Nonlinearity, Quasisymmetry, Differentiability, And Rigidity in One-Dimensional Dynamics

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(In memory of Professor Liao Shantao)

Abstract
In this article, we review some of our research in the study of one-dimensional dynamical systems, in particular, some technique and results in the study of the smooth classification of certain one-dimensional maps. The main results which we review are that if the conjugacy between two mixing and nice quasi-hyperbolic maps is differentiable at one point with uniform bound, then it is piecewise smooth and that furthermore, if the exponents of these two maps at all singular points are also the same, then the conjugacy is piecewise diffeomorphic. Other results on quasi-hyperbolic maps, on geometrically finite maps, on generalised Ulam-von Neumann transformations, and on expanding circle maps are also reviewed.

Introduction
Topological classification of one-dimensional dynamical systems can be defined as the equivalent relation that \( f \sim g \) if there is a homeomorphism \( h \) such that \( h \circ f = g \circ h \). Here \( h \) is called a conjugacy. An interesting question now is what kinds of geometric properties of a conjugacy can have? For a long time, the study of the Hölder continuity property of a conjugacy is pursued. Quasisymmetry (see §3 for the definition) is a stronger geometric property for a homeomorphism of an interval of the real line because a quasisymmetric homeomorphism is Hölder continuous (see [Ah]). We have been expecting that the conjugacy between two reasonable one-dimensional dynamical systems is quasisymmetric (refer to §3 and to [Su2, MS, Mc, GS, Ly, Ji1, Ji3, Ji8]). We would like to note that a \( C^1 \)-diffeomorphism is quasisymmetric but a quasisymmetric homeomorphism may be very singular meaning that it may map a set with positive measure into a set with zero measure and vice versa. Let us show two examples of conjugacies. The first one is the conjugacy \( h \) between \( f(x) = 1 - 2x^2 \) and \( g(x) = 1 - 2|x| \) on \([-1, 1] \). In this example, \( h \) is quasisymmetric (refer to §5), and more, \( C^1 \)

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on $(-1, 1)$ but not a $C^1$-diffeomorphism of $[-1, 1]$. The second one is the conjugacy $h$ between $g(x) = 1 - 2|x|$ on $[-1, 1]$ and $G(x)$ defined as $4x/3 + 1/3$ on $[-1, 1/2]$ and $-4x + 3$ on $[1/2, 1]$. Then $h$ is quasisymmetric but totally singular (refer to §5). We would like to know when is a conjugacy totally singular, differentiable at some points, $C^1$, or $C^1$-diffeomorphic? We would like to understand that does the properties like singular and differentiable coexist in a conjugacy? These questions are our motivation doing research in this direction. This article reviews a kind of complete answers to these questions in the space of mixing and nice quasi-hyperbolic maps.

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1. Quasi-hyperbolic one-dimensional maps

Let $M$ be the interval $[-1, 1]$ or the unit circle $\mathbb{R}/\mathbb{Z}$. Let $f : M \to M$ be a piecewise $C^1$ map. A point $c \in M$ is said to be singular if either $f'(c)$ does not exist or $f'(c)$ exists but $f'(c) = 0$. The later one is called a critical point. A singular point $c$ of $f$ is said to be power law if there is an interval $(c - \tau_c, c + \tau_c)$ for $\tau_c > 0$ such that $f$ on $(c - \tau_c, c)$ and on $(c, c + \tau_c)$ is $C^1$ and there is a real number $\gamma \geq 1$ such that the limits

$$\lim_{x \to c-} \frac{f'(x)}{|x - c|^\gamma-1} = B_- \quad \text{and} \quad \lim_{x \to c+} \frac{f'(x)}{|x - c|^\gamma-1} = B_+$$

exist and are non-zero. The number $\gamma$ is called the exponent of $f$ at $c$. For a power law singular point $c$, let

$$r_{c,-}(x) = \frac{f'(x)}{|x - c|^\gamma-1}, \quad x \in (c - \tau_c, c), \quad \text{and} \quad r_{c,+}(x) = \frac{f'(x)}{|x - c|^\gamma-1}, \quad x \in (c, c + \tau_c).$$

We would like to note that the exponent of $f$ at a power law singular point is $C^1$-invariant meaning that if $h$ is a $C^1$ diffeomorphism and $h \circ f = g \circ h$, then the exponent of $f$ and $g$ are the same at corresponding power law singular points.

Let $SP$ denote the set of all singular points of $f$ and let $CP$ denote the set of all critical points of $f$. Let $PSO = \bigcup_{i=1}^{\infty} f^i(SP)$ be the set of post-singular orbits. We use $H_0$ to denote the space of all piecewise $C^1$ map $f : M \to M$ such that (1) $SP$ is finite (could be empty) and (2) each singular point in $SP$ is power law type. For a critical point $c \in CP$ and a real number $\tau > 0$, let $U_c(\tau) = [c - \tau, c + \tau]$ and let $U(\tau) = \bigcup_{c \in CP} U_c(\tau)$. Let $V(\tau) = M \setminus U(\tau/2)$. We use $H_1$ to denote the subspace of
all maps $f$ in $\mathcal{H}_0$ satisfying that there are two real numbers $0 < \alpha \leq 1$ and $0 < \tau_1 \leq \tau_c$ for all critical points $c$ such that (1) $f$ restricted to every interval in $M \setminus SP$ is $C^{1+\alpha}$ and (2) for every $c \in CP$, $r_{c,\pm}$ restricted to $U_{c,\pm}(\tau_1) = U_c(\tau_1) \cap (c - \tau_1, c)$ and to $U_{c,\pm}(\tau_1) = U_c(\tau_1) \cap (c, c + \tau_1)$ are $\alpha$-Hölder continuous. A sequence of intervals \( \{I_i\} \) is said to be a $f$-chain if $I_i \subset M \setminus SP$ for all $0 \leq i \leq n$ and if $I_{i+1} = f(I_i)$ and $f : I_i \to I_{i+1}$ is $C^1$ and injective for all $0 \leq i \leq n - 1$.

**Definition 1.1** A map $f$ in $\mathcal{H}_1$ is said to be quasi-hyperbolic if there is a constant $0 < \tau \leq \tau_1$ such that (a) $\overline{PSO} \cap U(\tau) = \emptyset$; (b) there exist constants $C > 0$ and $0 < \mu < 1$ such that for any $f$-chain $\{I_i\}$ satisfying either $I_i \subseteq V(\tau)$ for all $0 \leq i \leq n - 1$ or $I_n \subseteq U(\tau)$, \( |I_0| \leq C\mu^n|I_n| \).

We use $\mathcal{H}$ to denote the space of all quasi-hyperbolic maps. The space $\mathcal{H}$ contains many interesting maps. Let us give three examples. A point $q$ in $M$ is called periodic of period $k$ of $f$ if $\overline{f^i(q)} = q$ for all $0 < i < k$ but $\overline{f^k(q)} = q$. For a periodic point $q$ of period $k$ of $f$, let $O = \{\overline{f^i(p)}\}^{k-1}_{i=0}$ be the periodic orbit and let $e_{O,f} = (f^k)'(q)$ be the eigenvalue of $f$ at $O$. The periodic orbit $O$ is called attractive if $|e_{O,f}| < 1$; parabolic if $|e_{O,f}| = 1$; expanding if $|e_{O,f}| > 1$. The first example comes from a theorem (see [MS, Theorem 6.3, pp. 261-262]). A critical point $c$ of a $C^2$ map is non-degenerate if $f'(c) = 0$ and if $f''(c) \neq 0$.

**Example 1.1** A $C^2$ map $f$ in $\mathcal{H}_1$ with only non-degenerate critical points such that $\overline{PSO} \cap SP = \emptyset$ and such that all periodic points are expanding.

The Schwarzian derivative of a $C^3$ map $f$ is

$$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$  

If $S(f)(x) < 0$ for all $x$ in $M$, we say $f$ has negative Schwarzian derivative. Singer (see [Si]) proved that if $f$ has negative Schwarzian derivative, then the immediate basin of every attractive or parabolic periodic orbit contains at least one critical orbit. Therefore, if $f$ has negative Schwarzian derivative and if $\overline{PSO} \cap SP = \emptyset$, and if $\overline{PSO}$ contains neither attractive nor parabolic periodic points, then all periodic points of $f$ are expanding. We have that

**Example 1.2** A $C^3$ map $f$ in $\mathcal{H}_1$ having negative Schwarzian derivative and satisfying that $\overline{PSO} \cap SP = \emptyset$ and that $\overline{PSO}$ contains neither attractive periodic points nor parabolic periodic points.

Suppose $f$ in $\mathcal{H}_1$ is $C^3$ and satisfies the condition that for every critical point $c$, $f^{\circ m}(c)$ is an expanding periodic point of $f$ for some integer $m \geq 1$, then $\overline{PSO} = PSO$ contains neither attractive nor parabolic periodic points. The condition here is called the preperiodic condition. Following Example 1.2, we have another example of a quasi-hyperbolic map.

**Example 1.3** A $C^3$ map $f$ in $\mathcal{H}_1$ having negative Schwarzian derivative and satisfying the preperiodic condition.
There are three more examples, which we call geometrically finite maps, generalized Ulam-von Neumann transformations, and circle expanding maps, will be discussed in later sections.

2. Nonlinearity of quasi-hyperbolic maps

The study of the nonlinearity of a map is an important step in dynamical systems. In one complex variable, Koebe’s distortion theorem (see [Bi]) represents a beautiful result showing how the nonlinearity can be controlled for all schlicht functions defined on the unit disk. In one-dimensional dynamical systems, the Denjoy distortion technique becomes an important tool to estimate the nonlinearity of a $C^2$ diffeomorphism of the circle (see [De]). In this section, we review some technique to study nonlinearity of quasi-hyperbolic maps.

Let us start from a higher smooth map. Let $D^3$ be the space of $C^3$-diffeomorphisms $f : I \to J$ where $I$ and $J$ are intervals of the real line $\mathbb{R}$. Let $S(f)$ be the Schwarzian derivative. We use $S(f) \geq 0$ (or $S(f) \leq 0$) to mean that $S(f)(x) \geq 0$ (or $S(f)(x) \leq 0$) for all $x$ in $I$. There is the chain rule for the Schwarzian derivatives: for any two maps $g : K \to I$ and $f : I \to J$ in $D^3$, $S(f \circ g) = (g')^2 \cdot S(f) \circ g + S(g)$. The chain rule implies that $S(f \circ g) \geq 0$ and $S(f^{-1}) \leq 0$ if $S(f) \geq 0$ and $S(g) \geq 0$ and that $S(f \circ g) \leq 0$ and $S(f^{-1}) \geq 0$ if $S(f) \leq 0$ and $S(g) \leq 0$.

The nonlinearity $N(f)$ is $N(f) = f''/f'$. It is easy to check that

$$S(f) = (N(f))' - \frac{1}{2}(N(f))^2.$$ 

The following lemma (see, for example, [Ji7] and [Ji1, Chapter 2] for the proof) shows that how one can control nonlinearity by Schwarzian derivative.

**Lemma 2.1 (C^2-Koebe Type Distortion Lemma)** Let $f : I = [a, b] \to J$ be a map in $D^3$. Suppose there is a constant $K \geq 0$ such that $S(f)(x) \geq -K$ for all $x$ in $I$. Then, for any $\epsilon > 0$,

$$\left|N(f)(x)\right| \leq \max\left\{\sqrt{2K + \epsilon}, \frac{2K + \epsilon}{\epsilon} \cdot \frac{2}{d(x, \partial I)}\right\}$$

for $x$ in $I$, where $d(x, \partial I) = \min\{|x - a|, |x - b|\}$. In particular, if $S(f) \geq 0$, then

$$\left|N(f)(x)\right| \leq \frac{2}{d(x, \partial I)}$$

for $x$ in $I$.

This lemma tells us a kind of magic because $f$ could be an arbitrary map in $D^3$ (compare to Koebe’s distortion theorem in one complex variable). By a detail analysis of the reason behind this magic, we can generalize the lemma to any quasi-hyperbolic map.
Suppose $f$ is a quasi-hyperbolic map. Let $0 < \alpha \leq 1$ be a number in §1 and $U(\tau)$ and $V(\tau)$ be sets in Definition 1.1 satisfied by $f$. Let $\gamma$ be the maximum among the exponents of $f$ at all singular points. A $f$-chain $I = \{I_i\}_{i=0}^n$ is said to be regulated if either $I_i \subseteq V(\tau)$ or $I_i \subseteq U(\tau)$ for all $0 \leq i \leq n$. For any $x$ and $y$ in $M$, let $x_n = f^{on}(x)$ and $y_n = f^{on}(y)$. The proof of the following lemma is not so hard, refer to [Ji1, Ji3, Ji4, Ji5, Ji6].

**Lemma 2.2 (First Distortion Lemma)** There is a constant $C_0 > 0$ such that for any regulated $f$-chain $I = \{I_i\}_{i=0}^n$ satisfying $I_i \subseteq V$ for all $1 \leq i \leq n - 1$,

$$\left| \log \left( \frac{|(f^{on})'(x)|}{|(f^{on})'(y)|} \right) \right| \leq C_0 |x_n - y_n|^\alpha.$$ 

for all $x$ and $y$ in $I_0$.

The following result is one of the key lemmas in our study and it generalizes the $C^3$-Koebe distortion lemma to quasi-hyperbolic maps. One can find the proof from [Ji1, Ji3, Ji4, Ji5, Ji6]. The proof uses Lemma 2.2.

**Lemma 2.3 ($C^{1+h}$-Denjoy-Koebe Type Distortion Lemma)** There are constants $C_1, C_2 > 0$ such that for any regulated $f$-chain $I = \{I_i\}_{i=0}^n$ with $I_n \subseteq U_0$,

$$\left| \log \left( \frac{|(f^{on})'(x)|}{|(f^{on})'(y)|} \right) \right| \leq C_1 |x_n - y_n|^\beta + C_2 \frac{|x_n - y_n|}{d(\{x_n, y_n\}, PSO)}.$$ 

for all $x$ and $y$ in $I_0$.

A special case of Lemma 2.3 is often used in our study.

**Lemma 2.4 (Second Distortion Lemma)** Suppose $U_0$ is an open set such that $d(U_0, \text{PSO}) > 0$. Then there is a constant $C_3 > 0$ such that for any regulated $f$-chain $\{I_i\}_{i=0}^n$ with $I_n \subseteq U_0$,

$$\left| \log \left( \frac{|(f^{on})'(x)|}{|(f^{on})'(y)|} \right) \right| \leq C_3 |x_n - y_n|^\alpha$$

for all $x, y \in I_0$.

Another similar result for higher dimensional dynamical systems is proved in [Ji9], which is called the geometric distortion theorem. Its proof is also done by a detail analysis of the reason behind Koebe’s distortion Theorem in one complex variable. It is a kind of generalizations of Koebe’s distortion lemma for a larger class of maps including many non-conformal maps. The reader who is interested in this result and its application to some higher dimensional dynamical systems may refer to [Ji1, Chapter 2] too.
3. Geometry of geometrically finite maps

In this section, we review some results on the geometric aspect of geometrically finite maps which form a special subspace of quasi-hyperbolic maps. Suppose $f$ is in $\mathcal{H}_1$. Let $SO = \cup_{n=0}^{\infty} f^n(SO)$ be the union of singular orbits of $f$. If $SO$ is non-empty and finite, let $\eta_1 = \{I_0, \ldots, I_{k-1}\}$ be the set of the closures of intervals in $M \setminus SO$, then $(f, \eta_1)$ is a Markov map, this means that

a. $I_0, \ldots, I_{k-1}$ have pairwise disjoint interiors,

b. the union $\cup_{i=0}^{k-1} I_i$ of all intervals in $\eta_1$ is $M$,

c. the restriction $f : I \to f(I)$ for every interval $I$ in $\eta_1$ is homeomorphic, and

d. the image $f(I)$ of every interval $I$ in $\eta_1$ is the union of some intervals in $\eta_1$.

Here $\eta_1$ is called a Markov partition. Let $g_i = (f|I_i)^{-1}$ be the inverse of $f : I_i \to f(I_i)$ for each $I_i \in \eta_1$. A sequence $w_n = i_0 \cdots i_{n-1}$ of 0's, $\ldots$, $(k-1)'$s is called admissible if the domain $f(I_{i_l})$ of $g_{i_l}$ contains $I_{i_{l+1}}$ for all $0 \leq l < n - 1$. For an admissible sequence $w_n = i_0 \cdots i_{n-1}$ of 0's, $\ldots$, $(k-1)'$s, we can define $g_{w_n} = g_{i_0} \circ \cdots \circ g_{i_{n-1}}$ and $I_{w_n} = g_{w_n}(I_{i_{n-1}})$. Let $\eta_n$ be the set of the intervals $I_{w_n}$ for all admissible sequences of length $n$. It is also a Markov partition of $M$ respect to $f$. We call it the $n^{th}$-partition of $M$ induced from $(f, \eta_1)$. Let $\kappa_n$ be the maximum of the lengths of intervals in $\eta_n$. Then $f$ in $\mathcal{H}_1$ is said to be geometrically finite if

1. the set of singular orbits $SO$ is non-empty and finite,

2. no critical point is periodic, and

3. there are constants $C > 0$ and $0 < \mu < 1$ such that $\kappa_n \leq C \mu^n$ for all $n > 0$.

Suppose $f$ is a geometrically finite map. Let $\eta = \{\eta_n\}_{n=1}^{\infty}$ be the sequence of nested Markov partitions we have constructed above. We call it the natural sequence of Markov partitions for $f$. Using this natural sequence, we define bounded geometry and bounded nearby geometry for $f$ as follows: $f$ is said to have bounded geometry if there is a constant $C > 0$ such that the ratio $|J_1|/|I|$ $\geq$ $C$ for every pair $J \subset I$ with $J \in \eta_{n+1}$ and $I \in \eta_n$ and every integer $n > 0$: $f$ is said to have bounded nearby geometry if there is a constant $C > 0$ such that the ratio $|J_1|/|J_2|$ $\geq$ $C$ for every pair $J_1$ and $J_2$ in $\eta_n$ with a common endpoint and for every $n > 0$. By applying the technique in §2, we have

**Theorem 3.1** Any geometrically finite map has bounded geometry and bounded nearby geometry.

A homeomorphism $h$ is called quasisymmetric if there is a constant $C > 0$ such that

$$C^{-1} \leq \frac{|h(x) - h(z)|}{|h(z) - h(y)|} \leq C$$

for $x, y \in M$ where $z = (x + y)/2$. Following the bounded geometry and bounded nearby geometry, we could easily get
Theorem 3.2. Suppose $f$ and $g$ are two conjugate geometrically finite maps and suppose $h$ is the conjugacy between $f$ and $g$, i.e., $h \circ f = h \circ g$. Then $h$ is quasisymmetric.

The proofs of Theorems 3.1 and 3.2 can be found in [Ji3] (see also [Ji1, Chapter 3]). The quasisymmetric classification for certain geometrically infinite one-dimensional maps by using a similar method has been done in [Ji8], where we also find a natural sequence of nested Markov partitions and prove that this sequence of partitions has bounded geometry and bounded nearby geometry.

Let $\Sigma = \{a = i_0 i_1 \ldots i_n \ldots \}$ be the set of all infinite admissible sequences and let $\sigma(a) = i_1 \ldots i_n \ldots$ be the shift from $\Sigma$ into itself. Then $(\Sigma, \sigma)$ is semi-conjugate to $(M, f)$, i.e., there is a continuous onto map $\pi : \Sigma \rightarrow M$ such that

\[ \pi \circ \sigma = f \circ \pi. \]

Moreover, $\pi$ is one-to-one on all but countably many points in $\Sigma$ and on these countably many points $\pi$ is two-to-one. We call $\Sigma$ the symbolic space of $f$.

Now let us define the dual symbolic space of $f$. Let $\Gamma_n$ be the set of all admissible sequences $w_n$ of length $n$. A $(n, m)$-right cylinder about $w_n^0 = i_{n-1}^0 \ldots i_0^0$ for $0 \leq m \leq n - 1$ is

\[ \{w_n = i_{n-1} \ldots i_0 \in \Gamma_n \mid i_l = i_l^0, l = 0, \ldots, m\}. \]

All the $(n, m)$-right cylinders form a topological basis of $\Gamma_n$. Let $\Gamma^*_n$ be the set $\Gamma_n$ with this topological basis and $(\Sigma^*, \sigma^*)$ be the inverse limit of the sequence $\{(\Gamma^*_n, \sigma^*_n)\}_{n=1}^\infty$ where $\sigma^*_n : \Gamma^*_n \rightarrow \Gamma^*_{n-1}$ is defined as $\sigma^*_n(i_n \ldots i_1 i_0) = i_n \ldots i_1$ (the inclusion). So $\sigma^* : \Sigma^* \rightarrow \Sigma^*$ is the shift. We call $\Sigma^* = \{a^* = \ldots i_0 \ldots \}$ the dual symbolic space of $f$.

Both symbolic and dual symbolic spaces are invariant under topological classification.

The scaling function of $f$ is a function defined on the dual symbolic space $\Sigma^*$ as follows: for $a^* = \ldots i_1 i_0$ and $w_n = i_{n-1} \ldots i_1 i_0$, we have $\sigma^*(a^*) = \ldots i_1$; we also use $\sigma^*(w_n)$ to denote $i_{n-1} \ldots i_1$. For any $a^* = \ldots w_n$ in $\Sigma^*$, define

\[ s(w_n) = \frac{|I_{w_n}|}{|I_{\sigma^*(w_n)}|}. \]

Definition 3.2. If $\lim_{n \rightarrow +\infty} s(w_n)$ exists for every $a^*$ in $\Sigma^*$, then we define the scaling function

\[ s_f(a^*) = \lim_{n \rightarrow +\infty} s(w_n) \]

on $\Sigma^*$.

A geometrically finite map $f$ is said to be non-critical if it has no critical point; and $f$ is said to be critical if it has critical points. A function $s$ on $\Sigma^*$ is said to be Hölder if there are constants $C > 0$ and $0 < \nu < 1$ such that

\[ |s(a_1^n) - s(a_2^n)| \leq C \nu^n \]

whenever the first $n$ digits of $a_1^n$ and $a_2^n$ in $\Sigma^*$ are the same. By applying Lemma 2.2, we have
Theorem 3.3 The scaling function $s_f : \Sigma^* \to \mathbb{R}$ of a non-critical geometrically finite map $f$ exists and is Hölder.

Because of the existence of critical points for a critical geometrically finite one-dimensional map, we have by applying Lemmas 2.2 and 2.3

Theorem 3.4 The scaling function $s_f : \Sigma^* \to \mathbb{R}$ of a critical geometrically finite map $f$ exists but is discontinuous. All discontinuities are jump discontinuities.

The proofs of Theorems 3.3 and 3.4 can be found in [Ji4] (see also [Ji1, Chapter 3]). Some study of scaling functions for generalized Ulam-von Neumann transformations (see §5) can be found in [Ji10] (see also [Ji1, Chapter 3]). Some other study of scaling functions for circle expanding maps and their relations with Gibbs theory and Teichmüller theory for circle endomorphisms can be found in [CJQ, CGJ].

4. Differentiability of conjugacies

Let $m$ mean the Lebesgue measure on $M$. A quasi-hyperbolic map $f$ is said to be nice if

(i) $m(\text{PSO}) = 0$ and

(ii) there is an open neighborhood $W$ such that $\text{PSO} \subset W$ and $M \setminus W \neq \emptyset$ and such that for any point $p$ in $M$ either $\{f^{\circ n}(p)\}_{n=N}^\infty \subseteq \text{PSO}$ for some $N > 0$ or there is a subsequence $\{f^{\circ n_i}(p)\}_{i=1}^\infty \subseteq M \setminus W$.

We also need the mixing condition on $f$ so that $\{f^{\circ n}\}_{n=0}^\infty$ can not be decomposed into several dynamical systems. We say that $f$ is mixing if for any intervals $I$ and $J$ of $M$, there is an integer $n \geq 0$ such that $f^{\circ n}(J) \supseteq I$. The mixing condition is invariant under topological conjugacy.

Denote $M_0 = M \setminus \text{PSO}$. For any point $p$ in $M$, let $BO(p)$ be the set of points $q$ such that there is an integer $n \geq 0$ such that $f^{\circ n}(q) = p$. The set $BO(p)$ is called the backward orbit of $p$. It is countable.

Suppose that $f$ and $g$ are two conjugate quasi-hyperbolic maps and that $h$ is the conjugacy. From the equation $h \circ f = g \circ h$, we have that if $h$ is differentiable at $p \in M_0$, then $h$ is differentiable at all points in $BO(p)$. We say that $h$ is differentiable at $p \in M_0$ with uniform bound if there are a small neighborhood $Z$ of $p$ and a constant $C > 0$ such that

$$C^{-1} \leq |h'(q)| \leq C$$

for all $q \in BO(p) \cap Z$. Let $x$ be a point in $M$ and let $\omega(x)$ be all accumulation points of the forward orbit $\{f^{\circ n}(x)\}_{n=1}^\infty$, that is, $y \in \omega(x)$ if and only if there is a subsequence of this forward orbit converging to $y$. A point $x$ is called self-recurrent if $x \in \omega(x)$. Let $\Lambda = \bigcup_{n=0}^\infty f^{-n}(\text{PSO})$. Then $m(\Lambda) = 0$. Let $\Omega$ be the set of all self-recurrent points in $M \setminus \Lambda$ of $f$. The proof of the following proposition can be found in [Ji5, Ji6].

Proposition 4.1 The set $\Omega$ has full measure in $M$. 

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By applying this proposition and the technique in §1, we have

**Theorem 4.1** Suppose that $f$ and $g$ are conjugate nice and mixing quasi-hyperbolic maps and suppose that $h$ is the conjugacy from $f$ to $g$, i.e., $h \circ f = g \circ h$. If $h$ is differentiable at one point $p$ in $M_0$ with uniform bound, then $h|_{M_0}$ is $C^1$.

Consider the conjugacy $h$ between $f(x) = 1 - 2x^2$ and $g(x) = 1 - 2|x|$ on $[-1, 1]$. Then $h$ and $h^{-1}$ are both $C^1$ on $(-1, 1)$. But, one of $h'$ and $(h^{-1})'$ is uniformly continuous. The reason is that the exponents of $f$ and $g$ at 0 are different. However, we have

**Theorem 4.2** Suppose $f$ and $g$ and $h$ are the same as those in Theorem 4.1. If $h$ is differentiable at one point in $M_0$ with uniform bound and all the exponents of $f$ and $g$ at the corresponding singular points are the same, then $h$ restricted on the closure of every interval of $M_0$ is $C^{1+\beta}$ for some fixed $0 < \beta \leq 1$.

Main ingredients in the proofs of Theorems 4.1 and 4.2 are the following two lemmas. Remember that $U_i(\tau) = [c_i - \tau, c_i + \tau]$ for each critical point $c_i$ of $f$ and that $V(\tau) = M \setminus \bigcup_{i=1}^{d} U_i(\tau/2)$. Let $0 < \alpha \leq 1$ be a number in §1 satisfied by both $f$ and $g$. Let $\gamma$ be the maximal number among all exponents of $f$ and $g$.

**Lemma 4.1** Suppose $f$ and $g$ and $h$ are the same as those in Theorem 4.1. If $h$ is differentiable at one point $p$ in $M_0$ with uniform bound, then $h|_{U_i(\tau)}$ is $C^{1+\alpha}$.

**Lemma 4.2** Suppose $f$ and $g$ and $h$ are the same as those in Theorem 4.1. If $h$ is differentiable at one point in $M_0$ with uniform bound and all the exponents of $f$ and $g$ at the corresponding singular points are the same, then $h|_{J}$ for every interval $J$ in $V \setminus T_{SO}$ is $C^{1+\frac{\alpha}{\gamma}}$.

The proof of these two lemmas can be found in [Ji6] (also refer to [Ji5]). The number $\beta$ in Theorem 4.2 can be well calculated as $\alpha/\gamma$. The condition that differentiable at $p \in M_0$ with uniform bound in Theorems 4.1 and 4.2 can be checked by eigenvalues at periodic points.

**Lemma 4.3** Suppose $f$ and $g$ and $h$ are the same as those in Theorem 4.1. If $h$ is differentiable at a point $p$ in $M_0$ with non-zero derivative and if there is an open interval $Y$ about $p$ such that the absolute values of the eigenvalues of $f$ and $g$ at periodic points in $Y$ and at corresponding periodic points in $h(Y)$ are the same, then $h$ is differentiable at $p$ with uniform bound.

The proof of this lemma can be found in [Ji6] (also refer to [Ji5]). Thus we have the follow corollaries in [Ji6].

**Corollary 4.1** Suppose $f$ and $g$ and $h$ are the same as those in Theorem 4.1. If $h$ is differentiable at one point $p$ in $M_0$ with non-zero derivative and if the absolute values of the eigenvalues of $f$ and $g$ at periodic points in a small neighborhood $Y$ about $p$ and at corresponding periodic points in $h(Y)$ are the same, then $h|_{M_0}$ is $C^1$. Furthermore, if all the exponents of $f$ and $g$ at the corresponding singular points are also the same, then $h$ restricted on the closure of every interval of $M_0$ is $C^{1+\beta}$, $0 < \beta \leq 1$. 

**Corollary 4.2** Suppose $f$ and $g$ and $h$ are the same as those in Theorem 4.1. If there is a small interval $Y$ of $M$ such that $h|Y$ is absolutely continuous, then $h|M_0$ is $C^1$. Furthermore, if all the exponents of $f$ and $g$ at the corresponding singular points are also the same, then $h$ restricted on the closure of every interval of $M_0$ is $C^{1+\beta}$, $0 < \beta \leq 1$.

Now let us apply the above results to the space of geometrically finite maps. Let $\mathcal{F}$ be a conjugacy class in the space of geometrically finite maps, this means that every two maps in $\mathcal{F}$ are conjugated by a homeomorphism. We say $\mathcal{F}$ is mixing if one map in it is mixing. Since the mixing condition is invariant under topological conjugacy, this is equivalent to say that any map in $\mathcal{F}$ is mixing. For a map $f$ in $\mathcal{F}$, the post-singular orbit $PSO$ is finite, so $m(PSO) = 0$. Moreover, every critical point $c \in CP$ lands at an expanding periodic point of $f$ (this means that $f^k(c) = p$ is an expanding periodic point for some $k > 0$.) So $f$ satisfies Conditions (i) and (ii) in the definition of nice, that is, $f$ in $\mathcal{F}$ is always nice. Note that in this case $M_0 = M \setminus PSO$ consists of finitely many intervals. Thus we have the following corollaries which are originally obtained in [Ji4, Ji5].

Suppose that $f$ and $g$ are in a mixing class $\mathcal{F}$ and suppose $h$ is the conjugacy between $f$ and $g$, i.e., $h \circ f = g \circ h$.

**Corollary 4.3** The map $h$ restricted on the closure of every interval of $M_0$ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if $h$ is differentiable at a point $p$ in $M_0$ with uniform bound and all the exponents of $f$ and $g$ at the corresponding singular points are the same.

**Corollary 4.4** The map $h$ restricted on the closure of every interval of $M_0$ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if $h$ is differentiable at a point $p$ in $M_0$ with non-zero derivative and the absolute values of the eigenvalues of $f$ and $g$ at periodic points in a small neighborhood $Y$ about $p$ and at corresponding periodic points in $h(Y)$ and all the exponents of $f$ and $g$ at the corresponding singular points are the same.

**Corollary 4.5** The map $h$ restricted on the closure of every interval of $M_0$ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if there is a small interval $Y$ of $M$ such that $h|Y$ is absolutely continuous and all the exponents of $f$ and $g$ at the corresponding singular points are also the same.

**Corollary 4.6** The map $h$ restricted on the closure of every interval of $M_0$ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if $s_f = s_g$ and all the exponents of $f$ and $g$ at the corresponding singular points are the same.

### 5. Generalised Ulam-von Neumann transformations

In this section, let $M = [-1, 1]$. We say $f : M \to M$ is a generalised Ulam-von Neumann transformation if
(a) $f$ is a geometrically finite map with only one singular point 0,

(b) $f(-1) = f(1) = -1$ and $f(0) = 1$,

(c) $f|[-1,0]$ is $C^1$ and increasing and $f|[0,1]$ is $C^1$ and decreasing.

One example of a generalised Ulam-von Neumann transformation is $f(x) = 1 - 2|x|^{2\gamma}$ for $\gamma \geq 1$. Another one is $g(x) = -1 + 2\cos(\pi x/2)$. For a generalised Ulam-von Neumann transformation $f$, let $I_0 = [-1,0]$ and $I_1 = [0,1]$. We then have that $f(I_0) = f(I_1) = M$. Thus $\mathcal{M}_0 = \{I_0, I_1\}$ is a Markov partition of $M$ with respect to $f$. The post-singular orbit $\text{PSO} = \bigcup_{n=1}^{\infty} f^n(0)$ of $f$ is the boundary $\{1,1\}$ of $M$.

Any two generalised Ulam-von Neumann transformations $f$ and $g$ are topologically conjugate by an orientation-preserving homeomorphism $h$. Following the results in §4, we have the following corollaries for generalised Ulam-von Neumann transformations which originally studied in [Ji11] with a published version [Ji2] (see also [Ji1, Chapter 3]).

Suppose $f$ and $g$ are two generalised Ulam-von Neumann transformations. Let $h$ be the conjugacy from $f$ to $g$, i.e., $h \circ f = g \circ h$.

**Corollary 5.7** The map $h|(-1,1)$ is $C^1$ if and only if $h$ is differentiable at a point $p$ in $(-1,1)$ with uniform bound. Moreover, $h$ is a $C^{1+\beta}$-diffeomorphism of $M$ for some $0 < \beta \leq 1$ if and only if $h$ is differentiable at a point $p$ in $(-1,1)$ with uniform bound and the exponents of $f$ and $g$ at $0$ are the same.

**Corollary 5.8** The map $h|(-1,1)$ is $C^1$ if and only if $h$ is differentiable at a point $p$ in $(-1,1)$ and eigenvalues of $f$ and $g$ at all periodic points in a small neighborhood $Y$ about $p$ and at corresponding periodic points in $h(Y)$ are the same. Moreover, $h$ is a $C^{1+\beta}$-diffeomorphism of $M$ for some $0 < \beta \leq 1$ if and only if $h$ is differentiable at a point $p$ in $(-1,1)$, eigenvalues of $f$ and $g$ at all periodic points in a small neighborhood $Y$ of $p$ and at corresponding periodic points in $h(Y)$ are the same, and the exponents of $f$ and $g$ at $0$ are the same.

**Corollary 5.9** The map $h|(-1,1)$ is $C^1$ if and only if there is a small interval $Y$ such that $h|Y$ is absolutely continuous. Moreover, $h$ is a $C^{1+\beta}$-diffeomorphism of $M$ for some $0 < \beta \leq 1$ if and only if there is a small interval $Y$ such that $h|Y$ is absolutely continuous and the exponents of $f$ and $g$ at $0$ are the same.

If the eigenvalues at all corresponding periodic points and the exponents at $0$ of two generalised Ulam-von Neumann transformations of $f$ and $g$ are the same, then $h$ is a bi-Lipschitz homeomorphism (see [Ji2]). So $h$ is an absolutely continuous homeomorphism. In [Ji2], we argued from bi-Lipschitz to $C^1$ by the Gibbs theory. Now it is just a consequence of Theorem 4.1 as follows.

**Corollary 5.10** The map $h$ is a $C^{1+\beta}$-diffeomorphism for some $0 < \beta \leq 1$ if and only if the eigenvalues of $f$ and $g$ at corresponding periodic points and the exponents of $f$ and $g$ at $0$ are the same.
6. Expanding circle endomorphisms

In this section, let $M = \mathbb{R}/\mathbb{Z}$ be the unit circle. Let $f : M \to M$ be a $C^{1+\alpha}$-orientation-preserving endomorphism for some $0 < \alpha \leq 1$. We say $f$ is expanding if there are constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| \geq C\lambda^n$ for all $x \in S^1$ and all $n \geq 1$. Let $d = \deg(f)$ be the topological degree of $f$. An example of an expanding circle endomorphism of degree $d$ is $q_d : x \mapsto dx \pmod 1$. Shub [Sh] proved that every degree $d$ such map is topologically conjugate to $q_d$. Thus any two $C^{1+\alpha}$-orientation-preserving expanding circle endomorphisms $f$ and $g$ of degree $d > 1$ are topologically conjugate by an orientation-preserving homeomorphism $h$ of $M$. A circle expanding map is a quasi-hyperbolic map without any singular point. We have

**Corollary 6.11** Let $f$ and $g$ be two $C^{1+\alpha}$-orientation-preserving expanding circle endomorphisms of degree $d > 1$ where $0 < \alpha \leq 1$ is a real number. Let $h$ be the conjugacy from $f$ to $g$, i.e., $h \circ f = g \circ h$. Then $h$ is a $C^{1+\alpha}$-diffeomorphism if and only if $h$ is differentiable at one point $p$ with uniform bound.

This gives two results due to Shub and Sullivan [SS] and to Sullivan [Su1].

**Corollary 6.12** Let $f$ and $g$ be two $C^{1+\alpha}$-orientation-preserving expanding circle endomorphisms of degree $d > 1$ where $0 < \alpha \leq 1$ is a real number. Let $h$ be the conjugacy from $f$ to $g$, i.e., $h \circ f = g \circ h$. Then $h$ is a $C^{1+\alpha}$-diffeomorphism if and only if $h$ is absolutely continuous.

**Corollary 6.13** Let $f$ and $g$ be two $C^{1+\alpha}$-orientation-preserving expanding circle endomorphisms of degree $d > 1$ where $0 < \alpha \leq 1$ is a real number. Let $h$ be the conjugacy from $f$ to $g$, i.e., $h \circ f = g \circ h$. Then $h$ is a $C^{1+\alpha}$-diffeomorphism if and only if the eigenvalues of $f$ and $g$ at all periodic points of $f$ are the same.

Another interesting topic in the rigidity of one-dimensional maps is to study the conjugacy between two diffeomorphisms of the interval $[0, 1]$ which have 0 as an expanding fixed point and 1 as an attracting fixed point and which have no other fixed points. In this case, eigenvalues at 0 and 1 are not enough to give a complete smooth classification of such diffeomorphisms. There is another intermediate smooth invariant. This intermediate invariant and the eigenvalues at 0 and 1 form a complete smooth invariants. Li and Zhang [LZ] studied in this direction by embedding the diffeomorphisms of $[0, 1]$ into some flows. This study relates to the smooth classification of those Cantor sets as the maximal invariant sets of $C^{1+h}$ expanding maps on $[0, 1]$ (see [Su3, BF]).

**References**


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