On geometrically finite branched coverings[*]
II. Realization of rational maps[†]

Guizhen Cui[‡], Yunping Jiang[§] and Dennis Sullivan[¶]

Abstract

Following the first part of our research, we prove in this paper that a sub-hyperbolic semi-rational map with infinite post-critical set is combinatorially and locally holomorphically equivalent to a rational map if and only if it has no Thurston obstruction. Moreover, the rational map is unique up to holomorphic conjugation.

Contents

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1 Introduction

We study geometrically finite branched coverings of the Riemann sphere. In the first part of this research [CJS], we studied a local combinatorial property of a geometrically finite branched covering at an accumulation point of its post-critical set. We showed that only the global condition, no Thurston obstruction, was not enough for a geometrically finite branched covering to be combinatorially equivalent to a rational map. The locally combinatorially attracting condition is necessary for this problem. Precisely, we show that a geometrically finite branched covering is combinatorially equivalent to a semi-rational map (see below for the definition) if and only if it is locally combinatorially attracting. The semi-rational map can be further chosen to be sub-hyperbolic (refer to Theorem 2 in [CJS]).

In this paper, we study whether a sub-hyperbolic semi-rational map is combinatorially equivalent to a rational map. Note that a geometrically finite rational map admits a quasiconformal deformation in general, that means a semi-rational map may be combinatorially equivalent to many rational maps up to holomorphic conjugation. To recover the rigidity theorem, we introduce CLH-equivalence (see below) between semi-rational maps instead of combinatorial equivalence.

Now let us give some definitions and state our main theorem. Suppose that f is a geometrically finite branched covering of the Riemann sphere $\hat{\mathbb{C}}$, that means the accumulation point set P'_f of the post-critical set P_f is finite. The set P'_f consists of finite number of periodic cycles. The map f is called *a semi-rational map* (see §3 in [CJS]) if (i) f is holomorphic in a neighborhood of P'_f; (ii) each cycle in P'_f is either attractive or super-attractive or parabolic; and (iii) each attracting petal associated with a parabolic cycle in P'_f contains a post-critical point. Furthermore, if all cycles in P'_f are either attractive or super-attractive, then we call f a *sub-hyperbolic* semi-rational map.

We call two sub-hyperbolic semi-rational maps f and g are *CLH-equivalent* (CLH means combinatorially and local holomorhically) if there is a pair of homeomorphisms (ϕ, ψ) of the Riemann sphere $\hat{\mathbb{C}}$ such that

- ψ is homotopic to ϕ rel P_f,
- $\phi f = g\psi$,
- ϕ is holomorphic in a neighborhood of P'_f.

The reader may refer to [DH] or [CJS] for the definition of a Thurston obstruction. He also can find a brief introduction to the history of research in this direction in [CJS]. Our main result in this paper is that

Theorem 1 (Main Theorem). *A sub-hyperbolic semi-rational map with infinite post-critical set is CLH-equivalent to a rational map if and only if it has no Thurston obstruction. Moreover, the rational map is unique up to holomorphic conjugation.*

We will give a proof of the Theorem 1 in §2 and §3. Here we outline our idea in the proof as follows.

We start with a sub-hyperbolic semi-rational map f. Pick a round disk in Koenig's (or Boettcher's) coordinates for each point of P_f' and denote by U_0 their union such that $f^{-1}(U_0)$ contains the closure of U_0. Let $U_n = f^{-n}(U_0)$ and $V_n = \hat{\mathbb{C}} - U_n$ for all integers $n \geq 0$. Then $\{U_n, V_n\}_{n=0}^{\infty}$ gives a sequence of partitions of $\hat{\mathbb{C}}$. This sequence of partitions has a similar flavor to the sequence of partitions used by Branner-Hubbard (see [BH]) in the studying of cubic polynomials and the sequence of partitions used by Yoccoz in the study of non-renormalizable quadratic polynomials (see [Hu, Ji]). We then classify the connected components of V_n into strictly essential components, semi-essential components, and non-essential components according to their relations with P_f homotopically (see §2 for the definition). We observe that the number of homotopic classes of strictly essential components of V_n is eventually a constant. Moreover, each strictly essential component of V_n contains exactly a strictly essential component of V_{n+1} and they are homotopically equivalent. This observation is formulated into Lemma 1 in §2.

Choose the standard complex structure on U_0. Pull it back by f^n, we get a complex structure on U_n. Take an integer $N > 0$ such that strictly essential components of V_n are stabilized for $n \geq N$. On each periodic strictly essential component E_N of V_N, (f, E_N) can be extended to a critically finite branched covering. Applying Thurston's Theorem, we know that there exists an invariant complex structure on E_N.

By the no Thurston obstruction condition, we can set a modulus for each semi-essential component of V_N such that V_{N+1} with the pullback complex structure from V_N by f can be holomorphically and homotopically embedded into V_N. Now pasting these invariant complex structures on V_N and U_N by quasiconformal surgery, we get a global invariant complex structure.

We discuss mainly the global combinatorial structure of the topological Julia set $J = \hat{\mathbb{C}} - \cup_{n=0}^{\infty} U_n$ of a sub-hyperbolic semi-rational map f. We can also use the sequence of partitions $\{U_n, V_n\}_{n=0}^{\infty}$ about J to study the topological property, like connectivity and local connectivity, of the Julia set of a sub-hyperbolic rational map (refer to [PT]).

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2 Partitions on the dynamical plane

We start with a sub-hyperbolic semi-rational map f with P'_f non-empty. Suppose that f is holomorphic on an open set $W \supset P'_f$. From Koenig's Theorem and Boettcher's Theorem (refer to [Mi]), for every point $a \in P'_f$, there is a round disk $D_a \subset W$ centered at a in Koenig's or Boettcher's coordinates such that their boundaries are disjoint pairwise and disjoint from P_f and their union U_0 satisfies that $f^{-1}(U_0) \supset \overline{U_0}$. Denote by $U_n = f^{-n}(U_0)$ and $V_n = \hat{\mathbb{C}} - U_n$ for all integers $n \geq 0$. Then $f^{-1}(V_n) = V_{n+1}$ and V_{n+1} is contained in the interior of V_n. To describe the topology of V_n with respect to the post-critical set, we introduce the following definition.

Definition 1. *A continuum $E \subset \hat{\mathbb{C}}$ is called non-essential (essential) if there is (not) a simply-connected domain $D \supset E$ such that $D \cap P_f = \emptyset$. An essential continuum $E \subset \hat{\mathbb{C}}$ is called semi-essential (strictly-essential) if there is (not) an annulus $A \supset E$ such that $A \cap P_f = \emptyset$. It is called peripheral (about $a \in P_f$) if there is a simply-connected domain $D \supset E$ such that $D \cap P_f = \{a\}$.*

Two essential Jordan curves in $\hat{\mathbb{C}} - P_f$ are called *parallel* if they are homotopic to each other rel P_f. The following properties can be deduced from the definition directly.

- Suppose that E_1 and E_2 are continua with $E_1 \supset E_2$. If E_2 is essential (strictly-essential), then E_1 ia also essential (strictly-essential).
- Each component of the preimage of a non-essential continuum is also non-essential.
- Each component of the preimage of a semi-essential continuum is either non-essential or semi-essential.

- Each component of the preimage of a peripheral continuum is either non-essential or peripheral.
- If a component of V_n is semi-essential, then its boundary contains exactly two essential Jordan curves. Moreover, these two essential curves are parallel.
- If a component of V_n is strictly essential, then its boundary contains at least one essential curve. Any two essential curves on its boundary (if there are) are not parallel.

Lemma 1. *There exists an integer $N > 0$ such that for any $n \geq N$, each strictly-essential component E of V_n contains exactly one strictly-essential component E_1 of V_{n+1}. Moreover, $E - E_1$ is disjoint from P_f and every essential curve on the boundary of E (or E_1) is parallel to a Jordan curve on the boundary of E_1 (or E).*

Proof. Let B_n be the collection of essential Jordan curves on the boundary of U_n. Since $V_0 \cap P_f$ is finite, the number of homotopy classes of curves in $\cup_{n \geq 0}^{m} B_n$ is bounded for all integer $m > 0$. This implies that there exists an integer $N_0 > 0$ such that each curve γ in B_n with $n > N_0$ is parallel to a curve δ in B_m with $m \leq N_0$.

Suppose E_1 is a strictly-essential components of V_{n+1} with $n > N_0$. Let E be the component of V_n with $E \supset E_1$, then E is also strictly-essential. Suppose γ_1 is an essential Jordan curves on the boundary of E_1. From the above argument, there exists a curve $\beta \in B_m$ with $m \leq N_0$ such that β is parallel to γ_1. Since $E \supset E_1$, there exists a unique Jordan curve γ on the boundary of E such that γ separates γ_1 from β. Hence γ is also parallel to γ_1.

Conversely, for any essential curve γ' on the boundary of E, let γ_1' be the Jordan curve on the boundary of E_1 such that γ_1' separates the interior of E_1 from γ, then γ_1' is essential. From the above discussion, γ_1' is parallel to γ'.

If there exists a strictly-essential component of V_{n+1} in $E - E_1$, say E_1'. Let γ_1 be the Jordan curve on the boundary of E_1 such that γ_1 separates the interior of E_1 from E_1', then γ_1 is essential. Let γ be the Jordan curve on the boundary of E such that γ is parallel to γ_1. Then E_1' is contained in the annulus bounded by γ and γ_1. It contradicts with that E_1' is strictly-essential. The same argument also implies that $E - E_1$ is disjoint from P_f.

Let s_n be the number of strictly-essential components of V_n, then s_n is non-increasing for $n \geq N_0$. So s_n is a constant as $n \geq N$ for some $N \geq N_0$. This completes the proof of the lemma. $\square$

Lemma 2. *Suppose that* $a \in P_f - P'_f$ *is a fixed point. Suppose that* γ *is a Jordan curve on the boundary of* V_n *with* $n \geq N$ *so that* γ *is peripheral about* a. *Let* δ *be the component of* $f^{-1}(\gamma)$ *which is parallel to* γ. *Then* δ *separates* γ *from* a.

Moreover, either δ *and* γ *are contained in different components of* V_n, *or both of them are contained in a peripheral and strictly-essential component of* V_n.

Proof. Since $a \in V_j$ for all $j \geq 0$, a is contained in a component of V_n, denoted by D_n. Let β_i, $0 \leq i \leq k$ be all the curves on the boundary of V_n which are parallel to γ, so that β_i separates β_{i+1} from a. Then $\beta_0 \subset \partial D_n$. Let α_i be the component of $f^{-1}(\beta_i)$ which is parallel to γ. Since the component of V_{n+1} which contains a is contained in the interior of D_n, α_0 separates β_0 from a.

We claim that α_i separates β_i from a for all $1 \leq i \leq k$. Otherwise, $A(\beta_i, \beta_{i-1}) \subset A(\alpha_i, \alpha_{i-1})$ for some $1 \leq i \leq k$, where $A(\beta_i, \beta_{i-1})$ denotes the annulus bounded by β_i and β_{i-1}. Note that both β_i and β_{i-1} are contained in the boundary of either a component W_n of U_n or a component of V_n. The later case can not happen since otherwise both α_i and α_{i-1} are contained in a component of V_{n+1}. This contradicts to $V_{n+1} \subset V_n$. In the former case, $f^{-1}(W_n) \cap W_n \neq \emptyset$ and hence $f^{-1}(W_n) \supset W_n$. Therefore $f^n(W_n) \subset W_n \subset A(\beta_i, \beta_{i-1})$. But $f^n(W_n)$ is a component of U_0, which contains a point in P'_f. That is a contradiction.

Now, if $\gamma = \beta_0$, then both γ and δ are contained in D_n, which is strictly-essential and peripheral. Otherwise, γ and δ are contained in different components of V_n. $\square$

To find an invariant complex structure on V_N, let us simplify the topology of components of V_N at first. For every component E of V_N, the *enclosure* of E id defined by the following: If E is non-essential, then there exists a unique non-essential closed Jordan domain, which is defined to be the enclosure of E, such that the boundary of E' is contained in the boundary of E. If E is essential, the enclosure of E is defined to be the domain bounded by essential Jordan curves on the boundary of E. It is easy to see that when E is semi-essential, then the enclosure of E is a closed annulus. When E is strictly essential, then either its enclosure meets post-critical points, or is bounded by at least three essential curves. When E is strictly-essential and peripheral around $a \in P_f$, then its closure is a closed Jordan domain which intersects with P_f on the single point a. Enclosures of essential components of V_N are disjoint pairwise.

Denote by V'_N the union of enclosures of essential components of V_N, V'_{N+1} the union of essential component of $f^{-1}(V'_N)$. Then V'_{N+1} is also contained in the interior of V'_N.

Proposition 1 (Holomorphic and homotopic embeddings). *Suppose that f has no Thurston obstruction. Then the restriction $f : V'_{N+1} \to V'_N$ satisfies the following condition: For any component E_k of V'_N, there exist an embedding ϕ_k from E_k into $\hat{\mathbb{C}}$ and a homeomorphism θ_k from E_k to itself such that:*

- *θ_k is homotopic to the identity rel $P_f \cap E_k$ and*
- *for each component E of V'_{N+1}, let E_i be the component of V'_N which contains E and $E_j = f(E)$, then $\phi_j \circ f \circ (\phi_i \circ \theta_i)^{-1}$ is holomorphic in the interior of $\phi_i \circ \theta_i(E)$*

We will prove Proposition 1 in the next section. Now we will use Proposition 1 in the proof of our main result.

Proof of Theorem 1 (Main Theorem). We assume that the semi-rational map f is a quasiregular map (a rational map composed with a quasiconformal map) and holomorphic in U_{N+1} and the boundary of U_n $(0 \leq n \leq N+1)$ consists of quasicircles. This can be done by making a CLH-equivalence as the following: Note that there exists a homeomorphism ζ_0 of $\hat{\mathbb{C}}$ so that $f \circ \zeta_0^{-1}$ is holomorphic. Since f is holomorphic on U_0, ζ_0 is holomorphic on U_0. Therefore, $f_0 = \zeta_0 \circ f \circ \zeta_0^{-1}$ is holomorphic in $\zeta_0(U_1)$. There is also a homeomorphism ζ_1 of $\hat{\mathbb{C}}$ so that $f_0 \circ \zeta_1^{-1}$ is holomorphic. Since f_0 is holomorphic in $\zeta_0(U_1)$, ζ_1 is holomorphic in $\zeta_0(U_1)$. Therefore, $f_1 = \zeta_1 \circ f_0 \circ \zeta_1^{-1}$ is holomorphic in $\zeta_1 \circ \zeta_0(U_2)$. Continuing this process, we get homeomorphisms ζ_n $(0 \leq n \leq N+2)$ of $\hat{\mathbb{C}}$ such that $f_{n-1} \circ \zeta_n^{-1}$ is holomorphic. ζ_n is holomorphic in $\zeta_{n-1} \circ \cdots \circ \zeta_0(U_n)$ and

$$f_n = \zeta_n \circ f_{n-1} \circ \zeta_n^{-1} = \zeta_n \circ \cdots \circ \zeta_0 \circ f \circ (\zeta_n \circ \cdots \circ \zeta_0)^{-1}$$

is holomorphic in $\zeta_n \circ \cdots \circ \zeta_0(U_{n+1})$.

Since $V_{N+1} \cap P_f$ is a finite set, there exists a quasiconformal map η of $\hat{\mathbb{C}}$ so that $\eta = \zeta_{N+2}$ in $\zeta_{N+1} \circ \cdots \circ \zeta_0(U_{N+1})$ is homotopic to ζ_{N+2} rel

$P_{f_{N+1}}$. Since $f_{N+1} \circ \zeta_{N+2}^{-1}$ is holomorphic, $f_{N+1} \circ \zeta_{N+2}^{-1} \circ \eta$ is quasiregular, which is CLH-equivalent to f by the pair

$$(\zeta_{N+1} \circ \cdots \circ \zeta_0, \quad \eta^{-1} \circ \zeta_{N+2} \circ \zeta_{N+1} \circ \cdots \circ \zeta_0).$$

Note that the boundary of $\zeta_{N+1} \circ \cdots \circ \zeta_0(U_n)$ $(0 \le n \le N+1)$ consists of analytic Jordan curves (holomorphic preimages of round circles in Koenig's or Boettcher's coordinates). Hence they are quasicircles.

From Proposition 1, there exist an embedding ϕ from V'_N into $\hat{\mathbb{C}}$ and a homeomorphism θ from V'_N to itself such that θ is homotopic to the identity rel $P_f \cap V'_N$ and $\phi \circ f \circ (\phi \circ \theta)^{-1}$ is holomorphic in the interior of $\phi \circ \theta(V'_{N+1})$.

Let V''_N be the union of enclosures of components of V_N. Then $V''_{N+1} = f^{-1}(V''_N)$ is contained in the interior of V''_N. Since any component of $V''_N - V'_N$ is non-essential, the embedding ϕ and the homeomorphism θ can be extended to V''_N such that θ is homotopic to the identity rel $P_f \cap V''_N$ and $\phi \circ f \circ (\phi \circ \theta)^{-1}$ is holomorphic in the interior of $\phi \circ \theta(V''_{N+1})$.

Moreover, both ϕ and θ can be chosen so that they are quasiconformal and θ is homotopic to the identity rel $(P_f \cap V''_N) \cup \partial V''_N$. This shows that θ can be extended to a homeomorphism of $\hat{\mathbb{C}}$ such that it is the identity on the complement of V''_N and is homotopic to the identity rel P_f.

Consider the semi-rational map $g = f \circ \theta^{-1}$. It is quasiregular and CLH-equivalent to f. Restriction to the complement of V''_N, it is holomorphic. Restriction to interior of V''_{N+1}, it is quasiconformally conjugated to a holomorphic map. Since every grand orbit of G passes over $V''_N - V''_{N+1}$ at most twice, it is quasiconformally conjugated to a rational map by a quasiconformal map which is holomorphic in the complement of V''_N. That means f is CLH-equivalent to a rational map.

Suppose that f is CLH-equivalent to rational maps R_1 and R_2 be pairs (ϕ_1, ψ_1) and (ϕ_2, ψ_2). Then R_1 and R_2 are CLH-equivalent by the pair $(\phi_1 \phi_2^{-1}, \psi_1 \psi_2^{-1})$. Apply Theorem 2 in [Cu], we see R_1 and R_2 are holomorphically conjugated. This is the uniqueness part of the theorem. □

3 Holomorphic and homotopic embeddings

This section is devoted to the proof of Proposition 1. Before to do that, we present some lemmas.

Lemma 3. *Let h be a critically finite branched covering of S^2. Suppose that the signature of $\mathcal{O}_h$ is not $(2,2,2,2)$, $P \supset P_h$ is a finite set and*

$H(P) \subset P$. If for any multicurve Γ on $S^2 - P$, $\lambda(\Gamma) < 1$, then there exist homeomorphisms $\phi, \psi : S^2 \to \hat{\mathbb{C}}$ such that ϕ is homotopic to ψ rel P and $\phi \circ h \circ \psi^{-1}$ is a rational map.

Here we use the homotopy rel P instead of P_h when we define multicurves and their transition matrices. One may verify the above lemma following the proof of Thurston's theorem (refer to [DH]).

Lemma 4. *Suppose (D_1, z_1) and (D_2, z_2) are disjoint simply-connected domains in $\hat{\mathbb{C}}$ with marked points $z_1 \in D_1$ and $z_2 \in D_2$. Let $D_1(r_1)$, $D_2(r_2)$ be hyperbolic disks in D_1 and D_2, respectively, with pseudo-hyperbolic radii r_1 and r_2, and centers z_1 and z_2. Then there exists a constant $C > 0$ which is independent of r_1 and r_2 such that*

$$mod(\hat{\mathbb{C}} - \overline{D_1(r_1)} - \overline{D_2(r_2)}) \leq mod(D_1 - \overline{D_1(r_1)}) + mod(D_2 - \overline{D_2(r_2)}) + C.$$

Proof. Let ζ be a Möbius transformation of $\hat{\mathbb{C}}$ with $\zeta(z_1) = 0$ and $\zeta(z_2) = \infty$, so that the conformal radius of $(\zeta(D_1), 0)$ is 1, which means the Riemann mapping from the unit disk to $\zeta(D_1)$ with the origin fixed having derivative 1 at the origin. Suppose the conformal radius of $(\zeta(D_2), \infty)$ is C_1. By the Koebe 1/4-Theorem, $\zeta(D_1(r_1))$ contains the disk with radius $r_1/4$ and center the origin, and $\zeta(D_2(r_2)) \supset \{z : |z| > 4/C_1 r_2\}$. Therefore

$$\begin{aligned} & \mathrm{mod}(\hat{\mathbb{C}} - \overline{D_1(r_1)} - \overline{D_2(r_2)}) \leq \mathrm{mod}(\{z : r_1/4 < |z| < 4/C_1 r_2\}) \\ \leq\ & \mathrm{mod}(D_1 - \overline{D_1(r_1)}) + \mathrm{mod}(D_2 - \overline{D_2(r_2)}) + (\log 16)/(2\pi C_1). \end{aligned}$$

□

Lemma 5. *Let A be a non-negative matrix, $\lambda(A)$ the spectral radius of A. Then for any $\epsilon > 0$, there is a positive vector v such that $Av \leq (\lambda(A) + \epsilon)v$.*

Proof. There exists a permutation matrix P such that

$$P^T A P = B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ 0 & B_{22} & \cdots & B_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_{mm} \end{pmatrix}$$

where the square matrices B_{ii}, $i = 1, 2, \cdots, k$, are irreducible with maximal eigenvalues

$$\lambda_1 = \lambda(A) \geq \lambda_2 \geq \cdots \geq \lambda_k,$$

and B_{jj}, $j = k+1, \cdots, m$, are 1×1 zero-matrices (refer [LT]). Let v_i, $i = 1, \cdots, k$, be the eigenvectors corresponding to the eigenvalues λ_i. They are positive vectors. Let

$$v = (v_1, \delta v_2, \delta^2 v_3, \cdots, \delta^{k-1} v_k, \delta^k, \cdots, \delta^k).$$

It is easy to see that $Av \leq (\lambda(A) + \epsilon)v$ for any given $\epsilon > 0$ whenever the constant $\delta > 0$ is small enough. □

Now we are going to complete the proof of Proposition 1. For the simplicity of the notation, we use V and V' to replace V'_N and V'_{N+1}. Let us classify the components of V as the following:

- V_{nt} is the union of non-peripheral and strictly-essential components of V.
- V_{ns} is the union of non-peripheral and semi-essential components of V.
- V_{pt} is the union of peripheral and strictly-essential components of V.
- V_{ps} is the union of peripheral and semi-essential components of V.

Note that each component of $V_{ns} \cup V_{ps}$ is a closed annulus; each component of V_{pt} is a closed disk which contains exactly one post-critical point. By Lemma 1, for each component E_i of V_{nt} and each $k > 0$, there exists exactly one strictly-essential component of $f^{-k}(V)$, denote by E_i^k, so that $E_i^k \subset E_i$. Therefore E_i is either periodic (that means $f^p(E_i^p) = E_i$ for some integer $p \geq 1$) or preperiodic (that means $f^k(E_i^k)$ is periodic for some $k \geq 1$ but E_i is not periodic).

Let Γ be the collection of non-peripheral boundary curves of V. Define the transition matrix A by the formula:

$$A_{\delta\gamma} = \sum_{\alpha} \frac{1}{\deg(f : \alpha \to \gamma)},$$

where the sum is taken over all components α of $f^{-1}(\gamma)$ which are parallel to δ and contained in the same component of V with δ.

Let $\Gamma' \subset \Gamma$ be a multicurve such that for any $\gamma \in \Gamma$, there is $\gamma' \in \Gamma'$ such that γ' is parallel to γ. Then

$$\sum_{\delta \sim \delta'} A_{\delta\gamma} = A'_{\delta'\gamma'}$$

where A' is the transition matrix of Γ'. If $v = \{v_\gamma\}_{\gamma\in\Gamma}$ is an non-negative eigenvector of A corresponding to the eigenvalue $\lambda(A)$, let $u_{\gamma'} = \sum_{\gamma\sim\gamma'} v_\gamma$, then

$$\begin{array}{rcl} (A'u)_{\delta'} & = & \sum_{\gamma'} A'_{\delta'\gamma'} u_{\gamma'} = \sum_{\gamma'} \sum_{\gamma\sim\gamma'} v_\gamma \sum_{\delta\sim\delta'} A_{\delta\gamma} \\ & = & \sum_{\delta\sim\delta'} \sum_\gamma v_\gamma A_{\delta\gamma} = \sum_{\delta\sim\delta'} \lambda(A) v_\delta = \lambda(A) u_{\delta'}. \end{array}$$

So $\lambda(A)$ is an eigenvalue of A' and hence $\lambda(A) \le \lambda(A') < 1$.

Let Γ'' be the collection of boundary curves of $V - V_{pt}$. Define the transition matrix A'' by the formula:

$$A''_{\delta\gamma} = \sum_\alpha \frac{1}{\deg(f : \alpha \to \gamma)},$$

where the sum is taken over all components of α of $f^{-1}(\gamma)$ which are parallel to δ and contained in the same component of V with δ. By Lemma 2, $\lambda(A'') = \lambda(A)$. From Lemma 5, we have:

Lemma 6. *There is a positive function $v : \Gamma'' \to \mathbb{R}_+$ such that*

$$\sum_\alpha \frac{v(f(\alpha))}{\deg(f|_\alpha)} \le \frac{\lambda(A)+1}{2} v(\gamma) < v(\gamma)$$

where the sum is taken over all Jordan curves on the boundary of V' which are parallel to γ and contained in the same component of V with γ.

Let us prove Proposition 1 now.

Proof of Proposition 1. For each component E_i of V_{nt}, pick a point in each component of $\hat{\mathbb{C}} - E_i$ and denote by P_i the union of $P_f \cap E_i$ with these points. Then $f : E_i^1 \to E_j$ can be extended to a branched covering (or a homeomorphism) h_i from $\hat{\mathbb{C}}$ to itself such that $\deg(h_i) = \deg(f|_{E_i^1})$, $h_i(P_i) \subset P_j$ and there is at most one critical point in each component of $\hat{\mathbb{C}} - E_i^1$ whose image is contained in P_j.

Suppose that $E_1^p \to E_2^{p-1} \to \cdots \to E_p^1 \to E_1$ is a cycle. Let γ^0 be a Jordan curve on the boundary of E_1. Then there exists a unique Jordan curve γ^p on the boundary of E_1^p which is parallel to γ^0 by Lemma 1. Therefore there are integers $k \ge 0$ and $q \ge 1$ such that $f^{kp}(\gamma^{kp+qp}) = \alpha^{qp}$ is parallel to $f^{qp}(\alpha^{qp}) = \alpha^0$. By Lemma 2, α^0 is non-peripheral. Hence $\deg(f^{qp} : \alpha^{qp} \to \alpha^0) > 1$ since f has no Thurston obstruction. So $\deg(f^p|_{E_1^p}) > 1$.

Let $h = h_p \circ \cdots \circ h_2 \circ h_1$. Then h is a critically finite branched covering with $\deg(h) > 1$, $P_h \subset P_1$ and $h(P_1) \subset P_1$. From the above discussion, we see that the orbit of each point in $P_1 - E_1$ converges to a periodic cycle in $P_1 - E_1$ whose degree is bigger than one. Therefore the signature of $\mathcal{O}_h$ is not $(2, 2, 2, 2)$. Let Γ be a multicurve on $\hat{\mathbb{C}} - P_1$. Since each component of $\hat{\mathbb{C}} - E_1$ contains exactly one point in P_1, each $\gamma \in \Gamma$ is homotopic to a curve $\beta \in E_1$ rel P_1. So the transition matrix of Γ under h is less than the one of f^p and hence its spectral radius is less than 1. By Lemma 3, there exists a pair of homeomorphisms $(\phi_1, \psi_1) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that ϕ_1 is homotopic to ψ_1 rel P_1 and $\phi_1 \circ h \circ \psi_1^{-1} = R$ is a rational map.

There is a homeomorphism ϕ_p from $\hat{\mathbb{C}}$ to itself such that $\phi_1 \circ h_p \circ \phi_p^{-1} = R_p$ is a rational map. Inductively, there are homeomorphisms ϕ_j $(1 < j < p)$ from $\hat{\mathbb{C}}$ to itself such that $\phi_{j+1} \circ h_j \circ \phi_j^{-1} = R_j$ is a rational map. In particular, $\phi_2 \circ h_1 \circ \psi_1^{-1} = R_1$ is a rational map and $R_p \circ \cdots \circ R_2 \circ R_1 = R$.

For each point $a \in P_1 - E_1$, since $\phi_1(a)$ is eventually a superattracting periodic point, it is contained in a Fatou domain W_a of R, which is called the *marked disk* corresponding to a.

Let γ be the Jordan curve on the boundary of E_1 so that γ separates a from the interior of E_1, then we can choose the homeomorphism ϕ_1 such that $\phi_1(\gamma)$ is the hyperbolic circle in W_a with center $\phi_1(a)$ and pseudo-hyperbolic radius $\exp(-v(\gamma)M)$, where $v(\gamma)$ is defined in Lemma 6 and $M > 0$ is a constant.

Now we have defined embeddings ϕ_i for periodic components of V_{nt}. Suppose that E_{j_1} is a preperiodic component of V_{nt}. Let $k \geq 1$ be the minimal number so that

$$f^k(E_{j_1}^k) = f^{k-1}(E_{j_2}^{k-1}) = \cdots = f^1(E_{j_k}^1) = E_{j_{k+1}}$$

is periodic. Let $\phi_{j_{k+1}}$ be the homeomorphism of $\hat{\mathbb{C}}$ defined as above for periodic components of V_{nt}. Then there exists a homeomorphism ϕ_{j_1} of $\hat{\mathbb{C}}$ such that

$$\phi_{j_{k+1}} \circ h_{j_k} \circ \cdots h_{j_1} \circ \phi_{j_1}^{-1} = R_{j_1}$$

is a rational map. For each point $a \in P_{j_1} - E_{j_1}$, let W be the marked disk corresponding to the point $f^k(a)$. Then there exists a component W_0 of $R_{j_1}^{-1}(W)$ such that W_0 contains $\phi_{j_1}(a)$. We also call W_0 the *marked disk* corresponding to a.

Let γ be the Jordan curve on the boundary of E_{j_1} so that γ separates a from the interior of E_{j_1}, then we can choose the homeomorphism ϕ_{j_1}

such that $\phi_{j_1}(\gamma)$ is the hyperbolic circle in W_0 with center $\phi_{j_1}(a)$ and pseudo-hyperbolic radius $\exp(-v(\gamma)M)$.

Now let us embed other components of V into $\hat{\mathbb{C}}$. If E_i is a semi-essential component of V, define $\phi_i : E_i \to \hat{\mathbb{C}}$ by

$$\phi_i(E_i) = \{z : \exp(-v(\gamma_1)M - v(\gamma_2)M) < |z| < 1\}$$

where γ_1 and γ_2 are the curves on the boundary of E_i. If E_i is strictly-essential and peripheral, define ϕ_i a homeomorphism from E_i to be the unit disk with $\phi_i(a) = 0$, where a the only point in $E_i \cap P_F$.

Now we begin to construct θ_i. Suppose E_i is a component of V_{nt} with $E_j = f(E_i^1)$. Then there exists a homeomorphism ψ_i of $\hat{\mathbb{C}}$ which is homotopic to ϕ_i rel P_i such that $\phi_j \circ h_i \circ \psi_i^{-1} = R_i$ is a rational map. It is easy to see that for each essential curve γ_1 on the boundary of E_i^1, $\psi_i(\gamma_1)$ is a hyperbolic circle in a marked disk with pseudo-hyperbolic radius $\exp[-v(f(\gamma_1))M/\deg(f|_{\gamma_1})]$.

Suppose $E_k^1 \subset E_i$ is a component of V^1. Then E_k^1 is semi-essential. Let α_k, β_k be the essential curves on the boundary of E_k^1. If $f(E_k^1) = E_l$ is strictly-essential (then it must be non-peripheral), there exists a branched covering h_k of $\hat{\mathbb{C}}$ such that $h_k|_{E_k^1} = f|_{E_k^1}$, $\deg(h_k) = \deg(f|_{E_k^1})$, and there is at most one critical point in each component of $\hat{\mathbb{C}} - E_k^1$ whose image is contained in P_l. Let ψ_k be a homeomorphism of $\hat{\mathbb{C}}$ such that $\phi_l \circ h_k \circ \psi_k^{-1}$ is a rational map. By Lemma 4, the modulus of the ψ_k-image of the interior of the enclosure of E_k^1, denote by $\mathrm{mod}\psi_k(E_k^1)$ for the simplicity, satisfies the following inequality.

$$2\pi \mathrm{mod}\psi_k(E_k^1) \le \frac{v(f(\alpha_k))M}{\deg(f|_{\alpha_k})} + \frac{v(f(\beta_k))M}{\deg(f|_{\beta_k})} + C$$

where $C > 0$ is a constant independent of M.

If $f(E_k^1) = E_l$ is semi-essential, let $\psi_k : E_k^1 \to \hat{\mathbb{C}}$ be an embedding such that $\phi_l \circ f\psi_k^{-1}$ is holomorphic, then the modulus of the ψ_k-image of the interior of the enclosure of E_k^1 is

$$2\pi \mathrm{mod}\psi_i(E_k^1) = \frac{v(f(\alpha_k)) + v(f(\beta_k))}{\deg(f|_{E_k^1})}M = \frac{v(f(\alpha_k))M}{\deg(f|_{\alpha_k})} + \frac{v(f(\beta_k))M}{\deg(f|_{\beta_k})}.$$

Suppose γ is a Jordan curve on the boundary of E_i. Let γ_1 be a Jordan curve on the boundary of E_i^1. By Lemma 6, when $M > 0$ is large enough,

$$\sum 2\pi \mathrm{mod}\psi_k(E_k^1) + \frac{v(f(\gamma_1))}{\deg(f|_{\gamma_1})}M < v(\gamma)M,$$

where the sum is taken over all the semi-essential components of V' which are contained in the component of $E_i - E_i^1$ which attached to E_i^1 over γ_1. Therefore there exists a homeomorphism θ_i of E_i which is homotopic the identity rel $P_f \cap E_i$ such that $\phi_i \circ \theta_i|_{E_i^1} = \psi_i|_{E_i^1}$ and such that $\phi_i \circ \theta_i \circ \psi_k^{-1}$ is holomorphic in the interior of $\psi_k(E_k^1)$. Hence $\phi_l \circ f(\phi_i \circ \theta_i)^{-1}$ is holomorphic in the interior of $\phi_i \circ \theta_i(E_k^1)$.

If E_i is semi-essential, the same argument also guarantees the existence of θ_i. If E_i is strictly-essential and peripheral, since the modulus of $\phi_i(E_i - P_f)$ is infinity, there is no obstruction for the construction of θ_i. $\square$

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Guizhen Cui, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, P. R. China
Email: gzcui@math08.math.ac.cn
Yunping Jiang, Department of Mathematics, Queens College and Graduate Center of CUNY, Flushing, NY 11367, USA and the 100 Man Program in the Academy of Mathematics and System Sciences, Beijing 100080, P. R. China
Email: yunqc@forbin.qc.edu
Dennis Sullivan, Department of Mathematics, CUNY Graduate School, 365 Fifth Ave., New York, NY 10016, USA
Email: dsullivan@gc.cuny.edu