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A Proof of the Existence and Simplicity of a Maximal Eigenvalue for Ruelle–Perron–Frobenius Operators

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Abstract. We give a new proof of a result due to Ruelle about the existence and simplicity of a unique maximal eigenvalue for a Ruelle–Perron–Frobenius operator acting on some Hölder continuous function space.

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1. Introduction

Ruelle's Theorem represents remarkable progress in the study of thermodynamical formalism. The theorem concerns transfer operators with positive weights associated with certain dynamical systems. These kind of operators are called Ruelle–Perron–Frobenius operators. Ruelle's theorem was first proved by Ruelle [8, 9] in the study of the existence and uniqueness of the Gibbs measure associated with a one-sided finite-type subshift and a Hölder continuous potential function. The result was extended to continuous positive expansive transformations by Walters [12] by using g-measures. It has become a standard technique in thermodynamical formalism. In the original proof (see [2, 12]), the dual operator acting on the space of measures was first studied and the Schauder–Tychonoff fixed point theorem was used with the dual operator and then some difficult analysis followed. There is another proof of Ruelle's theorem associated with a one-sided finite-type subshift by Fan [3] by applying an idea from probability theory. A key part of Ruelle's theorem deals with the existence and simplicity of a unique maximal eigenvalue for a Ruelle–Perron–Frobenius operator. A more geometric proof of this part is given by Ferrero and Schmitt [4] using the Hilbert projective metrics defined by Birkhoff [1] on convex cones in Banach spaces. They showed that Ruelle–Perron–Frobenius operator acting on a certain Hölder continuous function space contracts the Hilbert projective metrics of certain convex cones in this Hölder continuous function space. So the existence and

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simplicity of the unique maximal eigenvalue follows from the contracting fixed point theorem. Here we give a new proof of the existence and simplicity of a unique maximal eigenvalue for a Ruelle–Perron–Frobenius operator acting on a certain Hölder continuous function space without using any fixed-point theorems and any properties for convex cones. The existence and simplicity of a unique maximal eigenvalue for a Ruelle–Perron–Frobenius operator has many interesting applications. One of them is to Krzyzewski–Szlenk's theorem [5] (see also [6, 11]) concerning the existence and uniqueness of the smooth probability invariant measure for a $C^{1+\alpha}$ locally expanding map from a compact C^2 Riemannian manifold into itself.

The Letter is organized as follows. In Section 2, we define local expanding and mixing maps from a compact metric space into itself and Ruelle–Perron–Frobenius operators. In the same section, we give our proof of the main theorem due to Ruelle. Before presenting our new proof, we first prove several key lemmas (Lemmas 1 to 3). In Section 3, we provide an application (Corollary) to a result from Krzyzewski and Szlenk about smooth probability invariant measures in dynamical systems.

2. Existence and Simplicity of a Maximal Eigenvalue

Let (X, d) be a compact metric space and let $B(x, r)$ mean the open ball centered at x with radius $r > 0$. Let $f: X \to X$ be a locally expanding map, i.e., there are constants $\lambda > 1$ and $b > 0$ such that $f|B(x, b)$ is homeomorphic for any x in X and

$$d(f(x), f(x')) \geqslant \lambda d(x, x')$$

for any x and x' in X with $d(x, x') < b$. For a locally expanding map, there is a constant integer $N_0 > 0$ such that $\#(f^{-1}(x)) \leqslant N_0$ for all x in X. We say that f is mixing if, for any open set U of X, there is an integer $n > 0$ such that $f^n(U) = X$.

Henceforth, we suppose that f is a locally expanding and mixing map. Then for any x and y in X such that $f(x) = y$, there is a neighborhood V of y such that f has the local inverse $g: V \to g(V)$ and $fg =$ identity, $g(y) = x$. We can further choose V such that g is contracting, i.e.,

$$d(g(y), g(y')) \leqslant \lambda^{-1} d(y, y')$$

for any y and y' in V. Since X is compact, there is a fixed constant $0 < a_0 < 1$ such that V can be picked as $B(y, a_0)$.

Let $\mathbb{R}$ denote the real line and let $C^0(X, \mathbb{R})$ be the space of all continuous functions $\phi: X \to \mathbb{R}$ with the supremum norm $||\phi|| = \max_{x \in X}\{|\phi(x)|\}$. Let $C^\alpha(X, \mathbb{R})$, for $0 < \alpha \leqslant 1$, be the space of all α-Hölder continuous functions ϕ, i.e., ϕ is in $C^0(X, \mathbb{R})$ and satisfies

$$\sup_{x \neq y \in X} \frac{|\phi(x) - \phi(y)|}{d(x, y)^\alpha} < \infty.$$

We say two functions $\phi_1 \geqslant \phi_2$ if $\phi_1(x) \geqslant \phi_2(x)$ for all x in X. Let $C^\alpha_{K,s}(X, \mathbb{R})$, for positive constants K and s, be the subset of all functions ϕ in $C^{\alpha}(X, \mathbb{R})$ satisfying that $\phi \geqslant s$ and

$$\left|\log\left(\frac{\phi(y)}{\phi(y')}\right)\right| \leqslant Kd(y, y')^{\alpha}$$

for all y and y' in X with $d(y, y') < a_0$. Simply, we will denote the spaces and the subset we defined by C^0, C^{α}, and $C^{\alpha}_{K,s}$. The following lemma is a conclusion of the Ascoli–Arzela theorem.

LEMMA 1. *Any bounded sequence in* $C^{\alpha}_{K,s}$ *has a convergent subsequence in* C^0 *whose limit is in* $C^{\alpha}_{K,s}$.

Suppose ψ is an α_0-Hölder continuous function on X for some $0 < \alpha_0 \leqslant 1$ and is positive, i.e., $\psi > 0$. The Ruelle–Perron–Frobenius operator with weight ψ is defined as

$$\mathcal{L}\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x) = \sum_{i=1}^{k} \psi(x_i)\phi(x_i),$$

where $\{x_1, \cdots, x_k\} = f^{-1}(y)$. For any $0 < \alpha \leqslant \alpha_0$, we have a linear operator $\mathcal{L} : C^{\alpha} \to C^{\alpha}$.

For the purpose of the study of eigenvalues of $\mathcal{L}$, we can normalize the weight ψ such that $\min_{x \in X} \psi(x) = 1$. Henceforth, we will always assume that ψ is a normalized element in $C^{\alpha_0}_{K_0,1}$ for some constant $K_0 > 0$.

LEMMA 2. *Let* $0 < s < 1$ *and* $K > K_0/(\lambda^{\alpha} - 1) > 0$ *be two fixed constants. Then for any* $\phi \geqslant 0$ *in* C^{α} *with* $||\phi|| = 1$, *there is an integer* $N > 0$ *such that* $\mathcal{L}^N\phi$ *is in* $C^{\alpha}_{K,s}$.

Proof. Since $||\phi|| = 1$, there is a point y in X such that $\phi(y) = 1$. We thus have a neighborhood U of y so that $\phi(y') > s$ for all y' in U. Since f is mixing, there is an integer $n_0 > 0$ such that $f^n(U) = X$ for all $n \geqslant n_0$. Therefore, for any z in X, $f^{-n}(z) \cap U$ is nonempty for all $n \geqslant n_0$. Thus, we have $\mathcal{L}^n\phi(z) \geqslant s$.

For any y and y' in X with $d(y, y') < a_0$, let

$$\{x_1, \cdots, x_k\} = f^{-1}(y) \quad \text{and} \quad \{x'_1, \cdots, x'_k\} = f^{-1}(y')$$

be the corresponding inverse images of y and y'. Then $d(x_i, x'_i) \leqslant \lambda^{-1}d(y, y')$ for all $1 \leqslant i \leqslant k$. Let $K' > 0$ be a constant such that

$$\left|\log\left(\frac{\mathcal{L}^{n_0}\phi(y)}{\mathcal{L}^{n_0}\phi(y')}\right)\right| \leqslant K'd(y, y')^{\alpha}$$

for all y and y' in X with $d(y, y') < a_0$. Then

$$\begin{aligned}\mathcal{L}\big(\mathcal{L}^{n_0}\phi\big)(y') &= \sum_{i=1}^{k} \psi(x_i')\mathcal{L}^{n_0}\phi(x_i') \\ &\leqslant \sum_{i=1}^{k} \psi(x_i)\exp\big(K_0 d(x_i, x_i')^{\alpha_0}\big)\mathcal{L}^{n_0}\phi(x_i)\exp\big(K' d(x_i, x_i')^{\alpha}\big) \\ &\leqslant \exp\big((K_0\lambda^{-\alpha_0+\alpha} + K')\lambda^{-\alpha} d(y, y')^{\alpha}\big)\sum_{i=1}^{k}\psi(x_i)\mathcal{L}^{n_0}\phi(x_i) \\ &\leqslant \exp\big((K_0 + K')\lambda^{-\alpha} d(y, y')^{\alpha}\big)\mathcal{L}\big(\mathcal{L}^{n_0}\phi\big)(y)\end{aligned}$$

for all y and y' in X with $d(y, y') < a_0$. Inductively, for

$$K_n = K_0\left(\sum_{i=1}^{n}\lambda^{-\alpha i}\right) + K'\lambda^{-\alpha n},$$

we have

$$\mathcal{L}^n\big(\mathcal{L}^{n_0}\phi\big)(y') \leqslant \exp\big(K_n d(y, y')^{\alpha}\big)\mathcal{L}^n\big(\mathcal{L}^{n_0}\phi\big)(y)$$

for all y and y' in X with $d(y, y') < a_0$. It is clear that K_n tends to $K_0/(\lambda^{\alpha} - 1)$ as n goes to infinity. So there is an integer $n_1 > 0$ such that for any $n \geqslant n_1$

$$\mathcal{L}^n\big(\mathcal{L}^{n_0}\phi\big)(y') \leqslant \exp\big(K d(y, y')^{\alpha}\big)\mathcal{L}^n\big(\mathcal{L}^{n_0}\phi\big)(y)$$

for all y and y' in X with $d(y, y') < a_0$. Then $N = n_0 + n_1$ satisfies the lemma. □

Lemma 2 implies that if $\mu > 0$ is an eigenvalue of $\mathcal{L}\colon C^{\alpha} \to C^{\alpha}$ with an eigenfunction $\phi \geqslant 0$, then $\mathcal{L}$ also has an eigenfunction in $C^{\alpha}_{K,s}$ with respect to μ. Therefore, we can use $C^{\alpha}_{K,s}$ to find all positive eigenvalues of $\mathcal{L}\colon C^{\alpha} \to C^{\alpha}$. From the calculation in the proof of Lemma 2, we have that $\mathcal{L}(C^{\alpha}_{K,s}) \subseteq \mathcal{L}(C^{\alpha}_{K,s})$ because $(K_0 + K)\lambda^{-\alpha} < K$ for $K > K_0/(\lambda^{\alpha} - 1)$.

Let $0 < s < 1$ and $K > K_0/(\lambda^{\alpha} - 1)$ be two fixed constants in Lemma 2. Define S as the set consisting of positive real numbers such that for any $\mu > 0$ in S there is a ϕ in $C^{\alpha}_{K,s}$ satisfying $\mathcal{L}\phi \geqslant \lambda\phi$.

LEMMA 3. *The set S is nonempty bounded subset in the real line $\mathbb{R}$.*

Proof. First let us show that S is nonempty. Take a function ϕ in $C^{\alpha}_{K,s}$. Then for any x and y in X,

$$\mathcal{L}\phi(y) = \sum_{x\in f^{-1}(y)}\psi(x)\phi(x) = \left(\sum_{x\in f^{-1}(y)}\frac{\phi(x)}{\phi(y)}\psi(x)\right)\phi(y) \geqslant \frac{s}{||\phi||}\phi(y).$$

Thus $\mu = s/||\phi||$ is in S.

For any ϕ in $C^{\alpha}_{K,s}$, let $\phi(y) = ||\phi||$. Then

$$\mathcal{L}\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x) \leqslant \phi(y) \sum_{x \in f^{-1}(y)} \psi(x) \leqslant N_0||\psi||\phi(y)$$

Therefore, any $\mu > N_0||\psi||$ will not be in S. Thus, S is a bounded subset in $\mathbb{R}$. □

MAIN THEOREM (Ruelle). *The linear operator $\mathcal{L}\colon C^{\alpha} \to C^{\alpha}$, $0 < \alpha \leqslant \alpha_0$, has a unique maximal positive eigenvalue whose corresponding eigenspace is one-dimensional.*

Proof. Take $\mu_{\max} = \sup S > 0$. Then there is a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in S convergent to $\mu_{\max}$. Let ϕ_n be a corresponding function in $C^{\alpha}_{K,s}$ such that $\mathcal{L}\phi_n \geqslant \lambda_n\phi_n$. Let us normalize ϕ_n with $\min_{x\in X}\{\phi_n(x)\} = s$. Then $\Pi = \{\phi_n\}_{n=1}^{\infty}$ is a bounded sequence in $C^{\alpha}_{K,s}$. From Lemma 1, Π has a convergent subsequence in C^0 whose limit is in $C^{\alpha}_{K,s}$. Let us still denote this convergent subsequence as Π and ϕ_0 as its limit. Then $\mathcal{L}\phi_0 \geqslant \mu_{\max}\phi_0$.

We now show that $\mathcal{L}\phi_0 = \mu_{\max}\phi_0$. Suppose there is a point y in X such that

$$\mathcal{L}\phi_0(y) > \mu_{\max}\phi_0(y).$$

Then there is a neighborhood U of y such that

$$\mathcal{L}\phi_0(y') - \mu_{\max}\phi_0(y') > 0$$

for all y' in U. Since f is mixing, there is an integer $n > 0$ such that $f^n(U) = X$. Then

$$\mathcal{L}^n(\mathcal{L}\phi_0 - \mu_{\max}\phi_0) > 0,$$

i.e.,

$$\mathcal{L}\big(\mathcal{L}^n\phi_0\big) > \mu_{\max}\mathcal{L}^n\phi_0.$$

Therefore, for $\phi = \mathcal{L}^n\phi_0$, we have a $\mu > \mu_{\max}$ such that $\mathcal{L}\phi \geqslant \mu\mathcal{L}\phi$. This contradicts to the maximal property of $\mu_{\max}$. This has proved that $\mathcal{L}\phi_0 = \mu_{\max}\phi_0$.

Now let us show that the eigenspace

$$E_{\mu_{\max}} = \{\phi \in C^{\alpha}, \mathcal{L}\phi = \mu_{\max}\phi\}$$

corresponding to $\mu_{\max}$ is a one-dimensional space. Suppose ϕ is any function in $E_{\mu_{\max}}$. Let

$$a = \min_{x\in X}\{\phi(x)/\phi_0(x)\} \quad \text{and} \quad \phi_1 = \phi - a\phi_0.$$

Then ϕ_1 is in $E_{\mu_{\max}}$ and $\phi_1 \geqslant 0$. Moreover, there is a point y in X such that $\phi_1(y) = 0$. Then $\phi_1(x) = 0$ for all x in $f^{-1}(y)$. Inductively, we have $\phi_1 = 0$ on $X_y = \cup_{n=0}^{\infty} f^{-n}(y)$. Since f is mixing, X_y is a dense subset in X. So $\phi_1 \equiv 0$ on X, i.e., $\phi \equiv a\phi_0$.

All that remains is to prove that $\mu_{\max}$ is the biggest eigenvalue but this is easy as can be seen in what follows. Suppose $\mu \neq \mu_{\max}$ is an eigenvalue of $\mathcal{L}\colon C^{\alpha} \to C^{\alpha}$. Then there is a non-zero function ϕ in C^{α} with $||\phi|| = 1$ such that $\mathcal{L}\phi = \mu\phi$. So $\mathcal{L}|\phi| \geqslant |\mu||\phi|$.

There is an integer $N > 0$ such that $\mathcal{L}^N|\phi|$ is in $C^{\alpha}_{K,s}$ and also $\mathcal{L}(\mathcal{L}^N|\phi|) \geqslant |\mu|\mathcal{L}^N|\phi|$. Thus, $|\mu|$ is a number in S, so $|\mu| \leqslant \mu_{\max}$. If $|\mu| < \mu_{\max}$, then we have nothing to prove. If $|\mu| = \mu_{\max}$, by using the mixing property as we did in the previous two sections, we have $|\phi| = a\phi_0$ for some $a > 0$. This implies $\phi = \pm a\phi_0$ and $\mu = \mu_{\max}$. □

3. One Application

Suppose M is an m-dimensional compact Riemannian manifold where $m \geqslant 1$ is an integer. Let $f : M \to M$ be a C^1 map. We say f is $C^{1+\alpha}$ for $0 < \alpha \leqslant 1$ if the determinant $J(f)$ of the Jacobi matrix $\mathrm{Jac}(f)$ of f is an α-Hölder continuous function defined on M. A probability measure v on M is called f-invariant if $v(f^{-1}(A)) = v(A)$ for all Lebesgue measurable subsets A of M. Let $\mathrm{d}y$ denote the Lebesgue metric on M. A probability measure v is called a smooth measure if there is a continuous function defined on M such that

$$v(A) = \int_A \rho(y)\mathrm{d}y$$

for all Lebesgue measurable sets A in M. The function ρ is called the density function of the smooth measure v.

LEMMA 4. *Suppose that $f : M \to M$ is a C^1 map so that $J(f)(y) \neq 0$ for all y in M and suppose that v is a smooth probability measure with $v = \int \rho \mathrm{d}y$. Then v is a f-invariant measure if and only if*

$$\sum_{x\in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} = \rho(y)$$

for all y in M.

Proof. Since $J(f)(y) \neq 0$ for all y in M there is a constant $a_1 > 0$ such that for any connected domain U with a diameter less than or equal to $2a_1$, f on each component V of $f^{-1}(U)$ is injective and has the local inverse $g : V \to U$ such that $fg = \text{identity}$. If v is a f-invariant measure, then $v(f^{-1}(U)) = v(U)$ for all Lebesgue measurable subsets U of M. In particular, take U as the ball of centered y and radius $0 < \varepsilon < a_1$ and denote $V_1, \ldots, V_k$ as the components of $f^{-1}(U)$. Then

$$\sum_{i=1}^{k} \int_{V_i} \rho(x)\mathrm{d}x = \int_U \rho(y)\, \mathrm{d}y.$$

By the mean value theorem and letting ε tend to zero, we get

$$\sum_{x\in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} = \rho(y).$$

Now assume

$$\sum_{x\in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} = \rho(y)$$

for all y in M. For any ball U with radius a_1, let $V_1, \ldots, V_k$ be the components of $f^{-1}(U)$. Then f on each V_i is injective and has the local inverse. So

$$\begin{aligned} \nu(U) &= \int_U \rho(y)\mathrm{d}y = \int_U \sum_{x\in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)}\mathrm{d}y \\ &= \sum_{x\in f^{-1}(y)} \int_U \frac{\rho(x)}{J(f)(x)}\mathrm{d}y = \sum_{i=1}^{k} \int_{V_i} \rho(x)\,\mathrm{d}x \\ &= \sum_{i=1}^{k} \nu(V_i) = \nu(f^{-1}(U)). \end{aligned}$$

Now we see an application of the main theorem to Krzyzewski and Szlenk's theorem [5] (see also [6, 11]). The proof is straightforward by applying the main theorem. But for completeness, we write down the proof.

COROLLARY (Krzyzewski–Szlenk). *Suppose that $f: M \to M$ is a $C^{1+\alpha}$ locally expanding map. Then f has a unique smooth f-invariant probability measure with α-Hölder continuous density function.*

Proof. Since f is C^1 and locally expanding, $J(f)(y) \neq 0$ for all y in M. Let

$$||J(f)|| = \max_{y\in M}\{J(f)(y)\} \quad \text{and} \quad \psi = \frac{||J(f)||}{J(f)}.$$

Then ψ is an function in $C^{\alpha_0}_{K_0,1}$ for some $K_0 > 0$ and $\alpha_0 = \alpha$. Consider a Ruelle–Perron–Frobenius operator with weight ψ:

$$\mathcal{L}\phi(y) = \sum_{x\in f^{-1}(y)} \psi(x)\phi(x).$$

It is a linear operator from C^α into itself. Ruelle's theorem implies that there is a unique maximal positive eigenvalue $\mu_{\max}$ with a positive eigenfunction in C^α and the corresponding eigenspace is one-dimensional. Let ρ be the one in the eigenspace normalized by $\int_M \rho(y)\mathrm{d}y = 1$. Then $\mathcal{L}\rho(y) = \mu_{\max}\rho(y)$. Furthermore,

$$\sum_{x\in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} = \mu_0\rho(y),$$

where $\mu_0 = \mu_{\max}/||J(f)||$. Integrate on both sides of the last equation and we have that the right-hand side is μ_0. Now let us calculate the left-hand side. Cut M into path-connected pieces $M_1, \ldots, M_n$, such that

(1) $M = M_1 \cup M_2 \cup \cdots \cup M_n$,

(2) the Lebesgue measure of each $M_i \cap M_j$ is zero for $i \neq j$,
(3) f on each component of $f^{-1}(M_i)$ is injective, $1 \leqslant i \leqslant n$.

Let M_i^j, $1 \leqslant j \leqslant k_i$ be the components of $f^{-1}(M_i)$ for $1 \leqslant i \leqslant n$. Then $M = \cup_{i=1}^n \cup_{j=1}^{k_i} M_i^j$ and the Lebesgue measure of each $M_i^j \cap M_{i'}^{j'}$ is zero for $i \neq i'$. Let us use x_{ij} to denote the point in $f^{-1}(y) \cap M_i^j$ for any $y \in M_i$, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k_i$. Therefore,

$$\begin{aligned}
&\int_M \sum_{x \in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} \mathrm{d}y \\
&\quad = \sum_{i=1}^n \int_{M_i} \sum_{x \in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} \mathrm{d}y \\
&\quad = \sum_{i=1}^n \sum_{j=1}^{k_i} \int_{M_i} \frac{\rho(x_{ij})}{J(f)(x_{ij})} \mathrm{d}y = \sum_{i=1}^n \sum_{j=1}^{k_i} \int_{M_i^j} \rho(x_{ij})\, \mathrm{d}x_{ij} \\
&\quad = \int_M \rho(y)\, \mathrm{d}y = 1.
\end{aligned}$$

So we have $\mu_0 = 1$ (that is, $\mu_{\max} = ||J(f)||$) and $\nu = \int \rho(y)\mathrm{d}y$ is a smooth f-invariant measure following Lemma 4.

Uniqueness follows the fact that if $\nu = \int \rho \mathrm{d}y$ is a smooth f-invariant measure, then ρ is in the eigenspace of $\mathcal{L}$ with respect to the eigenvalue $\mu_{\max} = ||J(f)||$. □

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