin \omega(z)$  implies that the distance between  $K$  and  $\overline{\text{Orb}(Q, z)}$  is positive. Let  $W_n^i, i = 1, 2, 3, \dots, q(n)$ , be the same as in Lemma 4.6 for any  $n$ . We know that they cover  $K$  and their diameters go to zero as  $n \rightarrow \infty$ . So when  $n$  is large enough,  $\cup_{i=1}^{q(n)} W_n^i$  has no intersection with  $\overline{\text{Orb}(Q, z)}$ . Let  $y \in \omega(z)$ . So  $y \notin \cup_{i=1}^{q(n)} W_n^i$  when  $n$  is large enough. Fix such a large  $n$ . There exists a puzzle piece  $D_{a,i(y)}^n$  in a deep enough partition such that it contains  $y$  and is different from any  $W_n^i, i = 1, 2, \dots, q(n)$ . The distance from  $\overline{D_{a,i(y)}^n}$  to  $K$  is also positive, so we can find the a simply connected open domain  $E^n(y)$  containing  $\overline{D_{a,i(y)}^n}$  with the modulus of  $E^n(y) \setminus \overline{D_{a,i(y)}^n}$  greater than  $\delta > 0$  and  $E^n(y) \cap K = \emptyset$ .

Since  $y \in \omega(z)$ , the iterates of  $z$  under  $Q$  enter  $D_{a,i(y)}^n$  infinitely many times. Let us denote the entering times by  $l(z), l \in \mathbb{N}$ . Pullback  $D_{a,i(y)}^n$  and  $E^n(y)$  according to the returns of  $z$  to  $D_{a,i(y)}^n$ . These pullbacks, denoted by  $Q^{-l(z)}$ , are monotone on  $E^n(y)$ . Then  $Q^{-l(z)}(D_{a,i(y)}^n)$  gives an infinite sequence  $\{D_{a(l(z)),i(z)}^n\}_{l=1}^{\infty}$  of puzzle pieces containing  $z$ . Because of the definite modulus of  $E^n(y) \setminus \overline{D_{a,i(y)}^n}$ , monotone pullbacks and  $z \in J(Q)$ , by the argument of proving the claim in Lemma 4.6, we can show  $\cap_{l=1}^{\infty} D_{a(l(z)),i(z)}^n$  is the point  $z$ . Hence  $J(Q)$  is locally connected at the point  $z$ .

Lemmas 4.6 and 4.7 complete a proof of the Theorem 1.1.

## 5. A direct proof of Sullivan's complex bounds

Let  $T = [a, b]$  be a compact nonempty interval on the real line  $\mathbb{R}$ , and  $\mathbb{C}_T$  denote the complement of two half lines  $(-\infty, a]$  and  $[b, \infty)$  in the complex plane  $\mathbb{C}$ . Sullivan originated the idea of using hyperbolic neighborhoods of  $T$  in  $\mathbb{C}_T$  to study the complex bounds of renormalizations [Su1].

The Riemann surface  $\mathbb{C}_T$  admits a hyperbolic metric in which  $T$  is a geodesics. The hyperbolic  $r$ -neighborhood of  $T$  is the set of all points of  $\mathbb{C}_T$  which have hyperbolic distance to  $T$  less than  $r$ . It is easy to see that such a neighborhood is symmetric with respect to  $\mathbb{R}$  and its boundary consists of two partial Euclidean circumferences whose angles with  $\mathbb{R}$  are denoted by  $\theta = \theta(r)$ . Such a neighborhood is denoted by  $D_\theta(T)$ , it will also be called a *hyperbolic disk* (see Figure 1). The hyperbolic disk  $D_{\frac{\pi}{2}}(T)$  is simply denoted by  $D(T)$ . Sullivan introduced the following version of Schwartz lemma.

**Lemma 5.1 (Schwartz Lemma)** *Suppose that two intervals  $J' \subset J \subset \mathbb{R}$ ,  $\phi : \mathbb{C}_J \rightarrow \mathbb{C}_{J'}$  is an analytic injection and  $\phi(J) \subset J'$ . Then for any  $\theta \in (0, \pi)$ ,*

$$\phi(D_\theta(J)) \subset D_\theta(J').$$

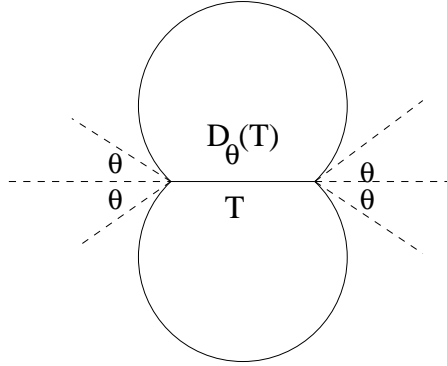


Figure 1: A hyperbolic disk  $D_\theta(T)$ .

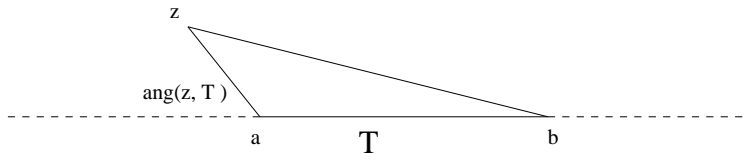


Figure 2: An illustration of  $\text{ang}(z, T)$ .

For a point  $z \in \mathbb{C}_T$ , the angle between  $z$  and  $T$ , denoted by  $\text{ang}(z, T)$ , is the least angle of the ones between the intervals  $[a, z], [b, z]$  and the corresponding rays  $[a, -\infty), [b, +\infty)$  of the real line, measured in the range from 0 to  $\pi$  (see Figure 2). Lyubich and Yampolsky introduced the following corollary of the Schwartz lemma [LY].

**Lemma 5.2** *Under the circumstances of the Schwarz lemma, let us consider a point  $z \in \mathbb{C}_J$  with  $\text{dist}(z, J) \geq \delta|J|$  and  $\text{ang}(z, J) \geq \varepsilon$ . Then*

$$\frac{\text{dist}(\phi(z), J')}{|J'|} \leq C \frac{\text{dist}(z, J)}{|J|},$$

for some constant  $C = C(\varepsilon, \delta)$ .

*Proof.* See the proof of Lemma 2.1 in [LY].

A collection (finite or infinite) of intervals on the real line  $\mathbb{R}$ , denoted by  $J_0 = J, J_{-1}, J_{-2}, \dots$ , is said to be a *string of monotone pull-backs* of an interval  $J \subset \mathbb{R}$  under the map  $Q$  if  $Q : J_{-i} \rightarrow J_{-i+1}$  is homeomorphic for each  $i = 1, 2, \dots$ . For any point  $z \in J$ , the backward orbit of  $z$  corresponding the above

pullbacks is clearly the sequence of the points  $z_0 = z, z_{-1}, z_{-2}, \dots$ , satisfying that  $z_{-i} \in J_{-i}$  and  $Q(z_{-i}) = z_{-i+1}$  for each  $i = 1, 2, \dots$ , that is the iterates of  $z$  under the inverse branch of  $Q$  which maps  $J_{-i}$  to  $J_{-i+1}$  for each  $i$ . For any point  $z \in C_J$ , we define the corresponding pullbacks of  $z$  as the iterates of  $z$  under the analytic continuation of the inverse branch of  $Q$  which maps  $J_{-i}$  to  $J_{-i+1}$  for each  $i = 1, 2, \dots$ .

Now let us consider the renormalizations of  $Q$ . For each  $n \in \mathbb{N}$ ,  $Q$  maps  $I_0^n$  onto  $I_1^n$  with a folding, then  $Q$  maps  $I_i^n$  monotonically onto  $I_{i+1}^n$  for  $i = 1, 2, \dots, q(n) - 1$ , and at last  $Q$  maps  $I_{q(n)-1}^n$  monotonically onto its image  $J = Q(I_{q(n)-1}^n)$ . Therefore  $J, I_{q(n)-1}^n, \dots, I_2^n, I_1^n$  are monotone pullbacks of  $J$  under  $Q$ . The string of these pullbacks can be extended to a maximal interval  $N_0^n \supset J$ . And in fact  $N_0^n \supset I_0^n$  and there exists  $\Delta > 0$  and  $\delta > 0$  (independent of  $n$ ) such that  $\Delta \geq |L^n|/|I_0^n| \geq \delta$  and  $\Delta \geq |R^n|/|I_0^n| \geq \delta$ , where  $L^n \cup R^n = N_0^n \setminus I_0^n$ . This is a part of the real bounds of the renormalizations of  $Q$ . Given any  $z \in \mathbb{C}_{N_0^n}$ , let  $z_0 = z, z_{-1}, z_{-2}, \dots, z_{-q(n)+1}$  be the pullbacks corresponding to the above pullbacks of  $N_0^n$  under  $Q$ . And let  $z_{-q(n)}$  denote one of the preimages of  $z_{-q(n)+1}$  under  $Q$ . The following lemma is a slightly changed version of a key lemma introduced by Lyubich and Yompolsky in [LY].

**Lemma 5.3** *For all  $z \in \mathbb{C}_{N_0^n}$  with  $\text{dist}(z, I_0^n) \geq |I_0^n|$ , one has*

$$\frac{\text{dist}(z_{-q(n)}, I_0^n)}{|I_0^n|} \leq C \left( \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|} \right)^{\frac{1}{2}}.$$

Through this lemma, one can easily prove Sullivan's complex bounds. In the next, we first give a somehow different proof of this lemma with the details. Then we provide a proof of Sullivan's complex bounds at the end of this section.

We first state three useful properties of square root functions. Let  $\Phi(z) = \sqrt{z}$  be the branch of the square roots which maps the slit plane  $\mathbb{C} \setminus \mathbb{R}^-$  into itself. The following geometric properties also hold for the other branches of the root functions with or without precomposition and/or postcomposition by affine maps except some slight modifications of the statements.

**Proposition 5.1** *Let  $K > 1$ ,  $\delta > 0$ ,  $K^{-1} < a < K$ . Suppose that  $T = [-ab, b]$ ,  $T' = [0, \sqrt{b}]$ , and  $J$  is an interval contained in  $[0, b]$  and  $J' = \Phi(J)$ . Then:*

- (1)  $\Phi(D_\theta(T)) \subset D_{\theta'}(T')$ , where  $\theta'$  only depends on  $\theta$  and  $K$ .
- (2) If  $z' \in \Phi(D(T)) \setminus D([-\delta\sqrt{b}, (1+\delta)\sqrt{b}])$ , then

$$\text{ang}(z', T') > \varepsilon(K, \delta) > 0,$$

$$C_1(K, \delta)^{-1} < \frac{\text{dist}(z', T')}{|T'|} < C_1(K, \delta),$$

and

$$\frac{\text{dist}(z', J')}{|J'|} \leq C_2(K, \delta) \frac{\text{dist}(z, J)}{|J|}.$$

**Proof.** (1) There exist  $\theta_1 > 0$  and  $\theta_2 > 0$ , depending only on  $\theta$  and  $K$ , such that  $D_\theta(T) \subset D_{\theta_1}([-ab, 0]) \cup D_{\theta_2}([0, b])$ . Applying  $\Phi$  to  $D_{\theta_2}([0, b])$ , by Schwartz lemma, one has  $\Phi(D_{\theta_2}([0, b])) \subset D_{\theta_2}(T')$ . When applying  $\Phi$  to  $D_{\theta_1}([-ab, 0])$ , one can think of applying another branch of the square root to it, and hence again by Schwartz lemma, one has  $\Phi(D_{\theta_1}([-ab, 0])) \subset D_{\theta_1}([0, \sqrt{abi}])$ . Therefore  $\Phi(D_\theta(T)) \subset D_{\theta_1}([0, \sqrt{abi}]) \cup D_{\theta_2}(T')$ , and hence there exists  $\theta' > 0$ , depending only on  $\theta$  and  $K$ , such that  $\Phi(D_\theta(T)) \subset D_{\theta'}(T')$ .

(2) Given  $z' \in \Phi(D(T)) \setminus D([- \delta\sqrt{b}, (1 + \delta)\sqrt{b}])$ , it is easy to see that  $\text{ang}(z', T') > \varepsilon(K, \delta) > 0$  and  $C_1(K, \delta)^{-1} < \frac{\text{dist}(z', T')}{|T'|} < C_1(K, \delta)$  because  $z'$  is in  $D_{\theta'}(T')$  from (1) for some constant  $\theta'$  but not in  $D([- \delta b, (1 + \delta)b])$ , where  $\theta'$  comes from the part (1) when  $\theta = \frac{\pi}{2}$ , and it only depends on  $K$ . We only need to show the last inequality. Clearly

$$\frac{\frac{\text{dist}(z', J')}{|J'|}}{\frac{\text{dist}(z, J)}{|J|}} = \frac{\frac{\text{dist}(z', J')}{|T'|}}{\frac{\text{dist}(z, J)}{|T|}} \cdot \frac{|T'|}{|J'|}.$$

Since  $z' \in D_{\theta'}(T')$ ,

$$\frac{\text{dist}(z', J')}{|T'|} \leq C_3(\theta').$$

Through the mean value theorem, one has

$$\frac{\frac{|T'|}{|T|}}{\frac{|J'|}{|J|}} = \frac{\frac{\sqrt{b}}{(1+a)b}}{\frac{1}{2\sqrt{x}}} = \frac{2}{1+a} \frac{\sqrt{x}}{\sqrt{b}} \leq 2,$$

where  $x \in [0, b]$ . And since  $z \notin \Phi^{-1}(D([- \delta b, (1 + \delta)b]))$ , one can show

$$\frac{\text{dist}(z, J)}{|T|} \geq C_4(\delta, \lambda) > 0.$$

Together we can see that the last inequality follows.

**Proposition 5.2** *Let  $\delta > 0$ . Let  $J = [a, b] \subset [0, +\infty)$ ,  $J' = \Phi(J) = [a', b']$ ,  $z \in \mathbb{C} \setminus \mathbb{R}^-$  and  $z' = \Phi(z)$ . Then:*

(1) *If  $\text{dist}(z, J) > \delta|J|$ , then*

$$\frac{\text{dist}(z', J')}{|J'|} < C(\delta) \frac{\text{dist}(z, J)}{|J|}.$$

(2) If  $\text{dist}(z', J') > \delta|J'|$ , then

$$\frac{\text{dist}(z', J')}{|J'|} < C(\delta') \frac{\text{dist}(z, J)}{|J|}.$$

(3) Let  $\theta$  denote the angle between the line segment  $[z, a]$  and the ray on the real line which starts at  $a$  and does not contain  $J$ , and  $\theta'$  the angle between  $[z', b']$  and the ray on the real line which starts at  $b'$  and does not contain  $J'$ . If  $\theta \leq \frac{\pi}{2}$ , then  $\theta' \geq \frac{\pi}{2}$ .

*Proof.* (1) Take a small angle  $0 < \theta_0 < \frac{\pi}{2}$ , the inequality follows from Lemma 5.2 if  $\text{ang}(z, J) > \theta_0$ . Since the root function  $\Phi$  is quasisymmetric on the whole real line (take any branch of the root function as the definition of  $\Phi$  on  $\mathbb{R}^-$ ), it is also quasisymmetric in the region  $\{z \in \mathbb{C}_{\mathbb{R}^-} : \text{dist}(z, J) > \delta|J| \text{ and } \text{ang}(z, J) \leq \theta_0\}$ . This shows the inequality for  $z$  if  $\text{ang}(z, J) \leq \theta_0$ .

(2) Since  $\text{dist}(z', J') > \delta'|J'|$ , then there exists  $\delta > 0$  such that  $\text{dist}(z, J) > \delta$ . Applying (1), one has the proof of (2).

(3) The proof is straight forward.

**Proposition 5.3** *Let  $\Delta > 0$ ,  $J = [0, b] \subset \mathbb{R}^+$ ,  $J' = \Phi(J)$ ,  $z \in \mathbb{C} \setminus \mathbb{R}^-$  and  $z' = \Phi(z)$ . If  $\text{dist}(z, J) \geq \Delta|J|$ , then*

$$\frac{\text{dist}(z', J')}{|J'|} < C(\Delta) \left( \frac{\text{dist}(z, J)}{|J|} \right)^{\frac{1}{2}}.$$

If  $\text{dist}(z', J') \geq \Delta'|J'|$ , then

$$\frac{\text{dist}(z', J')}{|J'|} < C(\Delta') \left( \frac{\text{dist}(z, J)}{|J|} \right)^{\frac{1}{2}}.$$

*Proof.* If  $\text{dist}(z, J) \geq \Delta|J|$ , then  $\text{dist}(z, J)$  is commensurate to  $|z|$  and  $\text{dist}(z', J') \geq \delta|J'|$  for a constant  $\delta$  only depending on  $\Delta$ . Therefore  $\text{dist}(z', J')$  is also commensurate to  $|J'|$ , and hence

$$\frac{\text{dist}(z', J')}{|J'|} < C(\Delta) \left( \frac{\text{dist}(z, J)}{|J|} \right)^{\frac{1}{2}}.$$

The proof for the second part is similar.

Let  $H \supset J$  be two nonempty open intervals with  $H \setminus J$  consisting of two connected components  $L$  and  $R$ . Suppose that  $L$  lies to the left of  $R$ . Let  $S_{\theta, \varepsilon}(H, J)$  denote the intersection of the hyperbolic disk  $D_{\theta}(H)$  with two disjoint  $\mathbb{R}$ -symmetric  $2\varepsilon$ -wedges with vertices at  $\partial J$ . Let  $B_{\varepsilon}(J)$  be the complement of the above two wedges in the complex plane  $\mathbb{C}$ , that is the set of all points looking at  $J$  with an angle at least  $\varepsilon$ . See Figure 3. If  $z_{-s} \in B_{\varepsilon}(I_{-s}^n)$  at some moment  $s \leq l$ , then we say that the backward orbit of  $z$  has an  $\varepsilon$ -jump.

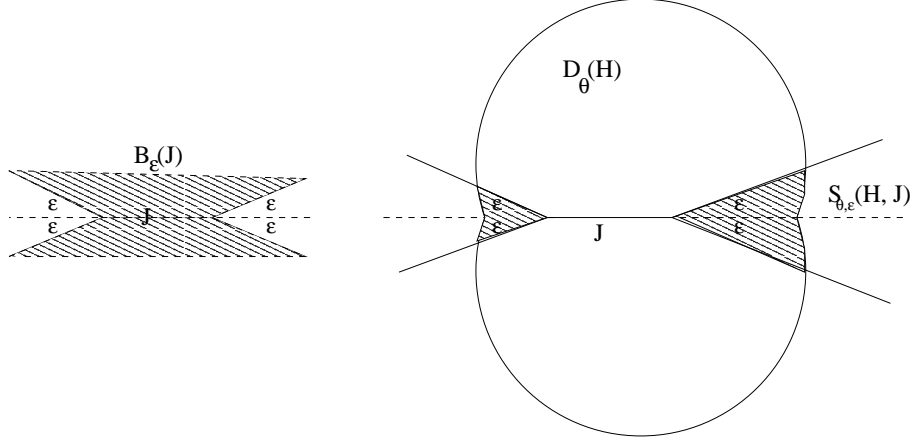


Figure 3: Domains  $B_\epsilon(J)$  and  $S_{\theta, \epsilon}(H, J)$ .

**Lemma 5.4** *If  $z_{-s}$  in the backward orbit of a point  $z \in \mathbb{C}_{N_0^n}$  has an  $\epsilon$ -jump at a moment  $s \in [0, q(n) - 1]$ , then either  $\frac{\text{dist}(z_{q(n)-1}, I_{q(n)-1}^n)}{|I_{q(n)-1}^n|} \leq C_1(\epsilon)$  or*

$$\frac{\text{dist}(z_{q(n)-1}, I_{q(n)-1}^n)}{|I_{q(n)-1}^n|} \leq C_2(\epsilon) \frac{\text{dist}(z_{-s}, I_{-s}^n)}{|I_{-s}^n|}.$$

*Proof.* If  $\text{dist}(z_{-s}, I_{-s}^k) < |I_{-s}^k|$ , then there exists  $\theta = \theta(\epsilon)$  such that  $z_{-s} \in D_\theta(I_{-s}^n)$ . Since  $Q^{-(q(n)-s-1)}$  is univalent on  $\mathbb{C}_{I_{-s}^n}$ , by Schwartz lemma,  $z_{q(n)-1} \in D_\theta(I_{-q(n)+1}^n)$ . Therefore there exists  $C_1 = C_1(\theta) = C_1(\epsilon)$  such that  $\frac{\text{dist}(z_{q(n)-1}, I_{q(n)-1}^n)}{|I_{q(n)-1}^n|} \leq C_1(\epsilon)$ .

If  $\text{dist}(z_{-s}, I_{-s}^k) \geq |I_{-s}^k|$ , then Lemma 5.2 shows the rest.

**Lemma 5.5** *Let  $J_0 = J, J_{-1}, \dots, J_{-l} = J'$  be the monotone pullbacks of an interval  $J$  under  $Q$ . Suppose  $T \supset J_0$  is the maximal interval on which the corresponding inverse branch of  $Q^l$  is well defined. Let  $z \in \mathbb{C}_T$  and  $z_0 = z, z_{-1}, \dots, z_{-l} = z'$  be the corresponding backward orbit of  $z$ . Then for all sufficiently small  $\epsilon > 0$  (independent of  $Q$ ), either  $z_{-s} \in B_\epsilon(J_s)$  at some moment  $s \leq l$ , or  $z' \in S_{\theta, \epsilon}(T', J')$  with  $\theta = \frac{\pi}{2} - O(\epsilon)$ , where  $T' = Q^l(T)$ , which is the maximal interval of monotonicity of  $Q^l$  containing  $J'$ .*

*Proof.* Let  $T_{0,i}$  be the maximal interval on which  $Q^{-i}$  is well defined and  $T_{-i} = Q^{-i}(T_{0,i})$ , where  $i = 1, 2, \dots, l$ .

Given a small  $\varepsilon > 0$ , after applying the first square root, either  $z_{-1} \in B_\varepsilon(J)$  or  $z_{-1} \in S_{\frac{\pi}{2}, \varepsilon}(T_{-1}, J_{-1})$ , where  $T_{-1}$  is a ray on  $\mathbb{R}^1$ . One connected component of  $S_{\frac{\pi}{2}, \varepsilon}(T_{-1}, J_{-1})$  is a union of two right triangle, denote it by  $\Delta_{-1}$ , its intersection with  $\mathbb{R}^1$  by  $R_{-1}$  and its vertex of the right angle by  $a_{-1}$ . The other component is an  $\mathbb{R}^1$ -symmetric  $2\varepsilon$  wedge. Let  $\theta = \frac{\pi}{2} - O(\varepsilon)$  be the smallest such that  $\Delta_{-1} \subset D_\theta(R_{-1})$ .

In the following pullbacks before the moment (denoted by  $-t$ ) that the square root function cuts the pullback of the  $\mathbb{R}^1$ -symmetric  $2\varepsilon$  wedge, by applying the Schwartz lemma to the domain  $D_\theta(R_{-1})$  in the monotone pullbacks, we have either  $z_{-s} \in B_\varepsilon(J_{-s})$  for some moment  $0 \leq s \leq t-1$  or  $z_{-(t-1)} \in D_\theta(R_{-(t-1)})$ , or  $z_{-(t-1)}$  is in an  $\mathbb{R}^1$ -symmetric  $2\varepsilon$  wedge, where  $R_{-(t-1)}$  is the finite component of  $T_{-(t-1)} \setminus J_{-(t-1)}$ . So either  $z_{-s} \in B_\varepsilon(J_{-s})$  for some moment  $0 \leq s \leq (t-1)$  or  $z_{-(t-1)} \in S_{\theta, \varepsilon}(T_{-(t-1)}, J_{-(t-1)})$ .

Now the next square root will cut the wedge component of  $S_{\theta, \varepsilon}(T_{-(t-1)}, J_{-(t-1)})$ . If  $z_{-(t-1)} \in S_{\theta, \varepsilon}(T_{-(t-1)}, J_{-(t-1)}) \cap D_\theta(R_{-(t-1)})$ , then by Schwartz lemma,  $z_{-t} \in D_\theta(Q^{-1}(R_{-(t-1)}))$ . If

$$z_{-(t-1)} \in S_{\theta, \varepsilon}(T_{-(t-1)}, J_{-(t-1)}) \setminus D_\theta(R_{-(t-1)}),$$

then either  $z_{-t} \in B_\varepsilon(J_{-t})$  or in a union of two right triangles with one sharp angle  $\varepsilon$  (denote it by  $\Lambda_{-t}$ , and its intersection with  $\mathbb{R}^1$  by  $L_{-t}$ ). Together if  $z_{-(t-1)} \in S_{\theta, \varepsilon}(T_{-(t-1)}, J_{-(t-1)})$ , then either  $z_{-t} \in B_\varepsilon(J_{-t})$  or  $z_{-t} \in D_\theta(L_{-t}) \cup D_\theta(R_{-t})$ , and hence  $z_{-t} \in S_{\theta, \varepsilon}(T_{-t}, J_{-t})$ , where  $L_{-t}$  and  $R_{-t}$  are two components of  $T_{-t} \setminus J_{-t}$ .

Inductively, in the further pullbacks, by applying Schwartz lemma to certain domains in the monotone pullbacks, we can see that if  $z_{-t} \in D_\theta(L_{-t}) \cup D_\theta(R_{-t}) \cap S_{\theta, \varepsilon}(T_{-t}, J_{-t})$  then either  $z_{-s} \in B_\varepsilon(J_{-s})$  for some moment  $t < s \leq l$  or  $z_{-l} \in D_\theta(L_{-l}) \cup D_\theta(R_{-l}) \cap S_{\theta, \varepsilon}(T_{-l}, J_{-l})$ , where  $L_{-l}$  and  $R_{-l}$  are the two components of  $T_{-l} \setminus J_{-l}$ .

For an interval  $J = [a, b]$  and  $\delta > 0$ , the  $(1 + \delta)$ -scaled nbhd of  $J$  is defined to be the interval  $[a - (b - a)\delta, b + (b - a)\delta]$ .

Now we are ready to show Lemma 5.3.

*Proof of Lemma 5.3.* Given a point  $z_0 \in \mathbb{C}_{N_0^n}$  with  $\text{dist}(z_0, I_0^n) \geq |I_0^n|$ , from (1) of Proposition 5.1 and Proposition 5.3, we only need to show the following claim: either

$$\text{dist}(z_{-q(n)+1}, I_{-q(n)+1}^n) \leq \Delta |I_{-q(n)+1}^n|$$

or

$$\frac{\text{dist}(z_{q(n)-1}, I_{q(n)-1}^n)}{|I_{q(n)-1}^n|} \leq C \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}$$

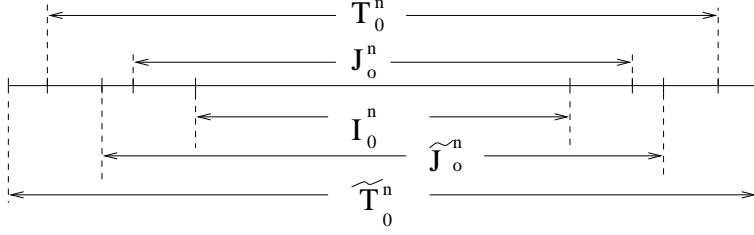


Figure 4: The nested intervals introduced in the proof of Lemma 5.3.

for some constants  $\Delta$  and  $C$ . Secondly through Lemma 5.4, we only need to show that either

$$\text{dist}(z_{-q(n)+1}, I_{-q(n)+1}^n) \leq \Delta |I_{-q(n)+1}^n|$$

or the backward iterate  $z_{-s}$  has an  $\varepsilon$ -jump at some moment  $s \in [0, -q(n) + 1]$  with

$$\frac{\text{dist}(z_{-s}, I_{-s}^n)}{|I_{-s}^n|} \leq C \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}$$

for some constants  $\varepsilon > 0$  and  $C > 0$ . We will show this statement through an induction on the renormalization level  $n$  by considering the pull-back  $z_{-s}$  following the reverse cycles of renormalizations at level 1, 2 and so on.

Before getting into the details, let us first introduce some notations.

Denote  $I_{0L}^n = I_0^n \cap (-\infty, 0]$  and  $I_{0R}^n = I_0^n \cap [0, \infty)$ . Clearly  $Q^{q(n)}$  is monotone on  $I_{0L}^n$  or  $I_{0R}^n$ .

Let  $\tilde{J}_{0L}^n$  (or  $\tilde{J}_{0R}^n$ ) be the maximal interval containing  $I_{0L}^n$  (resp.  $I_{0R}^n$ ) on which  $Q^{q(n)}$  is monotone. Clearly,  $\tilde{J}_0^n = \tilde{J}_{0L}^n \cup \tilde{J}_{0R}^n$  is a symmetric interval centered at 0.

Let  $\tilde{T}_0^n$  be the smallest symmetric interval centered at 0 containing  $Q^{q(n)}(\tilde{J}_0^n)$ . We also denote  $\tilde{T}_{0L}^n = \tilde{T}_0^n \cap (-\infty, 0]$  and  $\tilde{T}_{0R}^n = \tilde{T}_0^n \cap [0, \infty)$ .

From the real analysis of renormalizations, we know that  $\tilde{T}_0^n \supset \tilde{J}_0^n \supset I_0^n$ , and  $\tilde{T}_0^n$  contains a  $(1 + \delta_1)$ -scaled nbhd of  $\tilde{J}_0^n$ . Now let  $T_0^n$  be the  $(1 + \delta_1/2)$ -scaled nbhd of  $\tilde{J}_0^n$ ,  $T_{0L}^n = T_0^n \cap (-\infty, 0]$  and  $T_{0R}^n = T_0^n \cap [0, \infty)$ .

Let  $J_{0L}^n \subset \tilde{J}_{0L}^n$  such that  $Q^{q(n)}(J_{0L}^n) = T_{0L}^n$  and  $J_{0R}^n \subset \tilde{J}_{0R}^n$  such that  $Q^{q(n)}(J_{0R}^n) = T_{0R}^n$ . And denote  $J_0^n = J_{0L}^n \cup J_{0R}^n$ , which is again a symmetric interval centered at 0.

From the real bounds of the renormalizations,  $T_0^n \supset J_0^n \supset I_0^n$ , and  $T_0^n$  contains a  $(1 + \delta_2)$ -scaled nbhd of  $J_0^n$  and  $J_0^n$  contains a  $(1 + \delta_3)$ -scaled nbhd of  $I_0^n$ . In the above,  $\delta_1$ ,  $\delta_2$  and  $\delta_3$  are independent of  $n$ .

Together we have  $\tilde{T}_0^n \supset T_0^n \supset \tilde{J}_0^n \supset J_0^n \supset I_0^n$ , and any two intervals in between are commensurate to each other in length, see Figure 4.

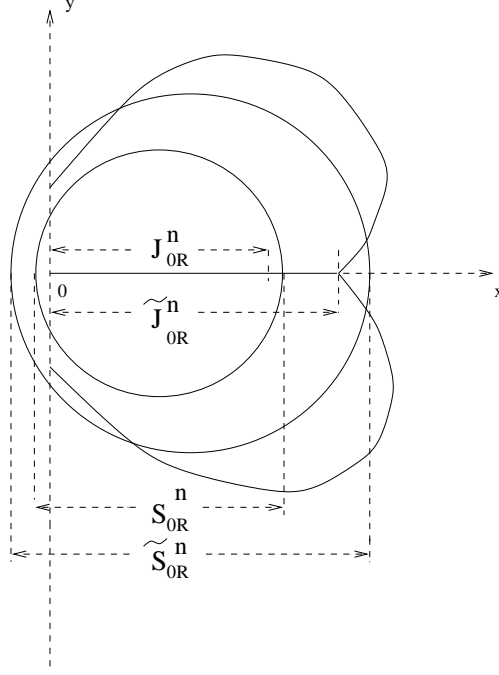


Figure 5: Some important domains introduced in the proof of Lemma 5.3.

Recall that  $N_0^n$  is the maximal interval on which the inverse branch  $Q^{-q(n)+1}$  is well-defined. Clearly either  $T_{0L}^n \subset \tilde{T}_{0L}^n \subset N_0^n$  or  $T_{0R}^n \subset \tilde{T}_{0R}^n \subset N_0^n$ . Again from the real bounds,  $N_0^n$  contains a  $(1 + \delta_4)$ -scaled nbhd of  $T_{0L}^n$  or  $T_{0R}^n$ , where  $\delta_4$  is independent of  $n$ .

Now suppose that  $S_{0L}^n$  (or  $S_{0R}^n$ ) is the  $(1 + \delta)$ -scaled nbhd of  $J_{0L}^n$  (resp.  $J_{0R}^n$ ), and  $\tilde{S}_{0L}^n$  (or  $\tilde{S}_{0R}^n$ ) is the  $(1 + \delta)$ -scaled nbhd of  $S_{0L}^n \cup \tilde{J}_{0L}^n$  (resp.  $S_{0R}^n \cup \tilde{J}_{0R}^n$ ). Clearly when  $\delta$  is small enough, we can require that  $S_{0L}^n$  and  $S_{0R}^n$  are well contained in  $\tilde{J}_0^n$ , and either  $\tilde{S}_{0L}^n$  or  $\tilde{S}_{0R}^n$  is contained in  $N_0^n$ , where  $\delta$  only depends on  $\delta_1$  and  $\delta_4$ .

Before carrying on further, let us first notice the following two observations.

**Observation 1:** The inverse branch  $Q^{-q(n)+1}$  is univalent on the disk  $D(\tilde{S}_{0L}^n)$  or  $D(\tilde{S}_{0R}^n)$ , and  $D(\tilde{S}_{0L}^n) \setminus D(S_{0L}^n)$ , respectively  $D(\tilde{S}_{0R}^n) \setminus D(S_{0R}^n)$ , has a definite modulus. Therefore  $Q^{-q(n)+1}$  has bounded nonlinearity in  $D(S_{0L}^n)$  or  $D(S_{0R}^n)$  by Koebe distortion theorem.

**Observation 2:** Denote  $\hat{S}_{0L}^n = Q^{q(n)}(D(S_{0L}^n) \cap \mathbb{H}_L) \cap \mathbb{R}$  and

$$\hat{S}_{0R}^n = Q^{q(n)}(D(S_{0R}^n) \cap \mathbb{H}_R) \cap \mathbb{R},$$

where  $\mathbb{H}_L$  (or  $\mathbb{H}_R$ ) denotes the closed left (resp. right) half plane. When  $\delta$  is small enough,  $N_0^n$  contains a  $(1 + \delta')$ -scaled nbhd  $\hat{S}_{0L}^n$  or  $\hat{S}_{0R}^n$ . And furthermore, there exists  $\theta_0 > 0$ , only depending on  $\delta$ , such that  $Q^{q(n)}(D(S_{0L}^n) \cap \mathbb{H}_L)$  and  $Q^{q(n)}(D(S_{0R}^n) \cap \mathbb{H}_R)$  are contained in  $D_{\theta_0}(\hat{S}_{0L}^n)$  and  $D_{\theta_0}(\hat{S}_{0R}^n)$  respectively. Since  $N_0^n$  contains a  $(1 + \delta')$ -scaled nbhd of  $\hat{S}_{0L}^n$  or  $\hat{S}_{0R}^n$ , the inverse branch  $Q^{-q(n)+1}$  is univalent on  $D_{\theta_0}(N_0^n)$  and  $D_{\theta_0}(N_0^n) \setminus D_{\theta_0}(\hat{S}_{0L}^n)$  (resp.  $D_{\theta_0}(N_0^n) \setminus D_{\theta_0}(\hat{S}_{0R}^n)$ ) has a definite modulus, again by Koebe distortion theorem, the inverse branch  $Q^{-q(n)+1}$  has bounded nonlinearity on  $D_{\theta_0}(\hat{S}_{0L}^n)$  (resp.  $D_{\theta_0}(\hat{S}_{0R}^n)$ ).

Now we begin to prove the claim: for any point  $z_0 \in \mathbb{C}_{N_0^n}$  with  $\text{dist}(z_0, I_0^n) \geq |I_0^n|$ , either

$$\text{dist}(z_{-q(n)+1}, I_{-q(n)+1}^n) \leq \Delta |I_{-q(n)+1}^n|$$

or

$$\frac{\text{dist}(z_{q(n)-1}, I_{q(n)-1}^n)}{|I_{q(n)-1}^n|} \leq C \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}$$

for some constants  $\Delta$  and  $C$ .

Without loss of generality, we assume  $n > 2$ .

Let  $2T_0^1$  denote the extension of the interval  $T_0^1$  to both sides by its half length.

If  $z_0 \notin D(2T_0^1)$ , then by applying Proposition 5.2 twice, there exists  $\varepsilon_1 > 0$  and  $C_1 > 0$  such that  $\text{ang}(z_{-2}, I_0^n) > \varepsilon_1$  and

$$\frac{\text{dist}(z_{-2}, I_{-2}^n)}{|I_{-2}^n|} \leq C_1 \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}.$$

Then the claim follows Lemma 5.4 for such  $z_0$ 's.

If  $z_0 \in D(2T_0^1)$ , then by applying Proposition 5.1 twice,  $z_{-2}$  is either in  $D(\tilde{S}_{0I}^1)$  or in  $Q^{-2}(D(2T_0^1)) \setminus D(\tilde{S}_{0I}^1)$ , where  $\tilde{S}_{0I}^1$  is either  $\tilde{S}_{0L}^1$  or  $\tilde{S}_{0R}^1$ , and furthermore there exists  $\varepsilon_2 > 0$  and  $C_2 > 0$  such that if  $z_{-2} \in Q^{-2}(D(2T_0^1)) \setminus D(\tilde{S}_{0I}^1)$  then  $\text{ang}(z_{-2}, I_{-2}^n) > \varepsilon_2$  and

$$\frac{\text{dist}(z_{-2}, I_{-2}^n)}{|I_{-2}^n|} \leq C_2 \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}.$$

Notice also that if  $z_{-2} \in D(\tilde{S}_{0I}^1)$ , then by applying (2) of Proposition 5.2 twice, we also have that if  $\text{dist}(z_0, I_0^n) \geq |I_0^n|$ , then

$$\frac{\text{dist}(z_{-2}, I_{-2}^n)}{|I_{-2}^n|} \leq C_3 \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}$$

for some constant  $C_3 > 0$ .

**Step 1:** In order to achieve the claim, by Lemma 5.4, we only need to take care of the points  $z_0$ 's with  $z_{-2} \in D(\tilde{S}_{0I}^1)$ . On the other hand, by applying Lemma 5.5, for any small  $\varepsilon > 0$ , either  $z_{-2} \in B_\varepsilon(I_{-2}^n)$  or  $z_{-2} \in S_{\theta,\varepsilon}(\tilde{J}_{0I}^1, I_{-2}^n)$  for some constant  $\theta = \frac{\pi}{2} - O(\varepsilon)$ . Clearly when  $\varepsilon$  is small enough,  $S_{\theta,\varepsilon}(\tilde{J}_{0I}^1, I_{-2}^n) \subset D(\tilde{S}_{0I}^1)$ , and we can also require that one connected component of  $S_{\theta,\varepsilon}(\tilde{J}_{0I}^1, I_{-2}^n)$  is contained in  $D(S_{0I}^1)$ . Therefore by Lemma 5.4, we only need to take care of the points  $z_0$ 's with  $z_{-2} \in S_{\theta,\varepsilon}(\tilde{J}_{0I}^1, I_{-2}^n) \subset D(\tilde{S}_{0I}^1)$ .

**Step 2:** Now we consider separately the points  $z_{-2} \in S_{\theta,\varepsilon}(\tilde{J}_{0I}^1, I_{-2}^n) \setminus D(S_{0I}^1)$  and the points  $z_{-2} \in S_{\theta,\varepsilon}(\tilde{J}_{0I}^1, I_{-2}^n) \cap D(S_{0I}^1)$ .

Let us first consider the points  $z_{-2} \in S_{\theta,\varepsilon}(\tilde{J}_{0I}^1, I_{-2}^n) \setminus D(S_{0I}^1)$ . Since  $Q^{-q(1)+1}$  is univalent on  $D(N_0^1)$  and  $D(N_0^1) \setminus D(\tilde{S}_{0I}^1)$  has a definite modulus, by Schwartz lemma and Koebe distortion theorem, we have

$$\begin{aligned} z_{-q(2)+1} &= z_{-2q(1)+1} = Q^{-q(1)+1}(z_{-q(1)}) = Q^{-q(n)+1}(z_{-2}) \\ &\in D_\theta(Q^{-q(1)+1}(\tilde{J}_{0I}^1)) \cap D(Q^{-q(1)+1}(\tilde{S}_{0I}^1)) \end{aligned}$$

and

$$\frac{\text{dist}(z_{-q(2)+1}, I_{-q(2)+1}^n)}{|I_{-q(2)+1}^n|} \leq C_4 \frac{\text{dist}(z_{-2}, I_{-2}^n)}{|I_{-2}^n|}.$$

If  $z_{-q(2)+1}$  has an  $\varepsilon$ -jump, then the claim follows Lemma 5.4. If

$$z_{-q(2)+1} \in S_{\theta,\varepsilon}(Q^{-q(1)+1}(\tilde{J}_{0I}^1), I_{-q(2)+1}^n) \setminus D(Q^{-q(1)+1}(S_{0I}^1)),$$

then after pulling back  $z_{-q(2)+1}$  one more time and by Proposition 5.1,  $z_{-q(2)}$  will have an  $\varepsilon'$ -jump, where  $\varepsilon'$  essentially depends on the real bounds only, and

$$\frac{\text{dist}(z_{-q(2)}, I_{-q(2)}^n)}{|I_{-q(2)}^n|} \leq C_5 \frac{\text{dist}(z_{-q(2)+1}, I_{-q(2)+1}^n)}{|I_{-q(2)+1}^n|}.$$

And hence the claim follows Lemma 5.4.

**Step 3:** Secondly we consider the points  $z_{-2} \in S_{\theta,\varepsilon}(\tilde{J}_{0I}^1) \cap D(S_{0I}^1)$ . Similar to Step 2, we know that  $z_{-q(2)+1} \in D(Q^{-q(1)+1}(S_{0I}^1))$  and

$$\frac{\text{dist}(z_{-q(2)+1}, I_{-q(2)+1}^n)}{|I_{-q(2)+1}^n|} \leq C_6 \frac{\text{dist}(z_{-2}, I_{-2}^n)}{|I_{-2}^n|}.$$

By (2) of Proposition 5.2, if  $\text{dist}(z_{-q(2)}, I_{-q(2)}^n) \geq |I_{-q(2)}^n|$ , then

$$\frac{\text{dist}(z_{-q(2)}, I_{-q(2)}^n)}{|I_{-q(2)}^n|} \leq C_7 \frac{\text{dist}(z_{-q(2)+1}, I_{-q(2)+1}^n)}{|I_{-q(2)+1}^n|}.$$

By Lemma 5.5, either  $z_{-q(2)}$  has an  $\varepsilon$ -jump or  $z_{-q(2)} \in S_{\theta, \varepsilon}(\tilde{J}_{0I}^2, I_{-q(2)}^n)$ , where we also have  $S_{\theta, \varepsilon}(\tilde{J}_{0I}^2, I_{-q(2)}^n) \subset D(\tilde{S}_{0I}^2)$  and one connected component of  $S_{\theta, \varepsilon}(\tilde{J}_{0I}^2, I_{-q(2)}^n)$  is contained in  $D(S_{0I}^2)$ . If the former case happens then the claim follow Lemma 5.4. If the later one happens, we need one more step to complete an inductive proof of the claim. From Observation 2, the inverse branch  $Q^{-q(2)+1}$  has bounded nonlinearity on the disk  $D_{\theta_0}(\hat{S}_{0I}^2)$ , therefore there exists a constant  $C_8 > 0$ , independent of  $n$ , such that

$$\frac{\text{dist}(z_{-q(2)+1}, I_{-q(2)+1}^n)}{|I_{-q(2)+1}^n|} \leq C_8 \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}.$$

And then by (2) of Proposition 5.2, there exists a constant  $C_9 > 0$ , independent of  $n$ , such that

$$\frac{\text{dist}(z_{-q(2)}, I_{-q(2)}^n)}{|I_{-q(2)}^n|} \leq C_9 \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}.$$

**Step 4:** Repeat the cycle of Steps 1, 2 and 3 by replacing  $q(k-1)$  and  $q(k)$  by  $q(k)$  and  $q(k+1)$  and doing corresponding changes of subscript indexes for  $k = 2, 3, 4, \dots, n-2$ , and then we can show that either  $z_{-s}$  has an  $\varepsilon$ -jump at some moment  $s \in [0, q(n-1)]$  and

$$\frac{\text{dist}(z_{-s}, I_{-s}^n)}{|I_{-s}^n|} \leq C_{10} \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}$$

if  $\text{dist}(z_{-s}, I_{-s}^n) \geq |I_{-s}^n|$ , where the constant  $C_{10}$  is independent of  $n$ , or

$$z_{-q(n-1)} \in S_{\theta, \varepsilon}(\tilde{J}_{0I}^{n-1}, I_{-q(n-1)}^n).$$

If the former case happens, then Lemma 5.4 shows the claim. Suppose that the later one happens. We also have  $S_{\theta, \varepsilon}(\tilde{J}_{0I}^{n-1}, I_{-q(n-1)}^n) \subset D(\tilde{S}_{0I}^{n-1})$ . Since  $Q^{-q(n-1)+1}$  is univalent on  $D(N_0^{n-1})$  and  $D(N_0^{n-1}) \setminus D(\tilde{S}_{0I}^{n-1})$  has a definite modulus, by Schwartz lemma and Koebe distortion theorem, we have

$$\begin{aligned} z_{-q(n)+1} &= z_{-2q(n-1)+1} = Q^{-q(n-1)+1}(z_{-q(n-1)}) \\ &\in D_{\theta}(Q^{-q(n-1)+1}(\tilde{J}_{0I}^{n-1})) \cap D(Q^{-q(n-1)+1}(\tilde{S}_{0I}^{n-1})) \end{aligned}$$

and

$$\frac{\text{dist}(z_{-q(n)+1}, I_{-q(n)+1}^n)}{|I_{-q(n)+1}^n|} \leq C_4 \frac{\text{dist}(z_{-q(n-1)}, I_{-q(n-1)}^n)}{|I_{-q(n-1)}^n|} \leq C_4 C_9 \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}.$$

Together we have shown that for any  $z_0 \in \mathbb{C}_{N_0^n}$  with  $\text{dist}(z_0, I_0^n) \geq |I_0^n|$ , either

$$\frac{\text{dist}(z_{-q(n)+1}, I_{-q(n)+1}^n)}{|I_{-q(n)+1}^n|} \leq C \frac{\text{dist}(z_0, I_0^n)}{|I_0^n|}$$

for a constant  $C > 0$ , or  $z_{-s}$  has an  $\varepsilon$ -jump with  $\text{dist}(z_{-s}, I_{-s}^n) < |I_{-s}^n|$ . If the later one happens, then  $z_{-s} \in D_\alpha(I_{-s}^n)$ , where  $\alpha = \alpha(\varepsilon)$ . By Schwartz lemma,  $z_{-q(n)+1} \in D_\alpha(I_{-q(n)+1}^n)$  and hence  $\text{dist}(z_{-q(n)+1}, I_{-q(n)+1}^n) \leq \Delta |I_{-q(n)+1}^n|$  for a constant  $\Delta > 0$ . We complete a proof of the claim.

Now we can give a simple proof of Sullivan's complex bounds of the renormalizations of  $Q$  by using the following lemma (see Lemma 2.4 in [LY] or Proposition 4.10 in [Mc2]).

**Lemma 5.6** *Let  $U \subset U' \subset \mathbb{C}$  be two topological disks which are conformally equivalent to the unit disk. Suppose that  $F : U \rightarrow U'$  is a complex analytic branched covering map of degree 2 with a non-escaping critical point and a compact filled-in Julia set  $K(f)$ . Then there are topological disks  $K(f) \subset V \subset V' \subset U'$  such that the restriction  $F : V \rightarrow V'$  is a polynomial-like mapping. Moreover, if  $\text{mod}(U' \setminus K(f)) \geq \varepsilon$  then  $\text{mod}(V' \setminus V) \geq \delta(\varepsilon) > 0$ .*

*Proof of Sullivan's complex bounds.* Let  $F_n = Q^{q(n)} : \mathbb{C}_{\tilde{M}_0^n} \rightarrow \mathbb{C}_{\tilde{N}_0^n}$  be the analytic extension of the  $n^{\text{th}}$  renormalization of  $Q$ . By Lemma 5.3, if  $z \in \mathbb{C}_{\tilde{M}_0^n}$  and  $\text{dist}(F_n(z), I_0^n) \geq 1$  then

$$\frac{\text{dist}(z, I_0^n)}{|I_0^n|} \leq C \left( \frac{\text{dist}(F_n(z), I_0^n)}{|I_0^n|} \right)^{\frac{1}{2}}$$

for some constant  $C > 0$ . Then the filled-in Julia set  $K(F_n)$  has its diameter commensurate to  $I_0^n$ . Hence  $K(F_n) \subset D(\tilde{N}_0^k)$  when  $n - k$  is larger than an integer  $N$ , which only depends on the above constant  $C$  and the real bounds.

$\forall z \in K(F_n)$ , let  $z' = F_n(z)$ . Consider the backward orbit

$$z_0 = z', z_{-1}, z_{-2}, \dots, z_{-q(n)} = z.$$

In the proof of Lemma 5.3, either  $z_{-j}$  has an  $\varepsilon$ -jump at some moment or  $z \in D(\tilde{N}_0^n)$ . If the former case happens, then by Koebe distortion theorem,

the property of  $\text{diam}(K(F_n))$  commensurate to  $I_0^n$  implies that  $\text{dist}(z_{-j}, I_{-j}^n)$  is commensurate to  $I_{-j}^n$ , and hence  $z_{-j} \in D_\theta(I_{-j}^n)$  for some  $\theta > 0$ , which depends on  $\varepsilon$  and  $N$ . By the Schwarz Lemma and (1) of Proposition 5.1,  $z \in D_{\theta'}(I_0^n)$  for some  $\theta' > 0$ , which only depends on  $\theta$  and the real bounds. Thus  $z \in D_{\theta'}(I_0^n) \cup D(\tilde{N}_0^n)$  and hence  $K(F_n) \subset D_{\theta'}(I_0^n) \cup D(\tilde{N}_0^n)$ . This implies that  $K(F_n)$  is compact and  $\text{mod}(\mathbb{C}_{\tilde{N}_0^n} \setminus K(F_n)) \geq \varepsilon$  for some constant  $\varepsilon > 0$ , which only depends on the real bounds of the renormalization intervals of  $Q$ . Finally Lemma 5.6 completes the proof.

**Remark 5.1** For an infinitely renormalizable real quadratic polynomial of an arbitrary combinatorial type, there also exists a complex bounds for analytic extensions of renormalizations (see [LS] or [LY] or [GS]), and hence the local connectivity of the Julia set follows to be true by a more general result in [Ji3].

## 6. Some open questions on the Julia set $J(Q_F)$

We finish this paper with three open problems on the study of the Julia set  $J(Q)$  of the Feigenbaum quadratic polynomial  $Q$ .

**Problem 6.1** *Is the Lebesgue measure of the Julia set  $J(Q)$  zero?*

**Problem 6.2** *What is the Hausdorff dimension of  $J(Q)$ ? Is it 2?*

A subset  $S$  of the complex plane is *conformally removable* provided that for any homeomorphism defined on the complex plane if it is conformal in the complement of  $S$  then it is conformal on the whole complex plane

**Problem 6.3** *Is the Julia set  $J(Q)$  conformally removable in the complex plane?*

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