Scaling Functions for Degree 2 Circle Endomorphisms

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Abstract. We prove that a continuous function on the dual Cantor set is the scaling function of a uniformly symmetric circle endomorphism if and only if it satisfies a summation condition and compatibility condition. We use this result to establish an isomorphism between the space of continuous functions on the dual Cantor set satisfying these conditions and a Teichmüller space.

Introduction

Scaling functions for Markov maps (see [**8, 12**] for the definition) have been used to describe fine-scale geometric structures of dynamical systems. For a Markov map one can construct a semi-conjugacy between the map and a shift map acting on a symbolic space of finite type. The scaling function, if it exists, is defined on a symbolic space dual to this space (see Section 1 for the definition). A natural problem is to determine *which functions defined on a given dual symbolic space are the scaling functions of a Markov map.*

A circle endomorphism of degree two is a special Markov map with a standard Markov partition. Using this partition, one obtains a semi-conjugacy between the one-sided full shift on two symbols and the circle endomorphism. If the endomorphism is expanding and C^{1+h}, then it has a scaling function which is Hölder continuous on the dual symbolic space (see [**8**, Chapter 3]). However, the scaling function exists for a larger class of circle expanding endomorphism, namely, for uniformly symmetric circle endomorphisms (see Section 1 for the definition). In this case the scaling function on the dual symbolic space is continuous only (see Theorem 1 in Section 1). However, not every continuous function on the dual symbolic space is the scaling function of an endomorphism. To be a scaling function, it must satisfy a trivial condition which we call the *summation condition* and a non-trivial convergence condition which we call the *compatibility condition*. Using Gibbs measures, it is proved in [**4**] that these two conditions are necessary and sufficient for a Hölder continuous function defined on the dual symbolic space to be the scaling function of a C^{1+h} circle endomorphism. Since the existence of Gibbs measures depends on the Hölder continuity condition (or the bounded summable variation condition) (see [**2, 4**]), there seems little hope that the same method could apply to the more general situation where the circle endomorphism is uniformly symmetric.

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In this paper we provide a different technique that applies to the more general class of all uniformly symmetric endomorphisms and extends the result of [**4**]. That is, we show that the summation and compatibility conditions are necessary and sufficient for a continuous function on the dual symbolic space to be the scaling function of a uniformly symmetric circle endomorphism. This result is shown in Theorem 3 of Section 3. The method of proof is based on linear models introduced by Douady and Hubbard in [**5**, pp.298] and on the necessary and sufficient criterion established in [**3**] for a circle endomorphism to be uniformly symmetric expressed in terms of a property of its linear model. The linear model is a smooth invariant in the space of C^{1+h} circle endomorphisms and symmetric invariant in the space of uniformly symmetric circle endomorphisms.

Our first main result, that is, Theorem 3 in Section 3, permits us to establish a correspondence between the Teichmüller space of uniformly symmetric circle endomorphisms and the space of scaling functions for these endomorphisms. Thus, the scaling functions, which are real-valued functions on a dual Cantor set, carry a natural complex manifold structure. This added structure is obtained simply by passing from the C^{1+h} class to the uniformly symmetric class by taking the closure of this class in the Teichmüller topology. The uniformly symmetric class concides with the class of circle endomorphisms which have quasiregular extensions to a neighborhood of the unit circle, with the property that the iterates of these extensions are are uniformly asymptotically conformal, [**7**].

The results on the complex structure on the space of scaling functions are contained in Theorems 6, 7 and 8 in Section 4. As a consequence, we see that the space of scaling functions is naturally embedded as a domain in a complex Banach space, and therefore provides a background for arguments involving holomorphic mappings and applications of Schwarz's lemma.

The paper is arranged as follows. In Section 1, we give the definition of the scaling function of a circle endomorphism, present some properties of scaling functions and define uniformly symmetric circle endomorphisms. In the same section, we show that scaling functions exist and are continuous (Theorem 1). In Section 2 we define and prove the existence of linear models. We also show the relation between an endomorphism and its linear model (Theorem 2). In Section 3, we prove the first main result (Theorem 3) that a continuous function on the dual symbolic space is a scaling function of a uniformly symmetric circle endomorphism if and only if it satisfies the summation and compatibility conditions. We verify the compatibility condition by showing it is a consequence of the *matching condition* on the solenoid function introduced in [**11**], where one can find a discussion of the solenoid and its properties.

Finally, in Section 4, we use our main result to study the Teichmüller space of uniformly symmetric circle endomorphisms. We show there is a bijective, bicontinuous map between the Teichmüller space of uniformly symmetric circle endomorphisms and the space of continuous functions on the dual symbolic space satisfying the summation and compatibility conditions (Theorems 6, 7, and 8).

Although this exposition deals only with degree 2 circle endomorphisms, we believe our discussion easily extends to endomorphisms with higher degree. In this case, we still have the summation condition but there is more than one compatibility condition. Therefore, for reasons of simplicity in exposition, we worked out the notation for what we called the matching condition only in the case $n = 2$. Hence the results presented here are restricted to that case.

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1. Scaling functions of circle endomorphisms.

Let $\mathbb{R}$ be the real line and $T(x) = x + 1$. Then $S^1 = \mathbb{R}/\{x \sim T(x)\}$ is the unit circle. The mapping $\pi(x) : \mathbb{R} \to S^1$ given by $\pi(x) = \{T^n(x), n = 0, \pm 1, \pm 2, \cdots\}$ is the universal covering. Let $f : S^1 \to S^1$ be a degree two orientation-preserving covering mapping and let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f, that is, F is an orientation-preserving homeomorphism of $\mathbb{R}$ such that $\pi \circ F = f \circ \pi$. It is easy to check that

$$FT = T^2 F. \tag{1}$$

There are many choices for a lift of f, but the lift F satisfying (1) and normalized by $F(0) = 0$ is unique. Since under these conditions each of f or F determines the other, in this paper we refer to either one interchangeably as a circle endomorphism.

If such an F is given, let a be the number between 0 and 1 such that $F(a) = 1$. We define two inverse branches

$$g_0 = F^{-1} : [0, 1] \to I_0 = [0, a], \quad \text{and} \quad g_1 = F^{-1}T : [0, 1] \to I_1 = [a, 1].$$

For any finite string $w_n = i_{n-1} \cdots i_1 i_0$ of 0's and 1's, let

$$g_{w_n} = g_{i_{n-1}} \cdots g_{i_1} g_{i_0} \quad \text{and} \quad I_{w_n} = g_{w_n}([0, 1]).$$

In other words, I_{w_n} is the maximal interval in S^1 such that $f^n|I_{w_n}$ is injective and $f^k(I_{w_n})$ is in $I_{w_{n-k}}$ for every $0 \le k < n$. Let

$$\sigma_n^*(w_n) = i_{n-1} \cdots i_1$$

be the right shift. Then

$$I_{\sigma_n^*(w_n)} \supset I_{w_n}.$$

Therefore, on letting $|I|$ denote the length of an interval I, we can consider the ratio

$$S(w_n) = \frac{|I_{w_n}|}{|I_{\sigma_n^*(w_n)}|}.$$

Let

$$\Sigma_n^* = \{w_n = i_{n-1} \cdots i_1 i_0\}$$

be the space of all strings of 0's and 1's of length n with the product topology. Then

$$\sigma_n^* : \Sigma_n^* \to \Sigma_{n-1}^*$$

is continuous. Let (Σ^*, σ^*) be the inverse limit of $\{\Sigma_n^*, \sigma_n^*\}$. It is called the dual symbolic space of f. For the reader interested in relationships between among Markov partitions, symbolic spaces, and dual symbolic spaces we refer to [**8**, 4-10, 73-77]. It is clear that $\Sigma^* = \{w = \cdots i_{n-1} \cdots i_1 i_0\}$ consists of all strings of 0's and 1's of length ∞ and $\sigma^*(w) = \cdots i_{n-1} \cdots i_1$ is the right shift.

Definition. Assume for any $w = \cdots w_n \in \Sigma^*$, the limit

$$\lim_{n \to \infty} S(w_n)$$

exists. Then we define the *scaling function* $S_f = S$ of f from Σ^* to $\mathbb{R}$ by

$$S(w) = \lim_{n \to \infty} S(w_n).$$

Two circle endomorphisms f and g are C^1-conjugate if there is a C^1-diffeomorphism h of S^1 such that

$$fh = hg.$$

Let H and F and G be the lifts of h and f and g such that $H(0) = F(0) = G(0) = 0$. Then the last equation means that

$$FH = HG.$$

Note that $HT = TH$. The following result is not hard to check. We refer to [**8**, Chapter 3] for details.

THEOREM A. *Suppose f and g are two circle endomorphisms with scaling functions S_f and S_g. If f and g are C^1-conjugate, then $S_f = S_g$.*

A homeomorphism h of $\mathbb{R}$ is called K-quasisymmetric for a real number $K > 1$ if for any $x \in \mathbb{R}$ and $t > 0$,

$$K^{-1} < \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} < K.$$

A homeomorphism h of $\mathbb{R}$ is called *symmetric* [**6**] if there is a bounded positive function $\epsilon(t)$ such that $\epsilon(t) \to 0$ as $t \to 0$ and

$$1 - \epsilon(t) \leq \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \leq 1 + \epsilon(t)$$

for any x in $\mathbb{R}$. The following lemma can be found in [**3**].

LEMMA 1. *Suppose h is a quasisymmetric homeomorphism of $\mathbb{R}$. Let*

$$\rho_h(x, t, k) = \frac{h(x+kt) - h(x)}{h(x) - h(x-t)}$$

for $x, t \in \mathbb{R}$ and $0 < k \leq 1$. The following conditions are equivalent:

(1) *there is a positive function $\epsilon_1(t) \to 0$, $t \to 0$, such that $|\rho_h(x, t, 1) - 1| < \epsilon_1(|t|)$, i.e., h is symmetric;*
(2) *there is a positive function $\epsilon_2(t) \to 0$, $t \to 0$, such that $|\mu(z)| < \epsilon_2(\Im z)$, where μ is the complex dilatation of Beurling-Ahlfors extension of h;*
(3) *there is a positive function $\epsilon_3(t) \to 0$, $t \to 0$, such that $|\rho_h(x, t, k) - k| < \epsilon_3(|t|)$.*

Suppose f is a circle endomorphism and F is its lift with $F(0) = 0$. Let

$$\rho(F^{-n}, t) = \sup_{x \in \mathbb{R}} \Big| \log \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \Big|$$

be the symmetric distortion of F^{-n}.

DEFINITION. The dynamical system consisting of the forward powers f^n of f is called *uniformly symmetric* if there is a bounded positive function $\epsilon(t)$ such that $\epsilon(t) \to 0$ as $t \to 0$ and

$$\rho(F^{-n}, t) \leq \epsilon(t)$$

for all n and all $t > 0$.

Applying Lemma 1, we have the following theorem.

THEOREM 1. *Suppose f is a uniformly symmetric circle endomorphism. Then the scaling function of f exists and is continuous.*

PROOF. First there is a constant $K > 0$ such that

$$K^{-1} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq K, \quad \text{for} \quad 0 < t \leq 1, \ x \in \mathbb{R}.$$

For any $w_{n-1} = i_{n-1}i_{n-2} \cdots i_1$, $n > 0$, we have

$$I_{w_{n-1}0} = g_{w_{n-1}0}([0,1]) \quad \text{and} \quad I_{w_{n-1}1} = g_{w_{n-1}1}([0,1]).$$

Then

$$K^{-1} \leq \frac{|I_{w_{n-1}0}|}{|I_{w_{n-1}1}|} \leq K.$$

Since $I_{w_{n-1}} = I_{w_{n-1}0} \cup I_{w_{n-1}1}$,

$$\frac{1}{1+K}|I_{w_{n-1}}| \leq |I_{w_{n-1}1}| \leq \frac{K}{1+K}|I_{w_{n-1}}|, \tag{2}$$

$$\frac{1}{1+K}|I_{w_{n-1}}| \leq |I_{w_{n-1}0}| \leq \frac{K}{1+K}|I_{w_{n-1}}|. \tag{3}$$

The right hand inequalities imply that for $0 < \lambda = K/(1+K) < 1$,

$$|I_{w_n}| \leq \lambda^n$$

for all $n > 0$ and $w_n = i_{n-1}i_{n-2} \cdots i_1 i_0$.

Take $w = \cdots w_n$ in Σ^*, and note that $\sigma^*(w_n) = w_{n-1}$. To show $\lim_{n\to\infty} S(w_n)$ exists, we show $\{S(w_n)\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since F^{-n} are uniformly symmetric, there is a positive function $\epsilon_1(t) \to 0$ as $t \to 0$ (refer to Lemma 1) such that

$$\rho(F^{-n}, t) \leq \epsilon_1(t), \quad n > 0, \ t > 0.$$

By Lemma 1, there is a positive function $\epsilon_3(t) \to 0$ as $t \to 0$ such that

$$|\rho_{F^{-n}}(x,t,k) - k| < \epsilon_3(|t|)$$

for all $n > 0$, $x, t \in \mathbb{R}$, and $0 < k \leq 1$.

Now for any $\epsilon > 0$, take $N_0 > 0$ such that $\epsilon_3(\lambda^n) < \epsilon$ for all $n \geq N_0$. Then for any $m > n > N_0$,

$$F^{m-n}(I_{\sigma_m^*(w_m)}) = I_{\sigma_n^*(w_n)} \pmod 1 \quad \text{and} \quad F^{m-n}(I_{w_m}) = I_{w_n} \pmod 1.$$

We get

$$|S(w_m) - S(w_n)| < \epsilon_3(\lambda^n) < \epsilon.$$

This implies that $\{S(w_n)\}_{n=0}^{\infty}$ is a Cauchy sequence.

Similarly, if $w = \cdots w_n$ and $w' = \cdots w_n$ are two elements in Σ^*, then

$$|S(w_m) - S(w_n)| < \epsilon \quad \text{and} \quad |S(w'_m) - S(w_n)| < \epsilon \quad \text{for} \quad m > n > N_0.$$

This implies that

$$|S(w) - S(w')| < 2\epsilon.$$

Thus S is continuous on Σ^*. □

2. Linear models of circle endomorphisms

Suppose f is a uniformly symmetric circle endomorphism and F is its lift with $F(0)=0$. Consider the sequence $\{\gamma_n = F^{-n}/F^{-n}(1)\}_{n=0}^{\infty}$ fixing the points 0 and 1. For any $n, m > 0$, let

$$\tau_{nm}(y) = \frac{F^{-m}[F^{-n}(1)y]}{F^{-n-m}(1)}. \tag{4}$$

Then τ_{nm} is symmetric. Because $F^{-n}(1)$ tends to 0 as n goes to infinity (see the proof of Theorem 1), for any $\epsilon > 0$, there is a $n > 0$ such that $\epsilon_3(F^{-n}(1)) < \epsilon$ (see Lemma 1), so

$$|\tau_{nm}(y) - y| < \epsilon_3(F^{-n}(1)) < \epsilon.$$

In particular, by putting $y = \gamma_n(x)$ in (4), for $x \in [-1,1]$, we have

$$|\gamma_{n+m}(x) - \gamma_n(x)| < \epsilon.$$

Thus $\{\gamma_n\}_{n=1}^{\infty}$ is a Cauchy sequence of homeomorphisms normalized to fix the points 0 and 1 and converging uniformly on $[-1,1]$ to a continuous function γ_∞. Since each of the function $\gamma_n(x)$ is symmetric on $[-1,1]$, so is the limit γ_∞.

Since

$$\gamma_n F^{-1} = \frac{F^{-n-1}(1)}{F^{-n}(1)}\gamma_{n+1}$$

and since the sequence γ_n converges uniformly, the sequence $\frac{F^{-n-1}(1)}{F^{-n}(1)}$ also converges, and by inequalities (2) and (3) we find that

$$\frac{F^{-n-1}(1)}{F^{-n}(1)} \to \delta^{-1} \quad \text{as} \quad n \to \infty,$$

where

$$\frac{1+K}{K} \le \delta \le 1+K.$$

We obtain

$$\gamma_\infty F^{-1} = \delta^{-1}\gamma_\infty$$

on $[-1,1]$. Define a homeomorphism γ from the real line onto itself by

$$\gamma(x) = \delta^n \gamma_\infty(F^{-n}(x)), \ \ x \in [-2^n, -2^{n-1}] \cup [2^{n-1}, 2^n], \ n > 0.$$

Since F is uniformly symmetric, γ is a symmetric homeomorphism of $\mathbb{R}$ conjugating F to $\gamma F \gamma^{-1}(x) = \delta x$. Let $P(x) = \delta x$ and $L = \gamma T \gamma^{-1}$. We call L the linear model for f. Under conjugation, the equation

$$FT = T^2 F$$

is transformed to

$$PL = L^2 P,$$

that is,

$$\delta L(x) = L^2(\delta x).$$

Moreover, since γ fixes 0 and 1 and $\gamma T \gamma^{-1} = L$, we have that

$$\gamma(k) = L^k(1).$$

On letting I be the unit interval $[0,1]$, we have

$$L^k(I) = [\gamma(k-1), \gamma(k)].$$

Definition. The *solenoid* function (see [**11**]) is defined on integers k by the following formula:

$$sol(k) = \frac{|L^k(I)|}{|L^{k-1}(I)|}.$$

The solenoid function can be defined on codes of the form

$$w = \cdots 000 i_{n-1} \cdots i_1 i_0$$

by using the correspondence between codes and nonnegative integers. For any such code w, we let

$$k = k(w) = \sum_{j=0}^{n-1} i_j 2^j \tag{5}$$

and for any nonnegative integer k we let $w = w(k)$ be the code $\cdots 000 i_{n-1} \cdots i_1 i_0$ where i_j is the unique value of 0 or 1 such that (5) is satisfied.

For $w = w(k)$, we will write $sol(w)$ in place of $sol(k)$. Using the facts that w of this form are dense in all possible codes and that $sol(w)$ is uniformly continuous (from the proof of Theorem 1 and equations (7) and (8) below), $sol(w)$ extends to a continuous function defined on the dual symbolic space Σ^*. For an arbitrary code $w = \cdots i_j \cdots i_0$ we let $w_n = 000 i_{n-1} \cdots i_0$ and call w_n the $n-$th truncation of w, and note that since the sequence w_n converges to w, the values of $sol(w_n)$ will determine the value of $sol(w)$.

We now use the linear model to calculate the relationship between the solenoid and scaling functions. From the definition of the scaling function and the fact that γ is symmetric, we have (refer to the proof of Theorem 1)

$$\frac{1}{S(w)} = \lim_{n\to\infty} \frac{|I_{\sigma^*(w_n)}|}{|I_{w_n}|} = \lim_{n\to\infty} \frac{|\gamma(I_{\sigma^*(w_n)})|}{|\gamma(I_{w_n})|}.$$

Let $w = \cdots i_j \cdots i_0$ be an arbitrary code and w_n be the $n-$th truncation of w. Assume $k = k(w_n)$ and

$$l = k(\sigma^*(w_n)) = \sum_{j=0}^{n-2} i_{j+1} 2^j.$$

Then $k = 2l + i_0$.

Note that

$$\begin{aligned} I_{w_n} &= g_{n-1} \circ \cdots \circ g_{i_0}([0,1]) = F^{-1} \circ T^{i_{n-1}} \circ \cdots \circ F^{-1} \circ T^{i_0} \\ &= F^{-n}(T^{i_0 + 2i_1 + \cdots + 2^{n-1} i_{n-1}}([0,1])) = F^{-n}([k, k+1]), \end{aligned}$$

and, similarly,

$$I_{\sigma^* w_n} = F^{-n+1}([l, l+1]).$$

Therefore, since $\gamma(F^{-n}(x)) = \delta^{-n}\gamma(x)$ and

$$\delta\gamma(l) = \gamma(F(l)) = \gamma(F(T^l(0))) = \gamma T^{2l}(F(0)) = \gamma(2l),$$

we have

$$\begin{aligned} \frac{|\gamma(I_{\sigma^*(w_n)})|}{|\gamma(I_{w_n})|} &= \frac{|\gamma(F^{-n+1}([l, l+1]))|}{|\gamma(F^{-n}([k, k+1]))|} = \frac{\delta|\gamma([l, l+1])|}{|\gamma([k, k+1])|} \\ &= \frac{|\gamma([k - i_0, k - i_0 + 2])|}{|\gamma([k, k+1])|}. \end{aligned} \tag{6}$$

Since $PL(0) = L^2P(0)$, $\delta = P(1) = L(L(0)) = L(1)$, we have

$$\delta I = [0, \delta] = I \cup L(I).$$

Also, since γ fixes 0 and 1 and $\gamma T^k \gamma^{-1} = L^k$, $\gamma(k) = L^k(1)$. We can therefore rewrite (6) as

$$\frac{|L^{k-i_0}(I + L(I))|}{|L^k(I)|} = \Big(1 + \frac{|L^{(-1)^{i_0}}(L^k(I))|}{|L^k(I)|}\Big).$$

Now suppose $w = w'i_0$ is a code of the form $000i_{n-1}\cdots i_0$ and consider the two cases corresponding to whether the last entry i_0 is equal to 0 or 1. In the first case we find

$$\frac{1}{S(w'0)} = \left(1 + \frac{|L^{k+1}(I)|}{|L^k(I)|}\right) = 1 + sol(w'1) \tag{7}$$

and, in the second case, we find

$$\frac{1}{S(w'1)} = \left(1 + \frac{|L^{k-1}(I)|}{|L^k(I)|}\right) = 1 + \frac{1}{sol(w'1)}. \tag{8}$$

These calculations are true for all codes w of the form $000i_{n-1}\cdots i_0$. Since the scaling function is defined on codes w in Σ^*, these formulas show that the solenoid function is defined on all codes in $\Sigma_1^* = \{\cdots i_n \cdots i_2 i_1 1\}$.

Now suppose L is an arbitrary orientation-preserving homeomorphism of $\mathbb{R}$ with $L(0) = 1$ and $\delta > 1$ be an arbitrary real number. Let $P(x) = \delta x$. Suppose $L^2P = PL$ on $\mathbb{R}$. We wish to know under what condition there exists a uniformly symmetric circle endomorphism so that its linear model is L. First we embed the non-negative integers into the dual symbolic space Σ^* (see [**9**]). For any non-negative integer $k \geq 0$, we have a unique expansion $k = \sum_{j=0}^{n-1} i_j 2^j$ where i_j is either 0 or 1. Then $w(k) = \cdots 000i_{n-1}\cdots i_0$ is a point in Σ^*. Let $d(\cdot,\cdot)$ be the usual metric for Σ^*. That is, let the distance between any two sequences of zeroes and ones be equal to $\frac{1}{2^n}$, where i_n is the first entry where the two sequences differ. As before, define the solenoid function to be

$$sol_L(w(k)) = \frac{L^k(I)}{L^{k-1}(I)},$$

where I is the unit interval. We call L *uniformly Cauchy* if for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\Big| sol_L(w(k)) - sol_L(w(m)) \Big| < \delta$$

for any k and m with $d(w(k), w(m)) < \delta$. It is clear that if L is uniformly Cauchy, then $sol(w(k))$ extends to a continuous function sol_L on Σ^*. The proof of the next theorem can be found in [**3**, Theorem 4.5].

THEOREM 2. *There exists a uniformly symmetric circle endomorphism f with the linear model L if and only if L is uniformly Cauchy.*

3. Characterization of the scaling functions of uniformly symmetric circle mappings

Let $\mathcal{US}$ be the space of uniformly symmetric circle endomorphisms. We have seen that any element of $\mathcal{US}$ induces a continuous scaling function S defined on Σ^*.

Any scaling function satisfies the following condition, which we call the *summation condition:*

$$S(w0) + S(w1) = 1$$

for all $w \in \Sigma^*$. This follows the fact that if an interval T is a disjoint union of two subintervals A and B, then obviously $|A|/|T| + |B|/|T| = 1$.

Suppose S is a continuous function on Σ^*. We define a sequence of functions on Σ^*:

$$C_{S,N}(w) = \prod_{n=0}^{N} \frac{S(w1\overbrace{0\cdots0}^{n})}{S(w0\underbrace{1\cdots1}_{n})}. \tag{9}$$

We say that S satisfies the *compatibility condition* if $C_{S,N}(w)$ converges uniformly on Σ^* to a positive constant as N approaches infinity.

REMARK 1. Note that we are assuming that the constant is independent of the code w. Moreover, since the general term in the product (9) converges to 1, we have

$$S(\cdots 000) = S(\cdots 111).$$

REMARK 2. The product $C_{S,N}(w)$ of the n terms in equation (9) can be viewed as the ratio of the lengths of two intervals in the same generation. It is the ratio of the lengths of the intervals whose codes are

$$w1\overbrace{0\cdots0}^{n} \text{ and } w0\underbrace{1\cdots1}_{n}.$$

In [**4**] we showed that the summation and compatibility conditions characterize those Hölder continuous functions S on Σ^* which are scaling functions of C^{1+h} expanding maps. The method employed thermodynamical formalism and Gibbs measures. Since the scaling function of a uniformly symmetric circle endomorphism need not be Hölder continuous, we must employ a different method here.

Under the assumption that we are given the solenoid function of a uniformly symmetric circle endomorphism defined in terms of its linear model L, we now establish a necessary condition on the solenoid function called the *matching condition*

Assume w is a code of the form

$$\cdots 000 i_{n-1} \cdots i_0$$

and $k = k(w) = \sum_{j=0}^{n-1} i_j 2^j$. We write $sol(w) = sol(k)$ and define the "add-one" function a on codes w by

$$a(w(k)) = w(k+1).$$

LEMMA 2. *The solenoid function satisfies the matching condition:*

$$\frac{sol(w)}{sol(w0)} = \frac{1 + sol(w1)}{1 + [sol(a^{-1}(w0))]^{-1}}. \tag{10}$$

PROOF. We begin the proof with the following motivation. Suppose A and B are contiguous intervals with A to the left of B and let A be the union of two subintervals a and b, with a to the left of b, and let B be the union of two subintervals

c and d, with c to the left of d. Then, if these letters also stand for the lengths of these intervals, one has the obvious identity

$$\frac{B/A}{c/b} = \frac{c+d}{a+b} \cdot \frac{b}{c} = \frac{1+(d/c)}{1+(b/a)^{-1}}. \tag{11}$$

On letting B be the interval corresponding to the code w, then c corresponds to the code $w0$, d corresponds to the code $w1$, and b corresponds to the code $a^{-1}(w0)$. Therefore, $sol(w) = B/A, sol(w1) = d/c$, and $sol(a^{-1}(w0)) = b/a$ and the identity (10) coincides with the identity (11).

Now we proceed to the formal proof based on the definition of sol in terms of the linear model L. Note that $sol(w)$ is defined by

$$sol(w) = \frac{L^k(I)}{L^{k-1}(I)}$$

for $k = k(w)$ and

$$\delta L^k(I) = L^{2k}(\delta I) = L^{2k}(I + L(I)).$$

Therefore,

$$sol(w) = \frac{\delta L^k(I)}{\delta L^{k-1}(I)} = \frac{L^{2k}(\delta I)}{L^{2k-2}(\delta I)} = \frac{L^{2k}(I) + L^{2k+1}(I)}{L^{2k-2}(I) + L^{2k-1}(I)}.$$

We find the following formula for the ratio:

$$\frac{sol(w)}{sol(w0)} = \frac{\frac{L^{2k}(I)+L^{2k+1}(I)}{L^{2k}(I)}}{\frac{L^{2k-2}(I)+L^{2k-1}(I)}{L^{2k-1}(I)}} = \frac{1+sol(w1)}{1+\frac{L^{2k-2}(I)}{L^{2k-1}(I)}}$$

and this formula is equivalent to the formula stated in the lemma. □

Lemma 3. *The matching condition on the solenoid function implies the compatibility condition on the scaling function.*

Remark 3. In the proof of the next theorem, Theorem 3, we shall also see that in the presence of the summation condition the compatibility condition on the scaling function implies the matching condition on the solenoid function.

Proof. We use repeatedly formulas (7), (8) and the matching condition of the previous lemma. First of all, by (7) and (8)

$$\frac{S(w1)}{S(w0)} = \frac{1+sol(w1)}{1+[sol(w1)]^{-1}} = sol(w1).$$

Secondly, by (7) and (8),

$$\frac{S(w10)}{S(w01)} = \frac{1+[sol(w01)]^{-1}}{1+[sol(w11)]},$$

and by the matching condition, this is equal to

$$\frac{sol(w10)}{sol(w1)}.$$

Similarly,

$$\frac{S(w100)}{S(w011)} = \frac{1+[sol(w011)]^{-1}}{1+[sol(w101)]} = \frac{1+[sol(a^{-1}(w100))]^{-1}}{1+sol(w101)}$$

and by the matching condition, this is equal to

$$\frac{sol(w100)}{sol(w10)}.$$

Proceeding by induction, we conclude

$$\mathrm{sol}(w1\underbrace{0\cdots0}_{n-1}) = \prod_{i=0}^{n-1} \frac{S(w1\overbrace{0\cdots0}^{i})}{S(w0\underbrace{1\cdots1}_{i})}.$$

Thus $C_{S,N}(w)$ converges to $sol(\cdots0\cdots0)$ uniformly for all $w \in \Sigma^*$.

Since the product in (9) converges, the general term must approach 1, and therefore $S(\cdots000) = S(\cdots111)$. □

THEOREM 3. *Let S be a positive continuous function on Σ^*. Then S is the scaling function of a map in $\mathcal{US}$ if and only if S satisfies the summation and compatibility conditions.*

PROOF. The necessity of the summation condition is obvious, and from Lemma 3 any scaling function of a uniformly symmetric circle endomorphism S must satisfy the compatibility conditions.

Conversely, suppose S is a positive continuous function on Σ^* satisfying the summation and compatibility conditions. From S, we first build a solenoid function $\mathrm{sol}(w)$ and then from this solenoid function a linear model L.

Construction of the solenoid function:

We first define sol on codes with last entry equal to 1 by

$$\mathrm{sol}(w1) = \frac{1}{S(w0)} - 1 = \frac{S(w1)}{S(w0)}.$$

It is a continuous function defined on $\Sigma_1^* = \{w = \cdots i_{n-1}\cdots i_1 1\}$. Let

$$\Sigma_0^* = \{w = \cdots i_{n-1}\cdots i_1 0\}.$$

We now extend sol continuously to $\Sigma_0^* \setminus \{\cdots000\}$ by

$$\mathrm{sol}(w1\underbrace{0\cdots0}_{n}) = \prod_{i=0}^{n} \frac{S(w1\overbrace{0\cdots0}^{i})}{S(w0\underbrace{1\cdots1}_{i})}.$$

Note that for $n > m$ the ratio

$$\frac{\mathrm{sol}(w1\overbrace{0\cdots0}^{n})}{\mathrm{sol}(w1\underbrace{0\cdots0}_{m})} = \prod_{i=m+1}^{n} \frac{S(w1\overbrace{0\cdots0}^{i})}{S(w0\underbrace{1\cdots1}_{i})}.$$

Since from the compatibility condition on S these partial products converge to 1, sol extends continuously to the point $\cdots000$ in Σ^*. So we have that $\mathrm{sol}(w) > 0$ and $\mathrm{sol}(w)$ is continuous on Σ^*. Moreover, we claim that sol satisfies the matching condition (10). Let us prove the claim: Suppose $w = w'1$. Then

$$\mathrm{sol}(w) = \mathrm{sol}(w'1) = \frac{S(w'1)}{S(w'0)},$$

$$\mathrm{sol}(w0) = \mathrm{sol}(w'10) = \frac{S(w'1)}{S(w'0)}\frac{S(w'10)}{S(w'01)},$$

$$\mathrm{sol}(w1) = \mathrm{sol}(w'11) = \frac{S(w'11)}{S(w'10)},$$

and

$$\mathrm{sol}(a^{-1}(w0)) = \mathrm{sol}(a^{-1}(w'10)) = \mathrm{sol}(w'01) = \frac{S(w'01)}{S(w'00)}.$$

Thus we have

$$\frac{\mathrm{sol}(w)}{\mathrm{sol}(w0)} = \frac{S(w'01)}{S(w'10)}$$

and

$$\frac{1+\mathrm{sol}(w1)}{1+[\mathrm{sol}(a^{-1}(w0))]^{-1}} = \frac{1+\frac{S(w'11)}{S(w'10)}}{1+\frac{S(w'00)}{S(w'01)}} = \frac{\frac{S(w'10)+S(w'11)}{S(w'10)}}{\frac{S(w'01)+S(w'00)}{S(w'01)}} = \frac{S(w'01)}{S(w'10)}.$$

Of course, the last equality follows from the summation condition. We see that the matching condition (10) holds for $w = w'1$.

Now suppose $w = w'1\underbrace{0\cdots0}_{n}$. Then

$$\mathrm{sol}(w) = \mathrm{sol}(w'1\underbrace{0\cdots0}_{n}) = \prod_{i=0}^{n}\frac{S(w'1\overbrace{0\cdots0}^{i})}{S(w'0\underbrace{1\cdots1}_{i})},$$

and

$$\mathrm{sol}(w0) = \mathrm{sol}(w'1\underbrace{0\cdots0}_{n+1}) = \prod_{i=0}^{n+1}\frac{S(w'1\overbrace{0\cdots0}^{i})}{S(w'0\underbrace{1\cdots1}_{i})}.$$

So

$$\frac{\mathrm{sol}(w)}{\mathrm{sol}(w0)} = \frac{S(w'0\overbrace{1\cdots1}^{n+1})}{S(w'1\underbrace{0\cdots0}_{n+1})}.$$

But

$$\mathrm{sol}(w1) = \mathrm{sol}(w'1\underbrace{0\cdots0}_{n}1) = \frac{S(w'1\overbrace{0\cdots0}^{n}1)}{S(w'1\underbrace{0\cdots0}_{n+1})}$$

and

$$\mathrm{sol}(a^{-1}(w0)) = \mathrm{sol}(a^{-1}(w'1\underbrace{0\cdots0}_{n+1})) = \mathrm{sol}(w'0\underbrace{1\cdots1}_{n+1})) = \frac{S(w'0\overbrace{1\cdots1}^{n+1})}{S(w'0\underbrace{1\cdots1}_{n}0)}.$$

We get

$$1+\mathrm{sol}(w1)=\frac{S(w'1\overbrace{0\cdots 1}^{n+1})+S(w'1\overbrace{0\cdots 0}^{n}1)}{S(w'1\underbrace{0\cdots 0}_{n+1})}=\frac{1}{S(w'1\underbrace{0\cdots 0}_{n+1})}$$

and

$$1+[\mathrm{sol}(a^{-1}(w0))]^{-1}=1+\frac{S(w'0\overbrace{1\cdots 1}^{n}0}{S(w'0\underbrace{1\cdots 1}_{n+1})}=\frac{1}{S(w'0\underbrace{1\cdots 1}_{n+1})}.$$

So

$$\frac{1+\mathrm{sol}(w1)}{1+[\mathrm{sol}(a^{-1}(w0))]^{-1}}=\frac{S(w'0\overbrace{1\cdots 1}^{n+1})}{S(w'1\underbrace{0\cdots 0}_{n+1})}.$$

The matching condition (10) holds for $w=w'1\underbrace{0\cdots 0}_{n}$.

Construction of the linear model L:

Embed the non-negative integers k into the dual symbolic space Σ^* by the formula $w(k)=\cdots 000i_{n-1}\cdots i_1 i_0$ where $k=i_0+i_1 2+\cdots+i_{n-1}2^{n-1}$. Then define $a_k=L^k(0)$ inductively by $a_0=0$, $a_1=1$ and a_k by

$$a_{k+1}-a_k=\mathrm{sol}(w(k))(a_k-a_{k-1}),\quad k\geq 1.$$

We must check that L on the set of points $\{a_k\}_{k\geq 0}$ satisfies the equation

$$\delta L^2(x)=L(\delta x),\quad \delta=\frac{1}{S(\cdots 000)},\quad x\in A.$$

From the matching condition (10),

$$\frac{\mathrm{sol}(k)}{\mathrm{sol}(2k)}=\frac{1+\mathrm{sol}(2k+1)}{1+[\mathrm{sol}(2k-1)]^{-1}},$$

we have

$$\frac{\frac{a_{k+1}-a_k}{a_k-a_{k-1}}}{\frac{a_{2k+1}-a_{2k}}{a_{2k}-a_{2k-1}}}=\frac{1+\frac{a_{2k+2}-a_{2k+1}}{a_{2k+1}-a_{2k}}}{1+\frac{a_{2k-1}-a_{2k-2}}{a_{2k}-a_{2k-1}}}=\frac{\frac{a_{2k+2}-a_{2k}}{a_{2k+1}-a_{2k}}}{\frac{a_{2k}-a_{2k-2}}{a_{2k}-a_{2k-1}}}.$$

Thus

$$\frac{a_{k+1}-a_k}{a_k-a_{k-1}}=\frac{\frac{a_{2k+2}-a_{2k}}{a_{2k+1}-a_{2k}}}{\frac{a_{2k}-a_{2k-2}}{a_{2k}-a_{2k-1}}}\cdot\frac{a_{2k+1}-a_{2k}}{a_{2k}-a_{2k-1}}=\frac{a_{2k+2}-a_{2k}}{a_{2k}-a_{2k-2}}.$$

One can check that $a_2=\delta$ and

$$a_2-a_1=\frac{a_4-a_2}{a_2}.$$

This implies that

$$a_4=\delta^2=\delta a_2.$$

Inductively, suppose $a_{2k}=\delta a_k$ holds for all $0\leq k\leq n$. Then

$$\frac{a_{2n+2}-a_{2n}}{a_{2n}-a_{2n-2}}=\frac{a_{n+1}-a_n}{a_n-a_{n-1}}=\frac{\delta a_{n+1}-\delta a_n}{\delta a_n-\delta a_{n-1}}=\frac{\delta a_{n+1}-a_{2n}}{a_{2n}-a_{2n-2}}.$$

Therefore, $a_{2n+2}=\delta a_{n+1}$ for all $n\geq 0$.

Since

$$\frac{a_{k+1}-a_k}{a_k-a_{k-1}}=\lim_{n\to\infty}\frac{a_{k+1+2^n}-a_{k+2^n}}{a_{k+2^n}-a_{k+2^n-1}}=\lim_{n\to\infty}\mathrm{sol}(w(k+2^n))$$

must be true for a linear model, we can use it to define a_k for all negative integers $k<0$. For example, $a_{-1}=[\mathrm{sol}(\cdots 000)]^{-1}$. Similarly, a_k for $k\le -2$ can be defined inductively.

Since L has been defined on the set $\{a_k\}_{k=-\infty}^{\infty}$ and it satisfies the equation $\delta L=L^2\delta$, we can extend $L|A$ to a unique orientation-preserving homeomorphism of $\mathbb{R}$. This homeomorphism is our linear model.

The linear model L is uniformly Cauchy since sol is continuous on Σ^*. Then Theorem 2 says that L is the linear model of a uniformly symmetric circle endomorphism f. For the map f, let S_f be its scaling function. Since both S_f and S are continuous, $S_f(w(k))=S(w(k))$ for all $w(k)$, and the set of all $w(k)$ is dense in Σ^*, $S_f=S$. □ □

4. Teichmüller space of circle endomorphisms and the space of scaling functions

Consider the space $\mathcal{US}$ of uniformly symmetric circle mappings. The mapping $p(x)=2x \pmod 1$ is in $\mathcal{US}$. Any f in $\mathcal{US}$ is conjugate to p, that is, there is a homeomorphism h of S^1 such that $hf=ph$.

Theorem 4. *Suppose f is in $\mathcal{US}$ and h is the conjugacy from f to p, i.e., $hf=ph$. Then h is quasisymmetric.*

Proof. The reader can find the full proof in [**8**, 88-90] where the result is proved for the more general class of geometrically finite maps. Here we only outline the proof. Since f is uniformly symmetric, the maps $\{g_{w_n}\}$ for all finite strings w_n of 0's and 1's are ρ-quasisymmetric for a fixed $\rho\ge 1$. It follows that f has bounded geometry in the sense that there is a constant $C>0$ such that

$$S(w_n)\ge C$$

for all finite strings w_n of 0's and 1's. Clearly p has bounded geometry. The statement that the conjugacy h is quasisymmetric is equivalent to the statement that both of f and p have bounded geometry (see [**8**, Chapter 3]). □ □

Theorem 5. *Suppose f_1 and f_2 are two mappings in $\mathcal{US}$. Let h be the conjugacy from f_1 to f_2, i.e., $hf_1=f_2h$. Then h is symmetric if and only if the scaling functions of f_1 and f_2 are the same.*

Proof. If h is symmetric, then by applying Lemma 1 and the fact that both f_1 and f_2 have bounded geometry, it is easy to see that the scaling functions of f_1 and f_2 are the same. Conversely, if the scaling functions of f_1 and f_2 are the same, then their linear models will be same. Therefore, they are symmetrically conjugate. □

Let $p(x)=2x \pmod 1$ be a base point in $\mathcal{US}$. We consider the pair (f,h) where $f\in\mathcal{US}$ and h is the quasisymmetric conjugacy from f to p, i.e. $hf=ph$. Two pairs (f_1,h_1) and (f_2,h_2) are equivalent, denoted by $(f_1,h_1)\sim(f_2,h_2)$, if $h_1^{-1}h_2$ is symmetric. Let $[(f,h)]$ denote the equivalence class of (f,h) in $\mathcal{US}$. The Teichmüller space $\mathcal{TUS}$ of $\mathcal{US}$ is the space of all equivalence classes in $\mathcal{US}$.

Write $\mathcal{C}(\Sigma^*)$ for the space of all continuous functions $S : \Sigma^* \to \mathbb{R}$ with the supremum norm. Let $\mathcal{S}F(\Sigma^*)$ be the subspace of all positive functions $S \in \mathcal{C}(\Sigma^*)$ that satisfy the summation and compatibility conditions. For any $\kappa = [f, h]$ in $\mathcal{T}US$, define $\iota(\kappa)$ to be the scaling function of f. From Theorems 3 and 5, we have

THEOREM 6. *The map* $\iota : \mathcal{T}US \to \mathcal{S}F(\Sigma^*)$ *is bijective.*

Now let us consider the universal Teichmüller space and the induced metric on the Teichmüller spaces $\mathcal{T}US$. Let $\mathcal{Q}S$ be the set of all quasisymmetric homeomorphisms of the circle factored by the space of all Möbius transformations of the circle. ($\mathcal{Q}S$ may be identified with the set of all quasisymmetric homeomorphisms of the circle fixing three points). Let $\mathcal{S}$ be the subset of $\mathcal{Q}S$ consisting of all symmetric homeomorphisms of the circle. In [**6**] it is shown that $\mathcal{S}$ is a subgroup of $\mathcal{Q}S$ closed in the Teichmüller topology. For any $h \in \mathcal{Q}S$, let $\mathcal{E}_h$ be the set of all quasiconformal extensions of h into the unit disk. Let $K_{\tilde{h}}$ be the quasiconformal dilatation of $\tilde{h} \in \mathcal{E}_H$ (see [**10**], Chapter 1). Using quasiconformal dilatation, one defines a distance in $\mathcal{Q}S$ by

$$d(h_1, h_2) = \frac{1}{2} \inf\{\log K_{\tilde{h}_1 \tilde{h}_2^{-1}} | \tilde{h}_1 \in \mathcal{E}_{h_1}, \tilde{h}_2 \in \mathcal{E}_2\}.$$

$(\mathcal{Q}S, d)$ is called universal Teichmüller space. It is a complete metric space and a complex manifold (refer to [**1**]) with complex structure compatible with the Hilbert transform. The topology coming from the metric d on $\mathcal{Q}S$ induces a topology on the factor space $\mathcal{Q}S$ mod $\mathcal{S}$. Given two cosets $\mathcal{S}f$ and $\mathcal{S}g$ in this factor space, define a metric by

$$\overline{d}(\mathcal{S}f, \mathcal{S}g) = \inf_{A,B \in \mathcal{S}} d(Af, Bg).$$

This metric is defined in [**6**] where the factor space $\mathcal{Q}S$ mod S is also shown to be complete metric space and a complex manifold. The topology on $(\mathcal{Q}S \text{ mod } \mathcal{S}, \overline{d})$ is the finest topology which makes the projection $\pi : \mathcal{Q}S \to \mathcal{Q}S/\mathcal{S}$ continuous, and π is also holomorphic. An equivalent topology can be defined as follows. For any $h \in \mathcal{Q}S$, let $\tilde{h}$ be a quasiconformal extension of h to a small neighborhood U of S^1 in the complex plane. Let $\mu_{\tilde{h}} = \frac{\tilde{h}_{\overline{z}}}{\tilde{h}_z}$, $k_{\tilde{h}} = \|\mu(z)\|_\infty$, and $C_{\tilde{h}} = (1 + k_{\tilde{h}})/(1 - k_{\tilde{h}})$. Then

$$C_h = \inf C_{\tilde{h}}$$

where the infimum is taken over all quasiconformal extensions of h near the circle. The number C_h is called the boundary dilatation of h. It is known that h is symmetric if and only if $C_h = 1$. Define

$$\tilde{d}(h_1, h_2) = \frac{1}{2} \log C_{h_2^{-1} h_1}.$$

It is shown in [**6**] that the metrics $\overline{d}$ and $\tilde{d}$ are equal.

The topology of the Teichmüller space $\mathcal{T}US$ can be introduced similarly and $\mathcal{T}US$ is a closed subspace of the complex manifold $\mathcal{Q}S$ mod $\mathcal{S}$. We give a proof of the following lemma from dynamical systems point of view (see also [**4, 7**]).

LEMMA 4. *The space* $\mathcal{T}US$ *is a complete space.*

PROOF. Take any Cauchy sequence $\{\kappa_n\}_{n=1}^\infty = \{[f_n, h_n]\}_{n=1}^\infty$ in $\mathcal{T}US$. Then $\overline{d}(h_n, h_m) \to 0$ as $n, m \to \infty$. We may assume by working modulo $\mathcal{S}$ that $h_n^{-1} h_m$ tends to the identity map as m, n go to infinity. Therefore, $\{h_n\}_{n=1}^\infty$ is a Cauchy sequence in the universal Teichmüller space and h_n tends to a quasisymmetric map h as n goes to infinity. Let $L_n = h^{-1}h_n$ and $f = h^{-1}ph$. Define maps f_n by $f = L_n f_n L_n^{-1}$ for all $n > 0$. Let $\rho(L_n)$ be the quasisymmetric constant of L_n. Then $\rho(L_n) \to 1$ as $n \to \infty$. Let g_{n,w_k} be the inverse of $f_n^k : I_{n,w_k} \to I = [0,1]$ and g_{w_k} be the inverse of $f^k : I_{w_k} \to I$. Let $\rho(g_{n,w_k}, t)$ be the symmetric distortion of g_{n,w_k}. Then there is a function $\epsilon_n(t) \to 1$ as $t \to 0$ such that $\rho(g_{n,w_k}, t) \leq \epsilon_n(t)$ for all w_k. Let $\rho(g_{w_k}, t)$ be the symmetric distortion of g_{w_k}. Since $g_{w_k} = L_n g_{n,w_k} L_n^{-1}$, $\rho(g_{w_k}, t) \leq (\rho(L_n))^2 \epsilon_n(t)$ for all w_k and all $n > 0$. Therefore, $\rho_f(t) \leq \inf_{n>0}\{(\rho(L_n))^2 \epsilon_n(t)\}$. So $\rho_f(t) \to 1$ as $t \to 0$. We may take I as any interval with length one in the above argument. This means that f is uniformly symmetric and $[f,h] \in \mathcal{TUS}$. So $\mathcal{TUS}$ is complete. □

LEMMA 5. *For any $[f,h] \in \mathcal{TUS}$ and any $\epsilon > 0$, there is an analytic expanding circle endomorphism f_ϵ and a conjugacy h_ϵ such that $h_\epsilon P = f_\epsilon h_\epsilon$ and such that $[f_\epsilon, h_\epsilon]$ is in the ϵ-neighborhood of $[f,h]$.*

PROOF. For the proof of this lemma we refer to the discussion following Lemma 5.2 in [**3**] and to Lemma 2 in [**7**]. □

Now let $\mathcal{C}^\omega$ be the space of analytic expanding circle endomorphism and $\mathcal{TC}^\omega = \{[f,h] | f \in \mathcal{C}^\omega\}$ where $[f]$ means the equivalent classes in $\mathcal{US}$. Then$\mathcal{TC}^\omega$ is a subspace of $\mathcal{TUS}$. Lemmas 4 and 5 imply that

THEOREM 7. *The completion of $\mathcal{TC}^\omega$ is $\mathcal{TUS}$.*

Now Theorem 3 and Theorem 6 imply that

THEOREM 8. *The map $\iota : \mathcal{TUS} \to \mathcal{SC}(\Sigma^*)$ is a continuous bijective map.*

PROOF. Theorems 3 and 6 imply

$$\iota : \mathcal{TUS} \to \mathcal{SC}(\Sigma^*)$$

is bijective and continuous. To complete the proof we must show that if a sequence of solenoid functions sol_n converges uniformly to a solenoid function sol, then the conjugacies H_n, where

$$H_n \circ L = L_n \circ H_n,$$

have quasisymmetry constants converging to 1. This is the content of Lemmas 4.7 and 4.8 in [**3**]. □

By Theorem 8, the metric on $\mathcal{TUS}$ induces a metric on $\mathcal{SC}(\Sigma^*)$ which we call Teichmüller metric. Thus $\mathcal{SC}(\Sigma^*)$ is a complex manifold.

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