Spectral theory of transfer operators*

Aihua Fan† and Yunping Jiang‡

Abstract

We give a survey on some recent developments in the spectral theory of transfer operators, also called Ruelle-Perron-Frobenius (RPF) operators, associated to expanding and mixing dynamical systems. Different methods for spectral study are presented. Topics include maximal eigenvalue of RPF operators, smooth invariant measures, ergodic theory for chain of markovian projections, equilibrium states, spectral gaps for RPF operators, spectral decomposition and perturbation theory, central limit theorem, Hilbert metric and convergence speeds of RPF operators, and dynamical determinants and zeta functions.

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1 Introduction.

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous map. This gives us a topological dynamical system $\{f^n\}_{n=0}^{\infty}$. We simply call f a dynamical system on X. Let $\psi : X \to \mathbb{C}$ be a function. For the given f and ψ, we can define an operator $\mathcal{L} = \mathcal{L}_{f,\psi}$ as

$$\mathcal{L}\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x)$$

for ϕ in a suitable function space on X. The operator we just defined is called a *transfer operator*. If ψ is positive, i.e., $\psi(x) > 0$ for all x in X, then the operator is a positive operator, this means that it maps a positive function to a positive function. A positive transfer operator is also called a Ruelle-Perron-Frobenius (RPF) operator.

When the given dynamical system f is the one-side shift on a symbolic space of finite type and when the given function ψ is positive and Hölder continuous, Ruelle proved in [Ru1, Ru2] (see also [Bo]) that the RPF operator acting on the Hölder continuous function space has a unique maximal positive eigenvalue ρ with a positive eigenfunction. He used this result when completing a mathematical understanding of the existence and uniqueness of the equilibrium measure (or the so-called Gibbs measure). Ruelle's theorem represents an important result in modern thermodynamical formalism. Since then RPF operators have become a standard tool in many different areas in dynamical systems and other branches of mathematics and mathematical physics as well. Walters [Wa] proved Ruelle's theorem in more general settings. In Walters' paper, a dynamical system can be any expansive and mixing map and a function can be a positive summable variational function. There are many textbooks and articles about Ruelle's theorem. We give a partial list [Ba1, Bo, Fa, FJ1, FJ2, FS, Ji, PP, Vi] in the literature.

Ruelle's theorem consists of two parts. The first part concerns the existence and simplicity of a unique maximal eigenvalue for an RPF operator acting on a Hölder continuous function space. The second part concerns the existence and uniqueness of the Gibbs measure. The proof of the second part is based on the first part. A geometric proof of the first part for a given expanding and mixing dynamical system on a compact metric space and for a given positive Hölder continuous function is given by Ferrero and Schmitt in [FS], where a remarkable point is that the Hilbert projective metrics introduced by Birkhoff in [Bi] are useful. In fact, Ferrero and Schmitt showed that the operator contracts the Hilbert projective metric on the convex cone of positive functions in the space of Hölder continuous functions. Therefore, the existence and simplicity of a maximal eigenvalue follows directly from the contracting fixed point theorem. In [Ji], we found an elementary but elegant proof of the first part of Ruelle's theorem. In [Fa], we gave a new probabilistic proof of the second part of Ruelle's theorem for symbolic dynamical systems. By combining the ideas in [Ji] and in [Fa] (see also [DF]), we give a proof of Ruelle's theorem in [FJ1, FJ2] for general dynamical systems with Dini continuous potentials. Furthermore, we studied in [FJ1, FJ2] the convergence speed of a RPF operators under general setting.

Ruelle's theorem becomes more and more important in the study of modern dynamical systems like searching Sinai-Bowen-Ruelle measures for deterministic and stochastic dynamical systems. In this article, we give a survey on some recent development on the spectral theory of transfer operators. Section 2 to Section 6 present a discussion of Ruelle's theorem. In Section 2, we discuss some geometry of an expanding and mixing dynamical system on a compact metric space. In Section 3, we present an elementary but elegant proof of the first part in Ruelle's theorem, that is, the existence and uniqueness of a maximal eigenvalue for a RPF operator. The study of the maximal eigenvalue for a RPF operator is not only beneficial for itself, it is also beneficial for the study of many other subjects. In Section 4, we give an application of the first part of Ruelle's theorem to Krzyzewski-Szlenk's Theorem [KS] which is about the existence and uniqueness of a smooth invariant measure for a $C^{1+\alpha}$ expanding and mixing dynamical system on a compact C^2 Riemannian manifold. In Sections 5 and 6, we present the theory of chains of markovian projections and apply it to give a proof of the existence and uniqueness of the Gibbs distribution for a given expanding and mixing dynamical system acting on a general metric space. Section 7 is devoted to the spectral decomposition of a bounded operator. In Section 8, we discuss the relation of the maximal eigenvalue and the rest of spectra for a RPF operator. This relation will be further discussed in Section 10 by a new method called Hilbert metric. The existence of a gap between the maximal eigenvalue and the rest of spectra for a RPF operator provides many important properties for the underline dynamical systems, two of which are discussed in Section 9 and Section 10, one is the central limit theorem and the other one is the convergence speed. Another topic closely related to transfer operators is dynamical determinants and dynamical zeta functions. In Sections 11-13, we give an introduction to this topic. The reader who is interested in this topic may refer to [Ba2, BL, Fr, PP, Rg, Ru3] for further study.

2 Geometry of expanding and mixing dynamical systems

Let X be a compact metric space with metric d. Let $f : X \to X$ be a continuous map. The iterates $\{f^n\}_{n=0}^{\infty}$ of f give rise to a dynamical system on X. We simply call f a dynamical system.

For each $n \geq 0$, we define d_n as

$$d_n(x, y) = \max_{0 \leq i \leq n} \{d(f^i(x), f^i(y))\}$$

and call it the *n-Bowen metric*. We also call

$$B_n(x, r) = \{y \in X \; ; \; d_n(x, y) < r\}$$

the *n-Bowen ball* centered at x of radius $r > 0$. The 0-Bowen metric and a 0-Bowen ball are just the original metric d and a ball for d. The Bowen metrics, which describe how orbits are close, will be very useful for studying the dynamics.

The dynamical system f is said to be *locally expanding* if there are constants $\lambda > 1$ and $b > 0$ such that

$$d(f(x), f(x')) \geq \lambda d(x, x'), \quad x, x' \in X \text{ with } d(x, x') \leq b.$$

The couple (λ, b) is called a *primary expanding parameter*. We say f is *mixing* if for any open set U of X, there is an integer $n > 0$ such that $f^n(U) = X$.

Remark 2.1. *In general one can define locally expanding as that there are three constants $C, b > 0$ and $\lambda > 1$ such that*

$$d(f^n(x), f^n(x')) \geq C\lambda^n d(x, x')$$

for any $x, x' \in X$ with $d_n(x, x') \leq b$.

Suppose f is locally expanding and mixing with a primary expanding constant (λ, b). We are going to present some geometric properties for such a dynamics f, as stated in the following propositions.

Proposition 2.1. *The map f is locally homeomorphic. More precisely,*

$$f : B(x, b) \to f(B(x, b))$$

is homeomorphic for any $x \in X$.

Proof. It is clear that $f|\overline{B(x, b)}$ is injective. Since f is continuous on the closed ball $\overline{B(x, b)}$ and since the closed ball is compact, the inverse of $f|\overline{B(x, b)}$ is also continuous. But $f : B(x, b) \to f(B(x, b))$ is bijective, so it is homeomorphic. □

Proposition 2.2. *There is a constant $0 < a < b$ such that for any $y \in X$ with $f^{-1}(y) = \{x_1, \cdots, x_n\}$, there are local inverses $g_1, \cdots, g_n$ of f defined on $\overline{B(y, a)}$ such that $g_i(y) = x_i$ and $\{g_i(\overline{B(y, a)})\}_{i=1}^n$ are pairwise disjoint. Moreover, there is a constant integer $n_0 \geq 1$ such that $\#(f^{-1}(y)) \leq n_0$ for all $y \in X$.*

Proof. First $\#(f^{-1}(y))$ is finite for each $y \in X$ because otherwise f would not be homeomorphic around a limit point of $f^{-1}(y)$. Let

$$d(y) = \inf_{1 \leq k \neq j \leq n} d(x_k, x_j)$$

be the shortest distance between the preimages of y. It is clear that $d(y) > b$. For $0 < r \leq b/2$, $f : B(x_i, r) \to f(B(x_i, r))$ is homeomorphic for each $1 \leq i \leq n$. Since the open set $\cap_{i=1}^n f(B(x_i, r))$ contains y, it must contain a sufficiently small closed ball $\overline{B(y, r_y)}$ with $r_y > 0$ such that the inverse g_{iy} mapping y to x_i satisfies

$$g_{iy} : \overline{B(y, r_y)} \to g_{iy}(\overline{B(y, r_y)}) \subset B(x_i, r).$$

Since $B(x_i, r)$ are disjoint, $g_{iy}(\overline{B(y, r_y)})$ are disjoint. Now take a finite number of balls $\{B(y_j, r_{y_j})\}$ such that $\{B(y_j, r_{y_j}/2)\}$ form a cover of X. Let

$$a = \frac{1}{2} \min_j \{r_{y_j}\}.$$

Then it satisfies the proposition. In fact, for any $y \in X$, we have $y \in B(y_j, r_{y_j}/2)$ for some j. Then $\overline{B(y,a)} \subseteq \overline{B(y_j, r_{y_j})}$. Let

$$g_i = g_{iy_j}|\overline{B(y_j, a)}, \quad 1 \le i \le n.$$

Then $g_1, \cdots, g_n$ are local inverses of f on $\overline{B(y,a)}$ such that $g_i(y) = x_i$ and $\{g_i(\overline{B(y,a)})\}_{i=1}^n$ are pairwise disjoint

Since $\#(f^{-1}(y))$ is a locally constant function of y. It is then bounded because of the compactness of X, that means that we have a constant integer $n_0 \ge 1$ such that $\#(f^{-1}(y)) \le n_0$ for all $y \in X$. □

We call (λ, a) where a is a number in Proposition 2.2 an *expanding parameter* for f. Note that any (λ', a') where $0 < a' < a$ and $1 < \lambda' < \lambda$ is also an expanding parameter. In this chapter (λ, a) will always mean an expanding parameter for f. For any $0 < r \le a$ and $y \in X$, let g be an inverse branch of f on $B(y,r)$. Let $x = g(y)$. For any $z, w \in B(y,r)$,

$$d(g(z), g(w)) \le \frac{1}{\lambda} d(z,w).$$

This implies that

$$g(B(y,r)) \subseteq B(x, \frac{r}{\lambda}).$$

Moreover, $B_1(x,r) = g(B(y,r))$ and $f : B_1(x,r) \to B(y,r)$ is homeomorphic. Furthermore, we have

Proposition 2.3. *For any $0 < r \le a$ and $x \in X$, the map*

$$f^n : B_n(x,r) \to B(f^n(x), r)$$

is homeomorphic.

Proof. Let $x, f(x), \cdots, f^n(x)$ be a finite orbit of f. Then we have the n local inverses, $h_1, \cdots, h_n$, of f,such that

$$x \xleftarrow{h_1} f(x) \xleftarrow{h_2} f^2(x) \xleftarrow{h_3} \cdots \xleftarrow{h_{n-1}} f^{n-1}(x) \xleftarrow{h_n} f^n(x).$$

From the argument before this proposition, we know that

$$h_n : B(f^n(x), r) \to B_1(f^{n-1}(x), r)$$

is homeomorphic. This implies that

$$h_n(B(f^n(x), r)) = B_1(f^{n-1}(x), r).$$

Suppose we already know that

$$h_{n-k+1} \circ \cdots \circ h_n(B(f^n(x), r)) = B_k(f^{n-k}(x), r)$$

for $1 \leq k < n-1$. Then

$$h_{n-k} \circ \cdots \circ h_n(B(f^n(x), r)) = h_{n-k}(B_k(f^{n-k}(x), r)).$$

This implies that $z \in h_{n-k}(B_k(f^{n-k}(x), r))$ if and only if $f(z) \in B_k(f^{n-k}(x), r)$ and if and only if

$$d(f^i(f(z)), f^i(n-k(x))) < r, \quad 0 \leq i \leq k.$$

So we have

$$d(f(z), f^{n-k}(x)) < r, \cdots, d(f^{k+1}(z), f^n(x)) < r.$$

This combining with the fact that

$$d(z, f^{n-k-1}(x)) \leq \frac{1}{\lambda} d(f(z), f^{n-k}(x)) < \frac{r}{\lambda}$$

gives us that

$$h_{n-k} \circ \cdots \circ h_n(B(f^n(x), r)) = B_{k+1}(f^{n-k-1}(x), r).$$

Now the induction implies

$$h_1 \circ h_2 \circ \cdots \circ h_n(B(f^n(x), r)) = B_n(x, r),$$

that is $f^n(B_n(x, r)) = B(f^n(x), r)$. Because f^n on $B_n(x, r)$ is also injective,

$$f^n : B_n(x, r) \to B(f^n(x), r)$$

is homeomorphic. □

Proposition 2.4. *For any $0 < r \leq a$, there is an integer $p = p(r) \geq 1$ such that $f^p(B(x, r)) = X$ for any $x \in X$.*

Proof. Let $\{B(y_i, r/2)\}$ be a finite ball cover of X. For each i, there is an integer $p_i = p(y_i) > 0$ such that

$$f^{p_i}(B(y_i, \frac{r}{2})) = X$$

Let

$$p = \max_i \{p_i\}.$$

For any $y \in X$, we have $y \in B(y_i, \frac{r}{2})$ for some i. Then

$$B(y,r) \supset B(y_i, \frac{r}{2})$$

and

$$f^p(B(y,r)) \supseteq f^{p-p_i}(f^{p_i}(B(y_i, \frac{r}{2}))) \supseteq f^{p-p_i}(X) = X.$$

□

Proposition 2.5. *For any $0 < r \leq a$, let $p = p(r)$ be the integer in Proposition 2.4 and let n_0 be the integer in Proposition 2.2, then*

$$1 \leq \#(f^{-(n+p)}(y) \cap B_n(x,r)) \leq n_0^p$$

for any $x, y \in X$ and any $n \geq 1$.

Proof. Since $f^n : B_n(x,r) \to B(f^n(x), r)$ is a homeomorphism,

$$f^{n+p}(B_n(x,r)) = f^p(B(f^n(x),r)) = X.$$

This implies that $f^{-(n+p)}(y) \cap B_n(x,r) \neq \emptyset$. On the other hand, $\#(f^{-p}(y)) \leq n_0^p$ and every $z \in f^{-p}(y) \cap B(f^n(x), r)$ has exact one preimage in $B_n(x,r)$ under f^n. So

$$1 \leq \#(f^{-(n+p)}(y) \cap B_n(x,r)) \leq n_0^p.$$

□

3 Maximal eigenvalues of RPF operators

Usually, one proves the existence of positive eigenfunction of a RPF operator by using the Schauder fixed point theorem. In this section, we present a direct proof just basing on the Ascoli-Arzela theorem.

Suppose f is a locally expanding and mixing map with an expanding parameter (λ, a). Let $\mathbb{R}$ denote the real line. Let $\mathcal{C}^0 = \mathcal{C}^0(X, \mathbb{R})$ be the space of all continuous functions $\phi : X \to \mathbb{R}$ with the supremum norm

$$||\phi|| = \max_{x \in X} |\phi(x)|.$$

Let $0 < \alpha \leq 1$ and let $\mathcal{C}^\alpha = \mathcal{C}^\alpha(X, \mathbb{R})$ be the space of all α-Hölder continuous functions ϕ in $\mathcal{C}^0$, that is, $\phi \in \mathcal{C}^0$ satisfying

$$[\phi]_\alpha = \sup_{0 < d(x,y) \leq a} \frac{|\phi(x) - \phi(y)|}{d(x,y)^\alpha} < \infty,$$

where $[\phi]_\alpha$ is called the local Hölder constant for ϕ. For two functions ϕ_1 and ϕ_2, we write $\phi_1 \geq \phi_2$ if $\phi_1(x) \geq \phi_2(x)$ for all x in X. A function ϕ is called *positive* if $\phi > 0$. A positive function in $\mathcal{C}^\alpha$ is called a *potential.*

For two constants $K, s > 0$, define

$$\mathcal{C}^\alpha_{K,s} = \mathcal{C}^\alpha_{K,s}(X, \mathbb{R}) = \{\phi \in \mathcal{C}^\alpha \ ; \ \phi \geq s, \ [\log \phi]_\alpha \leq K\}.$$

Recall that $[\log \phi]_\alpha \leq K$ means that

$$\phi(x) \leq \phi(y) e^{Kd(x,y)^\alpha}.$$

The following lemma is a consequence of Ascoli-Arzela Theorem.

Lemma 3.1. *Any bounded sequence in $\mathcal{C}^\alpha_{K,s}$ has a convergent subsequence in $\mathcal{C}^0$ whose limit is in $\mathcal{C}^\alpha_{K,s}$.*

Let ψ be a potential. The *Ruelle-Perron-Frobenius (RPF) operator* with weight ψ is defined as

$$\mathcal{L}\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x).$$

It is clear that $\mathcal{L}(\mathcal{C}^0) \subseteq \mathcal{C}^0$ and

$$\mathcal{L} : \mathcal{C}^0 \to \mathcal{C}^0$$

is a bounded linear operator. Moreover for any $\phi \in \mathcal{C}^\alpha$, consider $y, y' \in X$ with $d(y, y') \leq a$. Let $\{x_1, \cdots, x_k\} = f^{-1}(y)$ and $\{x'_1, \cdots, x'_k\} = f^{-1}(y')$ be the corresponding inverse images of y and y' such that $d(x_i, x'_i) \leq \lambda^{-1} d(y, y')$ for all $1 \leq i \leq k$. Then

$$\begin{aligned} |\mathcal{L}\phi(y) - \mathcal{L}\phi(y)| &= |\sum_{i=1}^k \psi(x_i)\phi(x_i) - \sum_{i=1}^k \psi(x'_i)\phi(x'_i)| \\ &= \Big| \sum_{i=1}^k \Big(\psi(x_i)(\phi(x_i) - \phi(x'_i)) + \phi(x'_i)(\psi(x'_i) - \psi(x_i)) \Big) \Big|. \end{aligned}$$

So

$$[\mathcal{L}\phi]_\alpha \leq \frac{n_0}{\lambda^\alpha}(||\psi||[\phi]_\alpha + ||\phi||[\psi]_\alpha) < \infty.$$

Thus $\mathcal{L}(\mathcal{C}^\alpha) \subseteq \mathcal{C}^\alpha$ and

$$\mathcal{L} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$$

is a bounded linear operator. It is well known that being a positive operator on $\mathcal{C}^0$ (i.e. $\mathcal{L}\phi \geq 0$ when $\phi \geq 0$), $\mathcal{L}$ has its spectral radius as a spectral point. We will prove that it is actually an eigenvalue of $\mathcal{L} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ and there is a corresponding strictly positive eigenfunction.

For the purpose of the study of eigenvalues of $\mathcal{L}$, we can normalize the weight ψ such that $\min_{x\in X} \psi(x) = 1$. Henceforth, we will always assume that ψ is a normalized element in $\mathcal{C}^\alpha_{K_0,1}$ for some constant $K_0 > 0$ in the rest of this section. Let $0 < s < 1$ and $K > K_0/(\lambda^\alpha - 1) > 0$ be two fixed constants.

Lemma 3.2. *For any $\phi \geq 0$ in $\mathcal{C}^\alpha$ with $||\phi|| = 1$, there is an integer $N = N(\phi) > 0$ such that*

$$\mathcal{L}^n\phi \in \mathcal{C}^\alpha_{K,s}, \quad \forall n \geq N.$$

Proof. Since $||\phi|| = 1$, there is a point y in X such that $\phi(y) = 1$. We thus have a neighborhood U of y such that $\phi(y') > s$ for all y' in U. Since f is mixing, there is an integer $n_1 > 0$ such that $f^n(U) = X$ for all $n \geq n_1$. Therefore for any z in X, $f^{-n}(z) \cap U$ is non-empty for all $n \geq n_1$. Thus $\mathcal{L}^n\phi(z) \geq s$.

For any y and y' in X with $d(y, y') \leq a$, let $\{x_1, \cdots, x_k\} = f^{-1}(y)$ and $\{x'_1, \cdots, x'_k\} = f^{-1}(y')$ be the corresponding inverse images of y and y' such that $d(x_i, x'_i) \leq \lambda^{-1}d(y, y')$ for all $1 \leq i \leq k$. Let

$$K' = [\log \mathcal{L}^{n_1}\phi]_\alpha.$$

Then

$$\begin{aligned}
\mathcal{L}(\mathcal{L}^{n_1}\phi)(y') &= \sum_{i=1}^{k} \psi(x'_i)\mathcal{L}^{n_1}\phi(x'_i) \\
&\leq \sum_{i=1}^{k} \psi(x_i) \exp\left(K_0 d(x_i, x'_i)^\alpha\right)\mathcal{L}^{n_1}\phi(x_i) \exp\left(K' d(x_i, x'_i)^\alpha\right) \\
&\leq \exp\left((K_0 + K')\lambda^{-\alpha} d(y, y')^\alpha\right)\mathcal{L}(\mathcal{L}^{n_1}\phi)(y)
\end{aligned}$$

for all y and y' in X with $d(y, y') \leq a$. Inductively, for

$$K_n = K_0\left(\sum_{i=1}^{n} \lambda^{-\alpha i}\right) + K'\lambda^{-\alpha n},$$

we have

$$\mathcal{L}^n(\mathcal{L}^{n_1}\phi)(y') \leq \exp\left(K_n d(y, y')^\alpha\right)\mathcal{L}^n(\mathcal{L}^{n_1}\phi)(y)$$

for all y and y' in X with $d(y, y') \leq a$. It is clearly that K_n tends to $K_0/(\lambda^\alpha - 1)$ as n goes to infinity. So there is an integer $n_2 > 0$ such that for any $n \geq n_2$

$$\mathcal{L}^n(\mathcal{L}^{n_1}\phi)(y') \leq \exp\left(K d(y, y')^\alpha\right)\mathcal{L}^n(\mathcal{L}^{n_1}\phi)(y)$$

for all y and y' in X with $d(y, y') \leq a$. Then $N = n_1 + n_2$ satisfies the lemma. □

Lemma 3.2 implies that if $t > 0$ is an eigenvalue of $\mathcal{L} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ with a non-zero eigenfunction $\phi \geq 0$. Then this eigenfunction must be in $\mathcal{C}^\alpha_{K,s}$. Therefore, we are led to find positive eigenvalues of $\mathcal{L} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ with non-negative eigenfunctions in $\mathcal{C}^\alpha_{K,s}$.

From the proof of Lemma 3.2, we have seen that

$$\mathcal{L}(\mathcal{C}^\alpha_{K,s}) \subseteq \mathcal{C}^\alpha_{K,s}$$

because $(K_0 + K)\lambda^{-\alpha} < K$ for $K > K_0/(\lambda^\alpha - 1)$. Define

$$S = \{t \in \mathbb{R}\ ;\ t > 0, \text{there is a } \phi \text{ in } \mathcal{C}^\alpha_{K,s} \text{ such that } \mathcal{L}\phi \geq t\phi\}$$

Lemma 3.3. *The set S is a nonempty bounded subset on the real line $\mathbb{R}$*

Proof. Take a function ϕ in $\mathcal{C}^\alpha_{K,s}$. Then for any x and y in X,

$$\mathcal{L}\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x) = \Big(\sum_{x \in f^{-1}(y)} \frac{\phi(x)}{\phi(y)}\psi(x)\Big)\phi(y) \geq \frac{s}{||\phi||}\phi(y).$$

Thus $s/||\phi|| \in S$. So, S is nonempty. Let

$$m = \sup_{y \in X} \sum_{x \in f^{-1}(y)} \psi(x).$$

For any ϕ in $\mathcal{C}^\alpha_{K,s}$, let $\phi(y) = ||\phi||$. Then

$$\mathcal{L}\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x) \leq \phi(y) \sum_{x \in f^{-1}(y)} \psi(x) \leq m\phi(y)$$

Therefore, any $t > m$ will not be in S. Thus S is bounded. □

Theorem 3.1 (Ruelle). *The linear operator $\mathcal{L} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ has a unique maximal positive eigenvalue whose corresponding eigenspace has dimension one.*

Proof. Take $\delta = \sup S > 0$. Then there is a sequence $\{t_n\}_{n=1}^{\infty}$ in S converging to δ. Let ϕ_n be a corresponding functions in $\mathcal{C}^{\alpha}_{K,s}$ such that $\mathcal{L}\phi_n \geq t_n\phi_n$. Let us normalize ϕ_n with $\min_{x\in X}\{\phi_n(x)\} = s$. Then $\{\phi_n\}_{n=1}^{\infty}$ is a bounded sequence in $C^{\alpha}_{K,s}$. From Lemma 3.1, $\{\phi_n\}_{n=1}^{\infty}$ has a convergent subsequence in $\mathcal{C}^0$ whose limit is in $\mathcal{C}^{\alpha}_{K,s}$. Let us assume, without loss of generality, that $\{\phi_n\}_{n=1}^{\infty}$ itself converges to ϕ_0. Then

$$\mathcal{L}\phi_0 \geq \delta\phi_0.$$

We now show that $\mathcal{L}\phi_0 = \delta\phi_0$. Suppose there is a point y in X such that

$$\mathcal{L}\phi_0(y) > \delta\phi_0(y).$$

Then there is a neighborhood U of y such that

$$\mathcal{L}\phi_0(y') - \delta\phi_0(y') > 0$$

for all y' in U. Since f is mixing, there is an integer $n > 0$ such that $f^n(U) = X$. Then

$$\mathcal{L}^n(\mathcal{L}\phi_0 - \delta\phi_0) > 0,$$

that is,

$$\mathcal{L}(\mathcal{L}^n\phi_0) > \delta\mathcal{L}^n\phi_0.$$

Therefore for $\phi = \mathcal{L}^n\phi_0$, we have a $t > \delta$ such that $\mathcal{L}\phi \geq t\phi$. This contradicts to the maximal property of δ. This proved that

$$\mathcal{L}\phi_0 = \delta\phi_0.$$

Now let us show that δ is simple, i.e., the eigenspace

$$E_\delta = \{\phi \in C^\alpha;\ \mathcal{L}\phi = \delta\phi\}$$

has dimension one. Suppose ϕ is any function in E_δ. Let

$$a = \min_{x\in X} \frac{\phi(x)}{\phi_0(x)}$$

and

$$\phi_1 = \phi - a\phi_0.$$

Then ϕ_1 is in E_δ and $\phi_1 \geq 0$. Moreover, there is a point y in X such that $\phi_1(y) = 0$. Then $\phi_1(x) = 0$ for all x in $f^{-1}(y)$ because

$$\mathcal{L}\phi_1(y) = \sum_{x\in f^{-1}(y)} \phi(x) = 0.$$

Inductively, we have $\phi_1 = 0$ on $X_y = \cup_{n=0}^{\infty} f^{-n}(y)$. Since f is mixing, X_y is a dense subset in X. So $\phi_1 \equiv 0$ on X, that is, $\phi \equiv a\phi_0$.

The reminder is to prove that δ is the biggest eigenvalue but it is easy as follows. Suppose $t \neq \delta$ is an eigenvalue of $\mathcal{L} : \mathcal{C}^{\alpha} \to \mathcal{C}^{\alpha}$. Then there is a non-zero function ϕ in $\mathcal{C}^{\alpha}$ with $||\phi|| = 1$ such that $\mathcal{L}\phi = t\phi$. So $\mathcal{L}|\phi| \geq |t||\phi|$. There is an integer $N > 0$ such that $\mathcal{L}^N|\phi|$ is in $\mathcal{C}^{\alpha}_{K,s}$ and also $\mathcal{L}(\mathcal{L}^N|\phi|) \geq |t|\mathcal{L}^N|\phi|$. Thus $|t|$ is a number in S, so $|t| \leq \delta$. If $|t| < \delta$, then we have nothing to prove. If $|t| = \delta$, by using the mixing property as we did in the previous two paragraphs, we have $|\phi| = a\phi_0$ for some $a > 0$. This implies $\phi = \pm a\phi_0$ and $t = \delta$. $\square$

4 Application: existence of smooth invariant measure

As an application of Theorem 3.1, we are going to prove the existence of smooth invariant measure for differentiable expanding dynamical systems. This result is due to Krzyzewski-Szlenk.

Suppose M is an m-dimensional compact C^2 Riemannian manifold where $m \geq 1$ is an integer. Let $f : M \to M$ be a C^1 map. We say f is $C^{1+\alpha}$ for $0 < \alpha \leq 1$ if the determinant $J(f)$ of the Jacobi matrix $Jac(f)$ of f is an α-Hölder continuous function defined on M. A probability measure ν on M is called f-invariant if $\nu(f^{-1}(A)) = \nu(A)$ for all Lebesgue measurable subsets A of M. Let dy denote the Lebesgue measure on M. A probability measure ν is called a *smooth measure* if there is a continuous function ρ defined on M such that

$$\nu(A) = \int_A \rho(y)dy$$

for all Lebesgue measurable sets A in M. The function ρ is called the density function of the smooth measure ν.

Lemma 4.1. *Suppose $f : M \to M$ is a C^1 map such that $J(f)(y) \neq 0$ for all y in M and suppose ν is a smooth probability measure with $\nu = \int \rho dy$. Then ν is a f-invariant measure if and only if*

$$\sum_{x \in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} = \rho(y)$$

for all y in M.

Proof. Since $J(f)(y) \neq 0$ for all y in M there is a constant $a_1 > 0$ such that for any connected domain U with diameter less than or equal $2a_1$, f on each component V of $f^{-1}(U)$ is injective and has the local inverse $g : V \to U$ such that $fg = identity$. If ν is a f-invariant measure, then $\nu(f^{-1}(U)) = \nu(U)$ for all Lebesgue measurable subsets U of M. In particular, take U as the ball of centered y and radius $0 < \epsilon < a_1$ and denote $V_1, \cdots, V_k$ as the components of $f^{-1}(U)$. Then

$$\sum_{i=1}^{k} \int_{V_i} \rho(x)dx = \int_U \rho(y)dy.$$

By the mean value theorem and let ϵ tend to zero, we get

$$\sum_{x \in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} = \rho(y).$$

Now assume

$$\sum_{x \in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} = \rho(y)$$

for all y in M. For any ball U with radius a_1 , let $V_1, \cdots, V_k$ be the components of $f^{-1}(U)$. Then f on each V_i is injective and has the local inverse. So

$$\begin{aligned}
\nu(U) &= \int_U \rho(y)dy = \int_U \sum_{x \in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} dy \\
&= \sum_{x \in f^{-1}(y)} \int_U \frac{\rho(x)}{J(f)(x)} dy = \sum_{i=1}^{k} \int_{V_i} \rho(x)dx \\
&= \sum_{i=1}^{k} \nu(V_i) = \nu(f^{-1}(U)).
\end{aligned}$$

□

Theorem 4.1 (Krzyzewski-Szlenk). *Suppose $f : M \to M$ is a $C^{1+\alpha}$ locally expanding map for some $0 < \alpha \leq 1$. Then f has a unique smooth f-invariant probability measure with an α-Hölder continuous density function.*

Proof. Since f is C^1 and locally expanding, $J(f)(y) \neq 0$ for all y in M. Let

$$||J(f)|| = \max_{y \in M} J(f)(y) \qquad \text{and} \qquad \psi = \frac{||J(f)||}{J(f)}.$$

Then ψ is an function in $C^{\alpha}_{K_0,1}$ for some $K_0 > 0$. Consider the RPF operator with weight ψ

$$\mathcal{L}\phi(y) = \sum_{x\in f^{-1}(y)} \psi(x)\phi(x).$$

Theorem 3.1 implies that there is a unique maximal positive eigenvalue δ with a positive eigenfunction in C^{α} and the corresponding eigenspace is one-dimensional. Let ρ be the one in the eigenspace normalized by $\int_M \rho dy = 1$. Then $\mathcal{L}\rho(y) = \delta\rho(y)$, that is,

$$\sum_{x\in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} = \delta_0 \rho(y),$$

for $\delta_0 = \delta/||J(f)||$. Integrate on both sides of the last equation, we have that the right hand side is δ_0. Now let us calculate the left hand side. Cut M into path connected pieces $M_1, \cdots, M_n$ such that

1. $M = M_1 \cup M_2 \cup \cdots \cup M_n$,
2. the Lebesgue measure of each $M_i \cap M_j$ is zero for $i \neq j$,
3. f on each component of $f^{-1}(M_i)$ is injective, $1 \leq i \leq n$.

Let M_i^j, $1 \leq j \leq k_i$ be the components of $f^{-1}(M_i)$ for $1 \leq i \leq n$. Then $M = \cup_{i=1}^n \cup_{j=1}^{k_i} M_i^j$ and the Lebesgue measure of each $M_i^j \cap M_{i'}^{j'}$ is zero for $i \neq i'$. Let us use x_{ij} to denote the point in $f^{-1}(y) \cap M_i^j$ for any $y \in M_i$, where $1 \leq i \leq n$ and $1 \leq j \leq k_i$. Therefore,

$$\begin{aligned}
\int_M \sum_{x\in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} dy &= \sum_{i=1}^n \int_{M_i} \sum_{x\in f^{-1}(y)} \frac{\rho(x)}{J(f)(x)} dy \\
&= \sum_{i=1}^n \sum_{j=1}^{k_i} \int_{M_i} \frac{\rho(x_{ij})}{J(f)(x_{ij})} dy \\
&= \sum_{i=1}^n \sum_{j=1}^{k_i} \int_{M_i^j} \rho(x_{ij}) dx_{ij} \\
&= \int_M \rho(y) dy = 1.
\end{aligned}$$

So we have that $\delta_0 = 1$ (that is, $\delta = ||J(f)||$) and that $\nu = \int \rho dy$ is a smooth f-invariant measure following Lemma 4.1.

Uniqueness follows the fact that if $\nu = \int \rho dy$ is a smooth f-invariant measure, than ρ is in the eigenspace of $\mathcal{L}$ with respect to the eigenvalue δ. □

5 Chains of markovian projections

In this section we present the theory of chains of markovian projections and G-measure theory [DF]. This theory includes RPF operators as a special case.

Let X be a compact Hausdorff space. Let $\mathcal{F}_X$ be the standard σ-algebra generated by all open sets in X. Let $\mathcal{C}^0 = \mathcal{C}(X, \mathbb{R})$ still be the space of all real valued continuous functions on X, equipped with the supremum norm $||\cdot||$. Let $\mathcal{M} = \mathcal{M}(X) = (\mathcal{C}^0)^*$ be the dual space of $\mathcal{C}^0$. By Riesz representation theorem, $\mathcal{M}$ is identical to the space of all measures on X with respect to $\mathcal{F}_X$. Let $\mathcal{M}_0 \subset \mathcal{M}$ be the family of all probability measures. We use

$$< \mu, \phi >= \int_X \phi \, d\mu$$

to mean the integral of a function ϕ with respect to a measure μ in $\mathcal{M}$.

Definition 5.1. *A linear map $P : \mathcal{C}^0 \to \mathcal{C}^0$ is said to be a* projection *if $P^2 = P$. It is said to be* markovian *if $P1 = 1$ and if $P\phi \geq 0$ whenever $\phi \geq 0$.*

Let us first give some basic properties about projections and markovian projections. Let $\mathcal{K}er(P) = \{\phi \in \mathcal{C}^0;\ P\phi = 0\}$ and let $\mathcal{I}m(P) = P(\mathcal{C}^0)$.

Proposition 5.1. *Let P and Q are two projections. Then*

(1) $\mathcal{C}^0 = \mathcal{K}er(P) \bigoplus \mathcal{I}m(P)$.

(2) $\phi \in \mathcal{I}m(P)$ *if and only if* $P\phi = \phi$.

(3) $PQ = Q$ *if and only if* $\mathcal{I}m(Q) \subseteq \mathcal{I}m(P)$.

(4) $QP = Q$ *if and only if* $\mathcal{K}er(P) \subseteq \mathcal{K}er(Q)$.

Proof. (1) means that for each $\phi \in \mathcal{C}^0$ we can write $\phi = \phi' + \phi''$ in a unique way with $\phi' \in \mathcal{K}er(P)$ and $\phi'' \in \mathcal{I}m(P)$. This is true because $\phi = (\phi - P\phi) + P\phi$. (2) is a consequence of (1). Suppose $\phi \in \mathcal{I}m(Q)$ and $PQ = Q$. By (2), we have $Q\phi = \phi$. So $P\phi = PQ\phi = Q\phi = \phi$, i.e., $\phi \in \mathcal{I}m(P)$. Conversely, for any $\phi \in \mathcal{C}^0$, there is another map $\phi' \in \mathcal{C}^0$ such that $Q\phi = P\phi'$. Then $PQ\phi = P^2\phi' = P\phi' = Q\phi$. For (4), suppose $QP = Q$ and $\phi \in \mathcal{K}er(P)$. Then it is clear that $Q\phi = 0$. Conversely, for $\phi \in \mathcal{C}^0$ we decompose it as $\phi = (\phi - P\phi) + P\phi$. By assumption, $\phi - P\phi \in \mathcal{K}er(Q)$, so $Q\phi = QP\phi$. □

For an operator $P : \mathcal{C}^0 \to \mathcal{C}^0$, let $P^* : \mathcal{M} \to \mathcal{M}$ be its dual operator. The proof of the next proposition is not hard. The reader may do it as an exercise.

Proposition 5.2. *Let P and Q be markovian projection on $\mathcal{C}^0$. We have*

1) $||P|| = ||P^*|| = 1$.

2) $P^{*2} = P^*$.

3) $P^*(\mathcal{M}_0) \subseteq \mathcal{M}_0$.

4) $PQ = Q$ *if and only if* $Q^*P^* = Q^*$.

A sequence of markovian projections $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ defined on $\mathcal{C}^0$ is called a *chain of markovian projections* (abbreviated as CMP) if it satisfies

$$P_m P_n = P_m, \qquad 1 \leq n \leq m.$$

For such a chain, define

$$\mathcal{G}_n = \{\mu \in \mathcal{M}_0 \ ; \ P_n^* \mu = \mu\}, \quad 1 \leq n < \infty$$

Because $\mathcal{M}_0$ is a weakly compact convex subset of $\mathcal{M}$ and $P_n^* : \mathcal{M}_0 \to \mathcal{M}_0$, from the Schauder-Tychonoff theorem (see [DS]), $\mathcal{G}_n \neq \emptyset$. Let

$$\mathcal{G}_\infty = \cap_{n=1}^{\infty} \mathcal{G}_n.$$

If $\mathcal{G}_\infty$ is non-empty, then every element in it is called a *G-measure* with respect to $\mathcal{P}$. If $\mathcal{G}_\infty$ contains only one element, then we call the given CMP *uniquely ergodic*.

Theorem 5.1. *For any CMP $\mathcal{P}$, $\mathcal{G}_\infty \neq \emptyset$. Actually, for any sequence $\{\mu_n\}_{n=1}^{\infty}$ in $\mathcal{M}_0$, any weak limit of $\{P_n^*\mu_n\}_{n=1}^{\infty}$ is an element in $\mathcal{G}_\infty$.*

Proof. Since $||P_n^*\mu_n|| = 1$, there is a weak limit of $\{P_n^*\mu_n\}_{n=1}^{\infty}$. Let ν be such a weak limit. Then there is a subsequence $P_{n_i}^*\mu_{n_i}$ tends to ν weakly as i goes to infinity. This implies that for any $\phi \in \mathcal{C}^0$,

$$\lim_{i\to\infty} < P_n^* P_{n_i}^* \mu_{n_i}, \phi >= \lim_{i\to\infty} < P_{n_i}^* \mu_{n_i}, P_n\phi >=< \nu, P_n\phi >=< P_n^*\nu, \phi > .$$

On the other hand,

$$< P_n^*\nu, \phi >= \lim_{i\to\infty} < P_{n_i}^*\mu_{n_i}, P_n\phi >= \lim_{i\to\infty} < \mu_{n_i}, P_{n_i}P_n\phi >$$

$$= \lim_{i\to\infty} < \mu_{n_i}, P_{n_i}\phi >= \lim_{i\to\infty} < P_{n_i}^*\mu_{n_i}, \phi >=< \nu, \phi > .$$

So $P_n^*\nu = \nu$ for all $n \geq 1$, that is, $\nu \in \mathcal{G}_\infty$. □

Theorem 5.1 says that $\mathcal{G}_\infty$ is weakly compact. It is clear that $\mathcal{G}_\infty$ is convex, i.e., $t\mu_1 + (1-t)\mu_2 \in \mathcal{G}_\infty$ if μ_1, $\mu_2 \in \mathcal{G}_\infty$ and $0 \le t \le 1$.

Theorem 5.2. *Suppose $\mathcal{P}$ is a CMP defined on $\mathcal{C}^0$. Then the following are equivalent.*

(1) *The CMP is unique ergodic.*

(2) *for every $\phi \in \mathcal{C}^0$, $P_n\phi$ converges uniformly on X to a constant.*

(3) *for every $\phi \in \mathcal{C}^0$, $P_n\phi$ converges pointwise on X to a constant.*

Proof. It is clear (2) implies (3). Assume (3), for any $\mu \in \mathcal{G}_\infty$, the constant is $< \mu, \phi >$ because by the Lebesgue theorem

$$< \mu, \phi >=< P_n^*\mu, \phi >= \lim_{n\to\infty} < \mu, P_n\phi >=< \mu, \lim_{n\to\infty} P_n\phi >= \lim_{n\to\infty} P_n\phi.$$

Therefore, for any $\mu, \nu \in \mathcal{G}_\infty$ and $\phi \in \mathcal{C}^0$,

$$< \mu, \phi >=< \nu, \phi >= \lim_{n\to\infty} P_n\phi.$$

It implies that $\mu = \nu$, so (1) holds. Now assume (1) and suppose μ is the unique element in $\mathcal{G}_\infty$. Suppose (2) is false. There exists $\phi \in \mathcal{C}^0$ such that $P_n\phi$ does not converge to $< \mu, \phi >$ uniformly. So we have a constant $\epsilon > 0$ and a subsequence $\{n_i\}_{i=1}^\infty$ of integers and a sequence of points $\{x_i\}_{i=1}^\infty$ such that

$$|P_{n_i}\phi(x_i)- < \mu, \phi > | \ge \epsilon$$

for all $i \ge 1$. Let δ_{x_i} be the Dirac measure concentrating at x_i. Then

$$P_{n_i}\phi(x_i) =< P_{n_i}^*\delta_{x_i}, \phi > .$$

By Theorem 5.1, any weak limit ν of $\{P_{n_i}^*\delta_{x_i}\}_{i=1}^\infty$ is in $\mathcal{G}_\infty$. But

$$| < \nu, \phi > - < \mu, \phi > | \ge \epsilon.$$

It contradicts with the assumption of (1). □

A CMP $\mathcal{P} = \{P_n\}_{n=1}^\infty$ is said to be *compatible* if it satisfies

(a) $P_n(\phi\chi) = \chi P_n\phi$ if $\chi \in \mathcal{I}mP_n$.

(b) $P_nP_m = P_m(= P_mP_n)$ if $m \ge n$.

Suppose $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ is a compatible CMP. Let $\mathcal{F}_0$ be the standard σ-algebra on $\mathbb{R}$ generated by all open sets. For a function $\phi : X \to \mathbb{R}$, let $\mathcal{F}_\phi = \phi^{-1}(\mathcal{F}_0)$ be the pull-back σ-algebra on X. Given a family of functions Γ, we use $\mathcal{F}_\Gamma$ to mean the minimal σ-algebra containing all σ-algebras $\mathcal{F}_\phi$ for $\phi \in \Gamma$. Let $\mathcal{F}_n = \mathcal{F}_{\mathcal{I}mP_n}$ for all $n \geq 1$. Then $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sub-σ-algebras in $\mathcal{F}_X$, i.e.,

$$\cdots \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_X.$$

This is because from Proposition 5.1

$$\cdots \subseteq \mathcal{I}mP_{n+1} \subseteq \mathcal{I}mP_n \subseteq \cdots \subseteq \mathcal{I}mP_1 \subseteq \mathcal{C}^0.$$

Let $\mathcal{F}_\infty$ be the σ-algebra generated by the limit of $\{\mathcal{F}_n\}_{n=1}^{\infty}$ which is defined as

$$\mathcal{F}_\infty = \cup_{n=1}^{\infty} \cap_{m\geq n} \mathcal{F}_m (= \cap_{n\geq 1} \cup_{m\geq n} \mathcal{F}_m).$$

A G-measure μ is called $\mathcal{P}$-ergodic if $\mu|\mathcal{F}_\infty$ is trivial, i.e., $\mu(A) = 0$ or 1 for any $A \in \mathcal{F}_\infty$.

Let $\mu \in \mathcal{M}_0$ be a probability measure. For any $\phi \in \mathcal{C}^0$ and $n \geq 1$, we have a measure μ_n defined on the sub-σ-algebra $\mathcal{F}_n$ by

$$\mu_n(A) = \int_A \phi d\mu, \qquad A \in \mathcal{F}_n.$$

It is clear that μ_n is absolutely continuous with respect to $\mu|\mathcal{F}_n$. By the Radon-Nikodym theorem there exists a unique (modulo sets of measure zero) $L^1(X, \mathcal{F}_n, \mu)$-function denoted by $E(\phi|\mathcal{F}_n)$ and called the conditional expectation of ϕ given $\mathcal{F}_n$, such that

$$\mu_n(A) = \int_A E(\phi|\mathcal{F}_n) d\mu, \qquad A \in \mathcal{F}_n.$$

The function is defined uniquely a.e. by

i $\int_A E(\phi|\mathcal{F}_n)d\mu = \int_A \phi d\mu$, $A \in \mathcal{F}_n$,

ii $E(\phi|\mathcal{F}_n) \in L^1(X, \mathcal{F}_n, \mu)$.

The operator $E(\cdot|\mathcal{F}_n)$ enjoys the following properties:

iii For all $\phi \in L^1(X, \mathcal{F}_X, \mu)$ and $\phi' \in L^\infty(X, \mathcal{F}_n, \mu)$,

$$E(\phi\phi'|\mathcal{F}_n) = \phi' E(\phi|\mathcal{F}_n).$$

The proof of the following theorem can be found in [Pa, pp. 30]

Theorem 5.3 (Decreasing martingale theorem). *If*

$$\cdots \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subseteq \mathcal{F}_X$$

is a decreasing sequence of sub-σ-algebras such that $\cap_{n=1}^{\infty}\mathcal{F}_n = \mathcal{F}_\infty$, then $E(\phi|\mathcal{F}_n) \to E(\phi|\mathcal{F}_\infty)$ a.e. and in $L^1(X, \mathcal{F}_X, \mu)$ when $\phi \in L^1(X, \mathcal{F}_X, \mu)$.

Now assume that $\mu \in \mathcal{G}_\infty$ is a G-measure for $\mathcal{P}$. Then we have that for any $\phi' \in \mathcal{I}mP_n$,

$$< \mu, \phi' P_n \phi > = < \mu, P_n(\phi\phi') > = < P_n^* \mu, \phi\phi' > = < \mu, \phi\phi' >,$$

so

$$P_n\phi = E(\phi|\mathcal{F}_n), \qquad \mu - \text{a.e.}.$$

Following Theorem 5.3, the limit of $P_n\phi$ exists μ-a.e. and, furthermore, a $\mathcal{P}$-invariant measure μ is $\mathcal{P}$-ergodic iff $\lim_{n\to\infty} P_n\phi = < \mu, \phi >$ μ-a.e. for any $\phi \in \mathcal{C}^0$ (see [Pa, pp. 21]). Then we have the following classical ergodicity theorem.

Theorem 5.4. *Suppose $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ is a compatible CMP.*

1. *If μ_1 and μ_2 in $\mathcal{G}_\infty$ are $\mathcal{P}$-ergodic, then either $\mu_1 = \mu_2$ or $\mu_1 \perp \mu_2$.*
2. *$\mu \in \mathcal{G}_\infty$ is $\mathcal{P}$-ergodic iff μ is an extremal point in $\mathcal{G}_\infty$.*

Proof. (1) Suppose $\mu_1 \neq \mu_2$. There exists $\phi \in \mathcal{C}^0$ such that

$$< \mu_1, \phi > \neq < \mu_2, \phi > .$$

Define

$$A_1 = \{x \in X; \lim_{n\to\infty} P_n\phi = < \mu_1, \phi >\} \text{ and } A_2 = \{x \in X; \lim_{n\to\infty} P_n\phi = < \mu_2, \phi >\}.$$

We have $\mu_1(A_1) = 1$ and $\mu_2(A_1) = 0$, and $\mu_1(A_2) = 0$ and $\mu_2(A_2) = 1$. This implies that $\mu_1 \perp \mu_2$.

(2) Suppose $\mu \in \mathcal{G}_\infty$ is $\mathcal{P}$-ergodic and $\mu = t\mu_1 + (1-t)\mu_2$ with μ_1, μ_2 in $\mathcal{G}_\infty$ and $0 < t < 1$. For any $A \in \mathcal{F}_\infty$, $\mu(A) = 0$ or 1 because of the ergodicity of μ. Then $\mu_1(A) = \mu_2(A) = 0$ or 1 because $0 < t < 1$. That means that μ_1 and μ_2 are also $\mathcal{P}$-ergodic. According to (1), if $\mu_1 \neq \mu_2$ we can find $A \in \mathcal{F}_\infty$ with $\mu_1(A) = 1$ and $\mu_2(A) = 0$. Consequently

$\mu(A) = t$. This contradicts the ergodicity of μ. Therefore $\mu_1 = \mu_2$ and μ is extremal.

Conversely, suppose $\mu \in \mathcal{G}_\infty$ is not $\mathcal{P}$-ergodic. Let $A \in \mathcal{F}_\infty$ such that $0 < t = \mu(A) < 1$. Define

$$\mu_1 = \frac{1}{t}\mu 1_A, \qquad \mu_2 = \frac{1}{1-t}\mu 1_{X\setminus A}.$$

We are now going to show μ_1, $\mu_2 \in \mathcal{G}_\infty$. For any $n \geq 1$

$$< P_n^* \mu_1, \phi >=< \mu, \frac{1}{t} 1_A P_n \phi >, \qquad \phi \in \mathcal{C}^0.$$

Since $A \in \mathcal{F}_\infty \subseteq \mathcal{F}_n$,

$$\begin{aligned} < \mu, \frac{1}{t} 1_A P_n \phi > &= \frac{1}{t} \sup < \mu, \phi' P_n \phi >= \frac{1}{t} \sup < P_n^* \mu, \phi\phi' > \\ &= \frac{1}{t} \sup < \mu, \phi'\phi >=< \frac{1}{t} 1_B \mu, \phi > \end{aligned}$$

where the supremum is taken over $\{\phi' \leq 1_A, \phi' \in \mathcal{I}m P_n\}$. This implies that $P_n^* \mu_1 = \mu_1$ for all $n \geq 1$, that is, $\mu \in \mathcal{G}_\infty$. Similarly, we can prove that $\mu_2 \in \mathcal{G}_\infty$. But

$$\mu = t\mu_1 + (1-t)\mu_2, \quad 0 < \alpha < 1$$

which implies that μ is not an extremal point. □

The space $\mathcal{M}$ is a locally convex topological space and metrizable. So $\mathcal{G}_\infty$ is a compact metrizable convex subset in $\mathcal{M}$. Let $\mathcal{E}\mathcal{G}_\infty$ be the set of $\mathcal{P}$-ergodic μ in $\mathcal{G}_\infty$. Then the above theorem says that $\mathcal{E}\mathcal{G}_\infty$ consists of all extremal points in $\mathcal{G}_\infty$. The relation between $\mathcal{G}_\infty$ and $\mathcal{E}\mathcal{G}_\infty$ can be obtained from the Choquet representation theorem (see [OR, pp. 1-32] for the proof).

Theorem 5.5 (Choquet representation theorem). *For each $\mu \in \mathcal{G}_\infty$, there exists a Borel probability measure m on $\mathcal{G}_\infty$, supported on the set $\mathcal{E}\mathcal{G}_\infty$ of extremal points, so that*

$$\mu = \int_{\mathcal{G}_\infty} \nu dm(\nu), \quad \mu \in \mathcal{G}_\infty.$$

6 Gibbs distributions

Suppose that a physical system of n states $\{1, 2, \cdots, n\}$ with their energies $E_1, \cdots, E_n$ and that the system is put in contact with a much larger "heat source" which is at temperature T. Energy is therefore allowed to pass between the original system and the heat source. Suppose the temperature T of the heat source remains constant. It is a physical fact derived in statistical mechanics that the probability p_j that state j occurs is given by the Gibbs distribution

$$p_j = \frac{e^{-\beta E_j}}{\sum_{i=1}^{n} e^{-\beta E_i}}$$

where $\beta = \frac{1}{kT}$ and k is a physical constant. This is the starting point for the thermodynamical formalism. In this section, we combine Theorem 3.1 and the theory of chains of markovian projections (Section 5) to develop a thermodynamical formalism for more general systems. We will keep use the same notations as those in Sections 2, 3 and 5.

Suppose f is a locally expanding and mixing dynamical system with an expanding parameter (λ, a) and suppose $\psi > 0 \in \mathcal{C}^\alpha$, $0 < \alpha \leq 1$, is a potential. Define

$$G_n(x) = \prod_{i=0}^{n-1} \psi(f^i(x)), \quad x \in X, \ n \geq 1.$$

Let

$$\mathcal{L}\phi(y) = \sum_{x \in f^{-1}(y)} \psi(x)\phi(x)$$

be the RPF operator with weight ψ. Let $\delta > 0$ be the maximal eigenvalue and $h > 0 \in \mathcal{C}^\alpha$ be a corresponding eigenvector of $\mathcal{L}$. Let $\mathcal{L}^* : \mathcal{M} \to \mathcal{M}$ be the dual operator of $\mathcal{L}$.

Theorem 6.1 (Ruelle). *There is a unique probability measure $\nu = \nu_\psi \in \mathcal{M}_0$ such that $\mathcal{L}^*\nu = \delta\nu$ and for any $0 < r \leq a/2$, there is a constant $C = C(r) > 0$ such that*

$$C^{-1} \leq \frac{\nu(B_n(x,r))}{\delta^{-n} G_n(x)} \leq C \qquad \textit{(Gibbs Property)}$$

for all $x \in X$ and $n \geq 1$. And moreover, take h such that $\int_X h \, d\nu = 1$. Then for any $\phi \in \mathcal{C}^0$,

$$\lim_{n\to\infty} \delta^{-n} \mathcal{L}^n \phi = <\nu, \phi> h$$

uniformly.

The inequality in the theorem is called the *Gibbs Property* and the probability measure $\mu = h\nu$ is called the *Gibbs measure* for (f, ψ). To prove Theorem 6.1, we first normalize the operator $\mathcal{L}$. Take

$$\tilde{\psi}(x) = \frac{h(x)}{\delta h(f(x))}\psi(x).$$

Then it is still a positive function in $\mathcal{C}^\alpha$. So we can consider the RPF operator

$$\tilde{\mathcal{L}}\phi(x) = \mathcal{L}_{\tilde{\psi}}\phi(x) = \sum_{y\in f^{-1}(x)} \tilde{\psi}(y)\phi(y).$$

The important feature of $\tilde{\mathcal{L}}$ is that

$$\tilde{\mathcal{L}}1 = 1.$$

We call it a normalized RPF operator. Let $\tilde{\mathcal{L}}^*$ be the dual operator of $\tilde{\mathcal{L}}$ acting on the space $\mathcal{M}$. Let

$$\tilde{G}_n(x) = \prod_{i=0}^{n-1} \tilde{\psi}(f^i(x)), \quad x \in X,\ n \geq 1.$$

Then we have

$$\tilde{G}_n = \frac{h}{\delta^n h \circ f^n} G_n$$

and relations between $\mathcal{L}$ and $\tilde{\mathcal{L}}$ and between $\mathcal{L}^*$ and $\tilde{\mathcal{L}}^*$,

$$\mathcal{L}^n\phi = \delta^n h\tilde{\mathcal{L}}^n(\phi h^{-1}) \qquad \text{and} \qquad \mathcal{L}^{*n}\nu = \delta^n h^{-1}\tilde{\mathcal{L}}^{*n}(h\nu)$$

From these relations, Theorem 6.1 follows from the following statement for normalized RPF operators.

Theorem 6.2. *Suppose $\mathcal{L}$ is a normalized RPF operator, i.e., $\mathcal{L}1 = 1$. Then there is a unique probability measure $\mu \in \mathcal{M}_0$ such that $\mathcal{L}^*\mu = \mu$ and for any $0 < r \leq a/2$, there is a constant $C = C(r) > 0$ such that the Gibbs property holds, i.e.,*

$$C^{-1} \leq \frac{\mu(B_n(x, r))}{G_n(x)} \leq C \qquad \textit{(Gibbs Property)}$$

for all $x \in X$ and $n \geq 1$. Moreover, for any $\phi \in \mathcal{C}^0$

$$\lim_{n\to\infty} \mathcal{L}^n\phi = <\mu, \phi>$$

Suppose $\mathcal{L}$ is normalized in the rest of this section. Define

$$P_n\phi(x) = \mathcal{L}^n\phi(f^n(x)) = \sum_{y\in f^{-n}(f^n(x))} G_n(y)\phi(y).$$

It is a linear operator from $\mathcal{C}^0$ into itself with $P_n 1 = 1$ and $P_n\phi > 0$ whenever $\phi > 0$. Moreover, we have the following facts.

Lemma 6.1. *For any $m \geq n \geq 1$, $P_nP_m = P_mP_n = P_n$.*

Proof. Let us show that $P_mP_n = P_m$.

$$\begin{aligned}
P_mP_n\phi(x) &= \sum_{y\in f^{-m}(f^m(x))} G_m(y)P_n\phi(y) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))}\sum_{y\in f^{-n}(w)} G_{m-n}(w)G_n(y)P_n\phi(y) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))}\sum_{y\in f^{-n}(w)} G_{m-n}(w)G_n(y)\sum_{z\in f^{-n}(f^n(y))} G_n(z)\phi(z) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))}\sum_{y\in f^{-n}(w)} G_n(y)\sum_{z\in f^{-n}(w)} G_{m-n}(w)G_n(z)\phi(z) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))}\Big(\sum_{y\in f^{-n}(w)} G_n(y)\Big)\Big(\sum_{z\in f^{-n}(w)} G_m(z)\phi(z)\Big) \\
&= \sum_{w\in f^{-(m-n)}(f^m(x))}\sum_{z\in f^{-n}(w)} G_m(z)\phi(z) \\
&= \sum_{z\in f^{-m}(f^m(x))} G_m(z)\phi(z) \\
&= P_m\phi(x).
\end{aligned}$$

We use the fact that $\sum_{y\in f^{-n}(w)} G_n(y) = 1$. This also implies that P_n is a projection, i.e., $P_n^2 = P_n$. Similar arguments imply that $P_nP_m = P_m$. □

Lemma 6.2. *For any $\phi \in \mathcal{C}^0$ and $\chi \in \mathcal{I}mP_n$, $P_n(\phi\chi) = \chi P_n\phi$.*

Proof. Suppose $\chi(x) = \sum_{y\in f^{-n}(f^n(x))} G_n(y)\beta(y)$. Then

$$\begin{aligned}
P_n(\phi\chi)(x) &= \sum_{z\in f^{-n}(f^n(x))} G_n(z)\phi(z)\sum_{y\in f^{-n}(f^n(z))} G_n(y)\beta(y) \\
&= \sum_{y\in f^{-n}(f^n(x))}\sum_{z\in f^{-n}(f^n(x))} G_n(y)G_n(z)\phi\beta
\end{aligned}$$

$$= \sum_{y\in f^{-n}(f^n(x))} G_n(y)\beta(y) \sum_{z\in f^{-n}(f^n(x))} G_n(z)\phi(z)$$
$$= \chi(x)P_n\phi(x).$$

□

Lemmas 6.1 and 6.2 say that $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ is a compatible CMP. So we can apply the theory of CMP in Section 5 to give a proof of Theorem 6.2.

Let P_n^* be the dual operator of P_n. Remember that $\mathcal{G}_\infty$ is the set of common fixed points of P_n^*'s and an element of $\mathcal{G}_\infty$ is called a *G-measure*. An element μ such that $\mathcal{L}^*\mu = \mu$ is called a *ψ-measure*. Because $\mathcal{L}(\phi\circ f) = \phi$, any ψ-measure μ is f-invariant, i.e., $\mu(f^{-1}(A)) = \mu(A)$ for any μ-measurable sets A. So a ψ-measure is a G-measure. Since $\mathcal{M}_0$ is a weakly compact convex subset of $\mathcal{M}$ and $\mathcal{L}^* : \mathcal{M}_0 \to \mathcal{M}_0$, by the Schauder-Tychonoff fixed point theorem there is at least one ψ-measure (and G-measure). Now we use the results in Section 5 to prove the uniqueness of G-measure.

Lemma 6.3. *For any ϕ in $\mathcal{C}^0$, $P_n\phi$ converges to a constant if and only if $\mathcal{L}^n\phi$ converges to the same constant. And moreover, the constant has to be $<\mu,\phi>$ for any ψ-measure (or G-measure) μ.*

Proof. Since

$$P_n\phi(x) = \sum_{y\in f^{-n}(f^n(x))} \psi(y)\phi(y) = (\mathcal{L}^n\phi)(f^n(x))$$

and $f: X\to X$ is surjective, it is clear that

$$\|P_n\phi(x) - c\| = \|\mathcal{L}^n\phi(x) - c\|.$$

Therefore, $P_n\phi$ converges to c if and only if $\mathcal{L}^n\phi$ converges to c. Suppose $P_n\phi$ converges to c. Then

$$c = \lim_{n\to\infty} P_n\phi = \lim_{n\to\infty} <\mu, P_n\phi> = \lim_{n\to\infty} <P_n^*\mu,\phi> = <\mu,\phi> .$$

□

Lemma 6.4 (Naive distortion lemma). *There is a constant $C > 0$ such that for any $n \geq 0$ and any $x, y \in X$ with $d_n(x,y) \leq a$,*

$$C^{-1} \leq \frac{G_n(x)}{G_n(y)} \leq C.$$

Proof. Let $x_i = f^i(x)$ and $y_i = f^i(y)$ for $0 \le i \le n$. Then

$$d(x_i, y_i) \le \lambda^{n-i} d(x_n, y_n).$$

So

$$\begin{aligned} |\log G_n(x) - \log G_n(y)| &\le \sum_{i=0}^{n-1} |\log \psi(x_i) - \log \psi(y_i)| \\ &\le \frac{[\psi]_\alpha}{A} \sum_{i=0}^{n-1} d(x_i, y_i)^\alpha \\ &\le \frac{[\psi]_\alpha}{A} \sum_{i=0}^{n-1} \lambda^{-\alpha(n-i)} d(x_n, y_n)^\alpha \\ &\le C_0 \end{aligned}$$

where $A = \min_{x \in X} \psi(x)$ and $[\psi]_\alpha$ is the Hölder constant for ψ, and

$$C_0 = \frac{[\psi]_\alpha a^\alpha \lambda^\alpha}{A(\lambda^\alpha - 1)}.$$

Therefore, let $C = e^{C_0}$, we have

$$C^{-1} \le \frac{G_n(x)}{G_n(y)} \le C.$$

□

Proof of Theorem 6.2. We prove the Gibbs Property first. Let μ be a G-measure. Let r be a real number such that $0 < 2r \le a$. For any $x \in X$, let ϕ be a function such that

$$1_{B_n(x,r)} \le \phi \le 1_{B_n(x,2r)}$$

where 1_B denotes the characteristic function of a set B. Then we have

$$\mu(B_n(x, r)) \le \int \phi \, d\mu = \int \phi \, dP_n^* \mu = \int P_n \phi \, d\mu$$

where

$$P_n \phi(y) = \sum_{z \in f^{-n}(f^n(y))} G_n(z) \phi(z) \le \sum_{z \in f^{-n}(f^n(y))} G_n(z) 1_{B_n(x,2r)}(z).$$

From Lemma 6.4 and Proposition 2.3, there is a constant $C > 0$ such that

$$\#(f^{-n}(f^n(y)) \cap B_n(x, 2r)) \le C$$

and

$$G_n(z) \le CG_n(x), \quad z \in B_n(x, 2r).$$

Thus we get

$$\mu(B_n(x, r)) \le C^2 G_n(x).$$

On the other hand, we have

$$\mu(B_n(x, 2r)) \ge \int \phi \, d\mu = \int \phi \, dP^*_{n+p}\mu \ge \int P_{n+p}\phi \, d\mu$$

where p is an integer in Proposition 2.4 in Section 2 and

$$\begin{aligned} P_{n+p}\phi(y) &= \sum_{z\in f^{-n-p}(f^{n+p}(y))} G_{n+p}(z)\phi(z) \\ &\ge \sum_{z\in f^{-n-p}(f^{n+p}(y))} G_{n+p}(z) 1_{B_n(x,r)}(z). \end{aligned}$$

Proposition 2.5 says that there is at least one term in the sum is non-zero. This and Lemma 6.4 imply that there is a positive constant, we still denote it as C, such that

$$\mu(B_n(x, 2r)) \ge CG_{n+p}(x) \ge CA^p G_n(x),$$

where $A = \min_{x\in X} \psi(x)$. Let s be the least integer such that $\lambda^s \ge 2$. Then we have $B_n(x, r) \supset B_{n+s}(x, \lambda^s r) \supset B_{n+s}(x, 2r)$. By the last inequality, we get

$$\mu(B_n(x, r)) \ge CA^{p+s} G_n(x).$$

Therefore, we have a positive constant depending on r only, which we still denote it as C, such that

$$C^{-1} \le \frac{\mu(B_n(x, r))}{G_n(x)} \le C.$$

Following Section 5 and the Gibbs Property, only remaining thing to be proven is that a G-measure is unique. From Theorem 5.5, we only need to prove that a $\mathcal{P}$-ergodic G-measure is unique. Theorem 5.4 says that any two $\mathcal{P}$-ergodic G-measures are either equal or totally singular. Now we use the Gibbs Property to prove that any two $\mathcal{P}$-ergodic G-measures

μ and ν are mutually absolutely continuous, that is, there is a constant $C > 0$ such that

$$C^{-1}\nu(U) \leq \mu(U) \leq C\nu(U)$$

for any open set U of X. Let us prove it as follows.

Fix a real number r, $0 < 2r \leq a$. Let $\{x_1, \cdots, x_m\}$ be a $2r$-net in (X, d), this means that the balls $\{B(x_i, r)\}_{1\leq i\leq m}$ are disjoint and the balls $\{B(x_i, 2r)\}_{1\leq i\leq m}$ form a cover of X. Define

$$\begin{aligned} A_1 &= B(x_1, 2r) \setminus (B(x_2, r) \cup \cdots \cup B(x_m, r)) \\ A_i &= B(x_i, 2r) \setminus (A_1 \cup \cdots \cup A_{i-1}), \quad 2 \leq i \leq m. \end{aligned}$$

Then we get a partition $\mathcal{Q}_0 = \{A_i\}_{i=1}^m$ of X satisfying

$$B(x_i, r) \subseteq A_i \subseteq B(x_i, 2r), \quad 1 \leq i \leq m.$$

For every $n \geq 1$ and every $1 \leq i \leq m$, denote $f^{-n}(x_i) = \{z_j\}_{j=1}^{k_{ni}}$. Let g_{jn} be the inverse of $f^n : B_n(z_j, 2r) \to B(x_i, 2r)$. Define $A_{nij} = g_{jn}(A_i)$. We call A_{nij} a n-component of $f^{-n}|\mathcal{Q}_0$. Let $\mathcal{Q}_n$ be the set of all n components of $f^{-n}|\mathcal{Q}_0$. It is again a partition of X and satisfies that for any $A \in \mathcal{Q}_n$,

$$B_n(c_A, r) \subseteq A \subseteq B_n(c_A, 2r)$$

where $c_A \in A$ such that $f^n(c_A) = x_j$. The point c_A is called the *center of* A. It is worth to note that for $n > k \geq 1$ and for any $A \in \mathcal{Q}_n$, $f^{(n-k)}(A) \in \mathcal{Q}_k$. However $\mathcal{Q}_k$ may not be a refinement of $\mathcal{Q}_n$. (So they are not Markov partitions.)

Let U be an arbitrary open set in X. For $n \geq 1$, let $\mathcal{Q}_n(U)$ be the family of all elements A of the partition $\mathcal{Q}_n$ such that the n-Bowen ball $B_n(c_A, r)$ is entirely contained in U. Let

$$V_n = \bigcup_{A\in\mathcal{Q}_n(U)} A.$$

This is a Borel subset of U which is a countable union of disjoints sets. From the Gibbs Property, we get

$$\begin{aligned} \mu(V_n) &= \sum_{A\in\mathcal{Q}_n(U)} \mu(A) \leq \sum_{A\in\mathcal{Q}_n(U)} \mu(B_n(c_A, 2r)) \\ &\leq C \sum_{A\in\mathcal{Q}_n(U)} G_n(c_A) \leq C^2 \sum_{A\in\mathcal{Q}_n(U)} \nu(B_n(c_A, r)) \\ &\leq C^2 \sum_{A\in\mathcal{Q}_n(U)} \nu(A) = C^2\nu(\bigcup_{A\in\mathcal{Q}_n(U)} A) = C^2\nu(V_n) \end{aligned}$$

Then we have $\mu(U) \le C^2\nu(U)$ by using Fatou lemma and the fact that

$$U = \liminf_{n\to\infty} V_n.$$

Similarly $\nu(U) \le C^2\mu(U)$. Therefore, a G-measure is unique.

Let μ be the unique G-measure. Then following Theorem 5.2, $P_n\phi \to <\mu, \phi>$ as $n \to \infty$ for any $\phi \in \mathcal{C}^0$. Therefore, $\mathcal{L}^n\phi \to <\mu, \phi>$ as $n \to \infty$ (Lemma 6.3). This completes the proof. □

7 Spectral gaps of RPF operators

It will be proved in this section that all the spectral points other than the maximal eigenvalue of $\mathcal{L}$ acting on α-Hölder continuous functions is contained in a disk centered at 0 of radius strictly smaller than the maximal eigenvalue. As we shall see, there are many consequences of this fact.

Let $\mathcal{L}$ be the RPF operator in the previous section. Recall that $\mathcal{C}^0_{\mathbb{C}} = \mathcal{C}^0(X, \mathbb{C})$ is the Banach space of all continuous complex valued functions $\phi : X \to \mathbb{C}$, equipped with the supremum norm

$$||\phi|| = \max_{x\in X} |\phi(x)|.$$

Let $0 < \alpha \le 1$ and let $\mathcal{C}^\alpha_{\mathbb{C}} = \mathcal{C}^\alpha(X, \mathbb{C})$ be the space of all α-Hölder complex valued continuous functions ϕ in $\mathcal{C}^0_{\mathbb{C}}$, that is, $\phi \in \mathcal{C}^0_{\mathbb{C}}$ satisfying

$$[\phi]_\alpha = \sup_{0<d(x,y)\le a} \frac{|\phi(x) - \phi(y)|}{d(x,y)^\alpha} < \infty,$$

where $[\phi]_\alpha$ is called the local Hölder constant for ϕ. For any $\phi \in \mathcal{C}^0_{\mathbb{C}}$, we can write it as

$$\phi = \phi_1 + i\phi_2, \quad \phi_1, \phi_2 \in \mathcal{C}^0.$$

Then ϕ is in $\mathcal{C}^\alpha_{\mathbb{C}}$ if and only if ϕ_1 and ϕ_2 are both in $\mathcal{C}^\alpha$. Since ψ is a real valued function, we have

$$\mathcal{L}\phi = \mathcal{L}\phi_1 + i\mathcal{L}\phi_2.$$

Thus we have that $\mathcal{L}_0 : \mathbb{C}^0_{\mathbb{C}} \to \mathbb{C}^0_{\mathbb{C}}$ is a bounded linear operator. The space $\mathcal{C}^\alpha_{\mathbb{C}}$ equipped with the norm

$$||\phi||_\alpha = ||\phi|| + [\phi]_\alpha$$

is a Banach space. Then $\mathcal{L}_\alpha = \mathcal{L} : \mathcal{C}^\alpha_{\mathbb{C}} \to \mathcal{C}^\alpha_{\mathbb{C}}$ is a bounded linear operator (refer to the argument between the statements of Lemma 3.1 and Lemma 3.2). The following is a directly consequence of Theorem 6.1.

Corollary 7.1. *The maximal eigenvalue δ of $\mathcal{L} : \mathcal{C}^0_{\mathbb{R}} \to \mathcal{C}^0_{\mathbb{R}}$ is the spectrum radius of*

$$\mathcal{L}_0 = \mathcal{L} : \mathcal{C}^0_{\mathbb{C}} \to \mathcal{C}^0_{\mathbb{C}}.$$

Proof. The spectrum radius can be calculated as

$$\rho(\mathcal{L}_0) = \limsup_{n\to\infty} ||\mathcal{L}_0^n||^{\frac{1}{n}}$$

where

$$||\mathcal{L}_0^n|| = \sup_{\phi\in\mathcal{C}^0_{\mathbb{C}}, ||\phi||\leq 1} ||\mathcal{L}_0^n \phi||.$$

For any $\phi \in \mathcal{C}^0_{\mathbb{C}}$ with $||\phi|| \leq 1$, from Theorem 6.1,

$$||\delta^{-n}\mathcal{L}_0^n\phi|| \leq | < \nu, \phi > | \cdot ||h|| + 1 \leq ||h|| + 1$$

for n large. So

$$\delta^{-n}||\mathcal{L}_0^n|| \leq ||h|| + 1$$

and

$$||\mathcal{L}_0^n||^{\frac{1}{n}} \leq (||h|| + 1)^{\frac{1}{n}}\delta$$

for n large. So $\rho(\mathcal{L}_0) \leq \delta$. But δ is a spectrum point, we have $\rho(\mathcal{L}_0) = \delta$. □

Furthermore, we have

Theorem 7.1. *The maximal eigenvalue δ is the spectrum radius of*

$$\mathcal{L}_\alpha = \mathcal{L} : \mathcal{C}^\alpha_{\mathbb{C}} \to \mathcal{C}^\alpha_{\mathbb{C}}.$$

The rest of spectrum is in a disk of center 0 with radius strictly less than δ.

This is a direct consequence of the relation between $\mathcal{L}$. More precisely, it follows from the next corollary which is a consequence of Theorem 6.2.

Corollary 7.2. *Suppose $\mathcal{L}$ is a normalized RPF operator, i.e., $\mathcal{L}1 = 1$. Then the maximal eigenvalue 1 is the spectrum radius of*

$$\mathcal{L}_\alpha = \mathcal{L} : \mathcal{C}^\alpha_{\mathbb{C}} \to \mathcal{C}^\alpha_{\mathbb{C}}.$$

The rest of spectrum is in a disk of center 0 with radius strictly less than 1.

Proof. The spectrum radius can be calculated as

$$\rho(\mathcal{L}_\alpha) = \limsup_{n\to\infty} ||\mathcal{L}_\alpha^n||_\alpha^{\frac{1}{n}},$$

where

$$||\mathcal{L}_\alpha^n||_\alpha = \sup_{\phi\in\mathcal{C}_\mathbb{C}^\alpha, ||\phi||_\alpha\le 1} ||\mathcal{L}_0^n\phi||_\alpha.$$

For any $\phi \in \mathcal{C}_\mathbb{C}^\alpha$ with $||\phi||_\alpha \le 1$ and $x, y \in X$ with $d(x,y) \le a$, let $f^{-1}(x) = \{x_1, \cdots, x_n\}$ and $f^{-1}(y) = \{y_1, \cdots, y_n\}$ such that $d(x_i, y_i) \le \frac{1}{\lambda d(x,y)}$. Then

$$\begin{aligned}
|\mathcal{L}_\alpha\phi(x) - \mathcal{L}_\alpha\phi(y)| &\le \sum_{i=1}^n |\psi(x_i)\phi(x_i) - \psi(y_i)\phi(y_i)| \\
&\le \sum_{i=1}^n |\psi(x_i) - \psi(y_i)||\phi(y_i)| + \sum_{i=1}^n \psi(x_i)|\phi(x_i) - \phi(y_i)|.
\end{aligned}$$

Since $d(x_i, y_i) \le \frac{1}{\lambda d(x,y)}$ and since $\sum_{i=1}^n \psi(x_i) = 1$, we have

$$[\mathcal{L}_\alpha\phi]_\alpha \le \frac{[\psi]_\alpha n_0}{\lambda^\alpha}||\phi|| + \frac{1}{\lambda^\alpha}[\phi]_\alpha.$$

It follows that

$$||\mathcal{L}_\alpha\phi||_\alpha \le C_1||\phi|| + \frac{1}{\lambda^\alpha}||\phi||_\alpha,$$

where $C_1 = 1 - 1/\lambda^\alpha + [\psi]_\alpha n_0/\lambda^\alpha$. Inductively, suppose

$$||\mathcal{L}_\alpha^{n-1}\phi||_\alpha \le C_{n-1}||\phi|| + \frac{1}{\lambda^{\alpha(n-1)}}||\phi||_\alpha.$$

Then

$$\begin{aligned}
||\mathcal{L}_\alpha^n\phi||_\alpha &\le C_{n-1}||\mathcal{L}_\alpha\phi|| + \frac{1}{\lambda^{\alpha(n-1)}}||\mathcal{L}_\alpha\phi||_\alpha \\
&\le C_{n-1}||\phi|| + \frac{1}{\lambda^{\alpha(n-1)}}C_1||\phi|| + \frac{1}{\lambda^{\alpha n}}||\phi||_\alpha \\
&= C_n||\phi|| + \frac{1}{\lambda^{\alpha n}}||\phi||_\alpha,
\end{aligned}$$

where $C_n = C_{n-1} + (1/\lambda^{\alpha(n-1)}C_1 \le C = C_1\lambda^\alpha/(\lambda^\alpha - 1)$. Therefore we have that

$$||\mathcal{L}_\alpha^n\phi||_\alpha \le C||\phi|| + \frac{1}{\lambda^{\alpha n}}||\phi||_\alpha$$

for all $n \geq 1$.

Let

$$\mathcal{C}_{\mathbb{C}}^{\alpha\perp} = \{\phi \in \mathcal{C}_{\mathbb{C}}^{\alpha} \; ; \; <\mu, \phi>= 0\}.$$

Then $\mathcal{C}_{\mathbb{C}}^{\alpha} = \mathcal{C}_{\mathbb{C}}^{\alpha\perp} \bigoplus \mathbb{C}$ because $\phi = (\phi - <\mu, \phi>) + <\mu, \phi>$. To prove the rest of spectrum of $\mathcal{L}_\alpha$ is in a disk of center 0 with radius less than 1, we only need to prove that the spectrum of

$$\mathcal{L}_\alpha | \mathcal{C}_{\mathbb{C}}^{\alpha\perp} : \mathcal{C}_{\mathbb{C}}^{\alpha\perp} \to \mathcal{C}_{\mathbb{C}}^{\alpha\perp}$$

is strictly less than 1. We prove this as follows.

Suppose $n, k > 0$. Then

$$\begin{aligned} ||\mathcal{L}_\alpha^{n+k}\phi||_\alpha &\leq C||\mathcal{L}_\alpha^k\phi|| + \frac{1}{\lambda^{\alpha n}}||\mathcal{L}_\alpha^k\phi||_\alpha \\ &\leq C||\mathcal{L}_\alpha^k\phi|| + C\frac{1}{\lambda^{\alpha n}}||\phi|| + \frac{1}{\lambda^{\alpha(n+k]}}||\phi||_\alpha. \end{aligned}$$

Since $\{\phi \in \mathcal{C}_{\mathbb{C}}^{\alpha\perp} \; ; \; ||\phi||_\alpha \leq 1\}$ is a compact set in $\mathcal{C}_{\mathbb{C}}^0$ (because it is a uniformly bounded and equicontinuous family), following Theorem 6.2, for any $0 < \tau < 1$ there are $m, k > 0$ such that

$$||\mathcal{L}_\alpha^{m+k}\phi||_\alpha \leq \tau$$

for all $\phi \in \mathcal{C}_{\mathbb{C}}^{\alpha\perp}$ with $||\phi||_\alpha \leq 1$. So $||\mathcal{L}_\alpha^{m+k}||_\alpha \leq \tau$. Therefore,

$$\limsup_{n\to\infty} ||\mathcal{L}^n|\mathcal{C}_{\mathbb{C}}^{\alpha\perp}||_\alpha^{\frac{1}{n}} \leq \tau^{\frac{1}{m+k}} < 1.$$

This completes the proof. □

In Section 10, we will prove an estimate on the gap between the maximal eigenvalue and the rest of spectrum of $\mathcal{L}_\alpha$ by using a new method (see Theorem 10.2 and Corollary 10.3).

8 Spectral decomposition and perturbation

In this section, we present two standard results on bounded linear operators. One is the spectral decomposition and the other is a perturbation theorem.

A *projection* on a Banach space $\mathcal{B}$ is a bounded linear operator P such that

$$P^2 = P.$$

It is clear that P is a projection if and only if $I - P$ is a projection. The space $\mathcal{B}$ is a direct-sum of two closed subspaces $\mathcal{B}_1$ and $\mathcal{B}_2$, written as $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are complementary in the following sense

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2, \qquad \mathcal{B}_1 \bigcap \mathcal{B}_2 = \emptyset.$$

To specify a projection P is essentially the same thing to specify a direct-sum decomposition $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$. In fact, if P is a projection, then

$$\mathcal{B} = R(P) \oplus N(P)$$

where $R(P)$ denotes the range of P and N(P) denotes the null space of P. Notice that $R(P)$ is closed because of the fact $P^2 = P$. So does $N(P) = R(1 - P)$.

If $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ is written into a direct-sum of two closed subspaces, then the operator $P : \mathcal{B} \to \mathcal{B}$ defined by

$$x = x_1 \oplus x_2 \to Px = x_1$$

is a projection having $\mathcal{B}_1$ as its range. Notice that P is continuous because of the closed graph theorem. We may speak of the *projection P onto $\mathcal{B}_1$ along $\mathcal{B}_2$*.

It is easy to see that a projection P commutes with a bounded operator A if and only if its range and null set are invariant under A, i.e.

$$AR(P) \subset R(P), \qquad AN(P) \subset N(P).$$

The main core of spectral decomposition is the following: to any splitting of the spectrum

$$\sigma(A) = S_1 \cup S_2$$

into two disjoint closed subsets, there corresponds a direct-sum splitting of the space

$$\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$$

into two closed invariant spaces, i.e.

$$A\mathcal{B}_1 \subset \mathcal{B}_1, \quad A\mathcal{B}_2 \subset \mathcal{B}_2$$

such that the spectra of the restrictions

$$A|_{\mathcal{B}_1} : \mathcal{B}_1 \to \mathcal{B}_1, \qquad A|_{\mathcal{B}_2} : \mathcal{B}_2 \to \mathcal{B}_2$$

are respectively S_1 and S_2. The splitting is not only invariant under A but also invariant under any operator which commutes with A. All these determine uniquely the splitting $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$.

The construction of the space splitting is trivial in the case of a diagonalizable operator in a finite-dimensional space: $\mathcal{B}_i$ is spanned by all eigenvectors with eigenvalues in S_i $(i = 1, 2)$. Unfortunately, this construction can not be generalized. However, the projection onto $\mathcal{B}_1$ along $\mathcal{B}_2$ can be written as follows

$$T = \frac{1}{2\pi i} \oint_\gamma (\lambda I - A)^{-1} d\lambda$$

where γ is a contour enclosing all points of the set S_1 but none of the points of S_2. In order to see that T is really the projection, we have only to verify

(i) $Tx = x$ for any eigenvectors with eigenvalue in S_1;

(ii) $Tx = 0$ for any eigenvectors with eigenvalue in S_2.

In fact, since A is diagonalizable, i.e. $A = QDQ^{-1}$ with D diagonal, we have

$$(\lambda I - A)^{-1} = Q(\lambda I - D)^{-1} Q^{-1}.$$

Now we may check (i) and (ii) by the Cauchy integral formula.

This representation of a projection in a space of finite dimension can be generalized to the general case as we will state in the following theorem.

Let γ be a cycle which is defined as a formal sum of closed paths in $\mathcal{C}$ and $z \in \mathbb{C}$ be a point not on γ. Define the winding number of γ around z as the integer

$$\frac{1}{2\pi i} \oint_\gamma \frac{d\lambda}{\lambda - z}.$$

Theorem 8.1. *Let A be a bounded linear operator on a Banach space $\mathcal{B}$. Let $\sigma(A) = S_1 \cup S_2$ be a decomposition of the spectrum of A into two disjoint closed subsets. For any cycle γ in the complement of the spectrum which winds once around each point of S_1 and zero times around each point of S_2, let*

$$P = \frac{1}{2\pi i} \oint_\gamma R(\lambda, A) d\lambda$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$, called the resolvent of A. Then

(1) The integral depends only on S_1 and S_2, not on γ;

(2) The operator P is a projection which commutes with every bounded operator commuting with A;

(3) The spectrum of $A|_{R(P)}$ is S_1 and the spectrum of $A|_{N(P)}$ is S_2;

(4) The properties of P listed above uniquely determines P.

Proof. (1) The independence of γ is a consequence of the Cauchy integral formula. By the way we point out that if S_2 is empty and γ is a large circle centered at the origin, for λ on γ we have the Neumann series

$$R(\lambda, A) = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}.$$

Integrating term by term gives $P = I$.

(2) Take another cycle γ' with the same properties as γ, such that γ' is inside γ in the sense that γ winds once around each point of γ' but γ' winds zero times around each point of γ. We may take γ' to lie in a small neighborhood of S_1. Then we have

$$\begin{aligned} P^2 &= \frac{1}{2\pi i}\oint_\gamma R(\lambda, A)d\lambda \cdot \frac{1}{2\pi i}\oint_{\gamma'} R(\lambda', A)d\lambda' \\ &= \frac{1}{(2\pi i)^2}\oint_\gamma\oint_{\gamma'} R(\lambda, A)R(\lambda', A)d\lambda d\lambda'. \end{aligned}$$

Applying the identity

$$U^{-1} - V^{-1} = U^{-1}(V - U)V^{-1}$$

to $U = \lambda I - A$ and $V = \lambda' I - A$ allows us to write

$$R(\lambda, A) - R(\lambda', A) = -(\lambda - \lambda')R(\lambda, A)R(\lambda', A).$$

So, we have

$$\begin{aligned} P^2 &= -\frac{1}{(2\pi i)^2}\oint_\gamma\oint_{\gamma'} \frac{R(\lambda, A) - R(\lambda', A)}{\lambda - \lambda'} d\lambda d\lambda' \\ &= \frac{1}{(2\pi i)^2}\oint_{\gamma'} R(\lambda', A)\oint_\gamma \frac{1}{\lambda - \lambda'} d\lambda d\lambda' - \frac{1}{(2\pi i)^2}\oint_\gamma R(\lambda, A)\oint_{\gamma'} \frac{1}{\lambda' - \lambda} d\lambda' d\lambda. \end{aligned}$$

Since γ' est inside γ, we have

$$\oint_\gamma \frac{1}{\lambda-\lambda'}d\lambda = 2\pi i, \qquad \oint_{\gamma'} \frac{1}{\lambda'-\lambda}d\lambda' = 0.$$

Finally we have $P^2 = P$, as desired. It is clear from the definition of P that P commutes with any bounded operator which commutes with A.

(3) In particular, P commutes with A, so the range and the null space of the projection P are invariant under A. Let $A_1 : R(P) \to R(P)$ be the restriction of A on $R(P)$ and A_2 the restriction of A on $N(P)$. The following inclusion is clear by the definition of spectrum

$$\sigma(A_1)\bigcup\sigma(A_2) \subset \sigma(A) = S = S_1\bigcup S_2.$$

We are going to prove $\sigma(A_1) \subset S_1$ by showing that $\lambda I - A_1$ is invertible for $\lambda \in S_2$. Actually we can construct its inverse explicitly as a contour integral:

$$B = \frac{1}{2\pi i}\oint_\gamma \frac{R(z,A)}{\lambda - z}dz$$

where γ as in the statement of the theorem. In fact, since B commutes with A, it also commutes with P. Hence

$$\begin{aligned}(\lambda I - A)B &= \frac{1}{2\pi i}\oint_\gamma \frac{(\lambda_z)I + (zI-A)}{\lambda - z}R(z,A)dz \\ &= \frac{1}{2\pi i}\oint_\gamma \left(R(z,A) + \frac{1}{\lambda - z}\right)dz \\ &= P + \frac{1}{2\pi i}\oint_\gamma \frac{1}{\lambda - z}dz \\ &= P.\end{aligned}$$

Thus if $Px = x$, $(\lambda I - A)Bx = x$. We also have $B(\lambda I - A)x = x$ because A and B commute. Thus the restriction of B on $R(P)$ is the inverse of the restriction on $R(P)$ of $\lambda I - A$. Interchanging the roles of S_1 and S_2, we get $\sigma(A_2) \subset S_2$.

(4) It remains to prove the uniqueness of P. Let $\widetilde{P}$ be another projection such that

(i) $\widetilde{P}$ commutes with every operator commuting with A;

(ii) the spectrum of A restricted to $R(\widetilde{P})$ (resp. $N(\widetilde{P})$) is S_1 (resp. S_2). It follows that

(a) $P(I - \widetilde{P})$ is also a projection;

(b) it commutes with A.

Since the range of $P(I - \widetilde{P})$ is contained in both $R(P)$ and $N(I - \widetilde{P})$, the spectrum of A restricted on the range of $P(I - \widetilde{P})$ is contained in both S_1 and S_2. So, the spectrum of $P(I - \widetilde{P})$ is empty since S_1 and S_2 are disjoint. So, $P = P\widetilde{P}$. Interchanging the roles of P and $\widetilde{P}$ gives $\widetilde{P} = \widetilde{P}P$. Finally we get $P = \widetilde{P}$. □

Assume now that $S_1 = \{\lambda\}$ consists of a single eigenvalue isolated (by γ) from the rest of the spectrum of A. We consider the projection P associated to $\{\lambda\}$. If A' is a bounded operator sufficiently close to A, then the spectrum of A' also consists of an eigenvalue $\{\lambda'\}$ isolated by γ from the rest of the spectrum of A'. The projection P' associated to $\{\lambda'\}$ is called the eigenprojection of λ'. The map $A' \to P'$ is analytic in a neighborhood of A. Let us state more precisely the following theorem for the case where λ is a simple isolated eigenvalue.

Theorem 8.2 (Perturbation theorem). *Let A be an element of the Banach algebra $L(\mathcal{B})$ of bounded linear operators on a Banach space $\mathcal{B}$. Suppose that A has a simple isolated eigenvalue λ with eigenvector v. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that if $B \in L(\mathcal{B})$ with $\|B - A\| < \delta$, then B has a simple isolated eigenvalue $\lambda(B)$ and a corresponding eigenvector $v(B)$ with the following properties*

(1) $\lambda(A) = \lambda$ and $v(A) = v$;

(2) The maps $B \to \lambda(B)$ and $B \to v(B)$ are analytic in $\|B - A\| < \delta$;

(3) for any B with $\|B - A\| < \delta$, $\lambda(B)$ is inside the open disk $D(\lambda, \epsilon)$ centered at λ of radius ϵ and $\sigma(B) \setminus \{\lambda(B)\}$ is outside the closed disk $\overline{D(\lambda, \epsilon)}$.

If λ is an isolated eigenvalue of A of finite multiplicity m, then for sufficiently close operator B of A the spectrum of B inside an neighborhood of λ (delimited by γ) will consist of eigenvalues $\lambda'_1, \cdots, \lambda'_m$, associated to which is the projection π' of B, and the map $B \to \pi$ will still be analytic. However the individual eigenvalues λ'_j are not well defined maps of B.

9 Application: central limit theorem

As application of the fact of spectral gaps (Theorem 7.1), we prove that for any Hölder continuous function ϕ and any Gibbs measure μ associated to a Hölder potential, the central limit theorem holds on the probability space (X, μ) for the process

$$S_n\phi(x) = \phi(x) + \phi(fx) + \cdots + \phi(f^{n-1}x)$$

(see [FSc]).

Recall that $f : X \to X$ is a continuous dynamical system on a compact metric space (X, d). Let $C(X) = C(X, \mathbb{C})$ be the space of real or complex continuous functions defined on X. Two functions $g_1, g_2 \in C(X)$ are said to be *cohomologous* if

$$g_1 = g_2 + u \circ f - u + c$$

for some $u \in C(X)$ and some constant c. Then we write $g_1 \asymp g_2$. Functions of the form $u \circ f - u + c$ are called *coboundaries*. The relation "$\asymp$" is an equivalence relation in $C(X)$. The quotient space $\tilde{C}(X) = C(X)/\asymp$, like $C(X)$, is also a linear space. If functions $\phi_1, \dots, \phi_n \in C(X)$, regarded as functions in the linear space $\tilde{C}(X)$, are linearly independent, we say that they are *cohomologously independent*. This is equivalent to saying that any linear combination $\xi_1\phi_1 + \cdots + \xi_d\phi_d$ (ξ_j's being not all zero) is not cohomologous to a constant.

Let μ be the Gibbs measure (also called equilibrium state) associated to a real valued α-Hölder function ψ. The measure μ depends only on the cohomology class of ψ. Thus we may assume that the potential ψ is normalized in the sense that

$$\sum_{f(y)=x} e^{\psi(y)} = 1 \qquad (\forall x \in X).$$

Let $\phi = (\phi_1, \dots, \phi_d)$ be a $\mathbb{R}^d$–valued α-Hölder continuous function. Let $z = (z_1, \dots, z_d) \in \mathbb{C}^d$. Consider the *transfer operator*

$$\mathcal{L}_z u(x) = \sum_{f(y)=x} e^{\psi(y)+\sum_{i=1}^d z_i\phi_i(y)} u(y).$$

For fixed z, the operator $\mathcal{L}_z$ is defined by a complex valued potential. It acts on the space of α-Hölder continuous functions. We denote its spectral radius by $\rho(z)$ and write

$$P(z) = \log \rho(z).$$

By Theorem 7.1 and Theorem 8.2, the function $P(z)$ is analytic in a neighborhood U of $0 \in \mathbb{C}^d$, the eigenspace associated to $\rho(z)$ is simple and it is possible to take an eigenvector $v(z)$ such that $v(z)$ is analytic in U and $v(0) = 1$. Thus we have

$$\mathcal{L}_z v(z) = e^{P(z)} v(z) \qquad (z \in U).$$

For a $\mathbb{R}^d$-valued function defined on X, we write

$$S_n\Phi(x) = \sum_{j=0}^{n-1} \Phi(f^j x).$$

Theorem 9.1. *Let μ be the equilibrium state associated to a normalized Hölder continuous potential ψ. Let $\Phi = (\phi_1, \ldots, \phi_d)$ be a $\mathbb{R}^d$–valued Hölder continuous function such that $\mathbb{E}_\mu \Phi = 0$. Suppose that the components of Φ are cohomologously independent. Then*

$$\frac{S_n\Phi}{\sqrt{n}} \to \mathcal{N}(0, P''(0)) \ \text{ in distribution}$$

where $P''(0)$ is the Hessian matrix of the second order derivative of $P(z)$ at zero, which is positive definite.

Proof. Recall that

$$\mathcal{L}_z v(z) = e^{P(z)} v(z) \qquad (z \in U).$$

The partial derivatives of the both sides of this equation are respectively

$$\begin{aligned} \frac{\partial}{\partial z_j} \mathcal{L}_z v(z) &= \mathcal{L}_z \left(\phi_j v(z) + \frac{\partial v}{\partial z_j}(z) \right) \\ \frac{\partial}{\partial z_j} e^{P(z)} v(z) &= e^{P(z)} \left(v(z) \frac{\partial P}{\partial z_j} + \frac{\partial v}{\partial z_j}(z) \right). \end{aligned}$$

Write $\mathcal{L} = \mathcal{L}_0$. Taking $z = 0$ leads to

$$\mathcal{L} \left(\phi_j + \frac{\partial v}{\partial z_j}(0) \right) = \frac{\partial P}{\partial z_j}(0) + \frac{\partial v}{\partial z_j}(0).$$

Integrating with respect to μ, we get

$$\frac{\partial P}{\partial z_j}(0) = \mathbb{E}_\mu \phi_j = 0.$$

Differentiate a second time:

$$\frac{\partial^2}{\partial z_k \partial z_j}\mathcal{L}_z v(z) = \mathcal{L}_z\left(\phi_k\phi_j v(z) + \phi_k \frac{\partial v}{\partial z_j}(z) + \phi_j \frac{\partial v}{\partial z_k}(z) + \frac{\partial^2}{\partial z_k \partial z_j}(z)\right)$$

$$\frac{\partial^2}{\partial z_k \partial z_j} e^{P(z)} v(z) = e^{P(z)}\left(v(z)\frac{\partial P}{\partial z_k}\frac{\partial P}{\partial z_j} + \frac{\partial P}{\partial z_k}\frac{\partial v}{\partial z_j} + \frac{\partial^2 P}{\partial z_k \partial z_j} + \frac{\partial^2 v}{\partial z_k \partial z_j}\right)$$

Taking $z = 0$ and then integrating, we get

$$\frac{\partial^2 P}{\partial z_k \partial z_j}(0) = \mathbb{E}_\mu(\phi_k\phi_j) + \frac{\partial v}{\partial z_k}\mathbb{E}_\mu\phi_j + \frac{\partial v}{\partial z_j}\mathbb{E}_\mu\phi_k.$$

Similarly, from the equality

$$\mathcal{L}_z^n v(z) = e^{nP(z)} v(z) \qquad (z \in U)$$

we can obtain

$$n\frac{\partial^2 P}{\partial z_k \partial z_j}(0) = \mathbb{E}_\mu(S_n\phi_k S_n\phi_j) + \frac{\partial v}{\partial z_k}\mathbb{E}_\mu S_n\phi_j + \frac{\partial v}{\partial z_j}\mathbb{E}_\mu S_n\phi_k.$$

Let $P''(0)$ be the matrix $(\frac{\partial^2 P}{\partial z_k \partial z_j}(0)\ )$. Let $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. Multiply both sides of the above equality by $\xi_k\xi_j$ and then sum over k and j. We obtain

$$n\xi^t P''(0)\xi = \mathbb{E}_\mu(S_n\langle\phi,\xi\rangle)^2 + 2\langle\nabla v,\xi\rangle \cdot \mathbb{E}_\mu S_n\langle\phi,\xi\rangle.$$

Divide by n then take the limit. According to the Birkhoff Theorem, we have

$$\xi^t P''(0)\xi = \lim_{n\to\infty}\mathbb{E}_\mu\frac{1}{n}(S_n\langle\phi,\xi\rangle)^2.$$

It can be proved that $\xi^t P''(0)\xi > 0$ if and only if $\langle\phi,\xi\rangle$ is not cohomologous to a constant. Consequently $P''(0)$ is positive definite if and only if $\langle\phi,\xi\rangle$ is not cohomologous to a constant for any $\xi \neq 0$.

Consider now the characteristic function of $\frac{1}{\sqrt{n}}S_n\phi$

$$f_n(\xi) := \int_X \exp i\langle\xi, (\sqrt{n})^{-1}S_n\phi\rangle d\mu = \int_X L^n_{i\xi/\sqrt{n}} 1 d\mu.$$

Write

$$1 = v\left(\frac{i\xi}{\sqrt{n}}\right) + 1 - v\left(\frac{i\xi}{\sqrt{n}}\right).$$

we get

$$f_n(\xi) = e^{nP(i\xi/\sqrt{n})} v\left(\frac{i\xi}{\sqrt{n}}\right) + o(1).$$

Taking the second order Taylor development of P(z) at zero yields that for any $\xi \in \mathbb{R}^d$, $f_n(\xi)$ tends to $\exp(-\frac{\xi^t P''(o)\xi}{2})$. Thus we can conclude by using the Lévy continuity theorem. □

10 Hilbert metric and convergence speed

In this section, we will present a new method, called Hilbert projective metric method, to prove Theorem 3.1. This method further provides us an estimate on the spectral gap as well. See [Bi, FS].

Let ν be the unique probability measure in Theorem 6.1 and define

$$\rho(\phi) =< \nu, \phi >, \quad \phi \in \mathcal{C}^0_{\mathbb{C}}.$$

Then ρ is a functional from $\mathcal{C}^0_{\mathbb{C}}$ to $\mathbb{C}$. From Theorem 6.1, we know that for $\phi \in \mathcal{C}^0_{\mathbb{C}}$, $\delta^{-n}\mathcal{L}^n\phi$ converges to $\rho(\phi)h$ uniformly as n goes to infinity. An important question in thermodynamical formalism is how fast does $\delta^{-n}\mathcal{L}^n\phi$ converge to $\rho(\phi)h$? We discuss this question in this section (refer to [FJ1, FJ2] for some further study).

Define

$$\mathcal{C}^\alpha_K = \{\phi \in \mathcal{C}^\alpha \ ; \ \phi > 0 \text{ and } [\log\phi]_\alpha \leq K\} \cup \{0\}.$$

One can check that $\mathcal{C}^\alpha_K$ is a convex cone in $\mathcal{C}^\alpha$, this means that (1) for any $\phi \in \mathcal{C}^\alpha_K$ and any real number $t \geq 0$, $t\phi \in \mathcal{C}^\alpha_K$ and (2) for any ϕ_1 and ϕ_2 in $\mathcal{C}^\alpha_K$ and any $0 \leq t \leq 1$, $t\phi_1 + (1-t)\phi_2$ is in $\mathcal{C}^\alpha_K$.

Suppose ψ is in $\mathcal{C}^\alpha_{K_0}$ for some $K_0 > 0$. Let $K > K_0/(\lambda^\alpha - 1)$ be a fixed constant and define $\tau = (K+K_0)\lambda^{-\alpha}/K < 1$. Then we have

Lemma 10.1.

$$\mathcal{L}(\mathcal{C}^\alpha_K) \subseteq \mathcal{C}^\alpha_{\tau K}.$$

Proof. Suppose $\phi \neq 0 \in \mathcal{C}^\alpha_K$ and $x, y \in X$ with $d(x,y) \leq a$. Then

$$\mathcal{L}\phi(x) = \sum_{z\in f^{-1}(x)} \psi(z)\phi(z) \qquad \text{and} \qquad \mathcal{L}\phi(y) = \sum_{w\in f^{-1}(y)} \psi(w)\phi(w).$$

We can arrange z's and w's such that $d(z,w) \leq \frac{d(x,y)}{\lambda}$. So

$$\begin{aligned} \mathcal{L}\phi(x) &= \sum_{z\in f^{-1}(x)} \psi(z)\phi(z) \\ &\leq \sum_{w\in f^{-1}(y)} \psi(w)e^{K_0 d(z,w)^\alpha}\phi(w)e^{K d(z,w)^\alpha} \\ &\leq \Big(\mathcal{L}\phi(y)\Big)e^{(K_0+K)\lambda^{-\alpha}d(x,y)^\alpha} \\ &= \Big(\mathcal{L}\phi(y)\Big)e^{\tau K d(x,y)^\alpha}. \end{aligned}$$

This implies that $\mathcal{L}\phi \in \mathcal{C}^\alpha_{\tau K}$. □

The real valued function space $\mathcal{C}^\alpha$ equipped with the norm

$$||\phi||_\alpha = ||\phi|| + [\phi]_\alpha$$

is a Banach space. The convex cone $\mathcal{C}_K^\alpha$ is closed in it. So we can define a so-called Hilbert projective metric with respect to the convex cone $\mathcal{C}_K^\alpha$ as follows (refer to [Bi])

First let $\preceq$ be the partial order in $\mathcal{C}^\alpha$ defined as $\phi_1 \preceq \phi_2$ if $\phi_2 - \phi_1 \in \mathcal{C}_K^\alpha$. The partial order $\preceq$ is integral, meaning that if $\phi_n \preceq \phi$ for all n and ϕ_n converges uniformly to ϕ_0, then $\phi \preceq \phi_0$. Let

$$A = A(\phi_1, \phi_2) = \sup\{t > 0;\ t\phi_1 \preceq \phi_2\}$$

and

$$B = B(\phi_1, \phi_2) = \inf\{t > 0;\ \phi_2 \preceq t\phi_1\}.$$

Note that A may be 0 and B may be ∞, however, if both ϕ_1 and ϕ_2 are not zero, then both A and B are finite numbers. The *Hilbert projective metric* with respect to $\mathcal{C}_K^\alpha$ is defined as

$$\Theta(\phi_1, \phi_2) = \log \frac{B}{A}.$$

The following lemma gives an explicit formula for Θ.

Lemma 10.2. *Let $\phi_1, \phi_2 \in \mathcal{C}_K^\alpha$. If both ϕ_1 and ϕ_2 are not zero, then*

$$\Theta(\phi_1, \phi_2) = \log \sup_{d(x,y), d(z,w) \le a} \frac{e^{Kd(x,y)^\alpha}\phi_1(x) - \phi_1(y)}{e^{Kd(z,w)^\alpha}\phi_1(z) - \phi_1(w)} \frac{e^{Kd(z,w)^\alpha}\phi_2(z) - \phi_2(w)}{e^{Kd(x,y)^\alpha}\phi_2(x) - \phi_2(y)}.$$

Proof. By the definition,

$$\Theta(\phi_1, \phi_2) = \log \frac{B}{A}$$

where

$$A\phi_1(x) \le \phi_2(x) \le B\phi_1(x), \quad x \in X,$$

and

$$\phi_2(y) - A\phi_1(y) \le e^{Kd(x,y)^\alpha}(\phi_2(x) - A\phi_1(x)), \quad d(x,y) \le a,$$

and

$$B\phi_1(w) - \phi_2(w) \le e^{Kd(z,w)^\alpha}(B\phi_1(z) - \phi_2(z)), \quad d(z,w) \le a.$$

Then

$$A \leq \frac{\phi_2(x)}{\phi_1(x)} \leq B, \quad x \in X,$$

and

$$A \leq \frac{e^{Kd(x,y)^\alpha}\phi_2(x) - \phi_2(y)}{e^{Kd(x,y)^\alpha}\phi_1(x) - \phi_1(y)}, \quad d(x,y) \leq a,$$

and

$$B \geq \frac{e^{Kd(z,w)^\alpha}\phi_2(z) - \phi_2(w)}{e^{Kd(z,w)^\alpha}\phi_1(z) - \phi_1(w)}, \quad d(z,w) \leq a.$$

We have

$$A = \min\{\inf_{x \in X} \frac{\phi_2(x)}{\phi_1(x)}, \inf_{d(x,y) \leq a} \frac{e^{Kd(x,y)^\alpha}\phi_2(x) - \phi_2(y)}{e^{Kd(x,y)^\alpha}\phi_1(x) - \phi_1(y)}\}$$

and

$$B = \max\{\sup_{z \in X} \frac{\phi_2(z)}{\phi_1(z)}, \sup_{d(z,w) \leq a} \frac{e^{Kd(z,w)^\alpha}\phi_2(z) - \phi_2(w)}{e^{Kd(z,w)^\alpha}\phi_1(z) - \phi_1(w)}\}.$$

Let $x_0 \in X$ such that

$$\frac{\phi_2(x_0)}{\phi_1(x_0)} = \min_{x \in X} \frac{\phi_2(x)}{\phi_1(x)}.$$

Then for $x \in X$ with $d(x, x_0) \leq a$,

$$\frac{e^{Kd(x,x_0)^\alpha}\phi_2(x_0) - \phi_2(x)}{e^{Kd(x,x_0)^\alpha}\phi_1(x_0) - \phi_1(x)} = \frac{e^{Kd(x,x_0)^\alpha}\frac{\phi_2(x_0)}{\phi_1(x_0)}\phi_1(x_0) - \frac{\phi_2(x)}{\phi_1(x)}\phi_1(x)}{e^{Kd(x,x_0)^\alpha}\phi_1(x_0) - \phi_1(x)} \leq \frac{\phi_2(x)}{\phi_1(x)}.$$

Take a sequence $\{x_n\}$ tending to x_0 in X as n goes to infinity. Then

$$\inf_{d(x,y) \leq a} \frac{e^{Kd(x,y)^\alpha}\phi_2(x) - \phi_2(y)}{e^{Kd(x,y)^\alpha}\phi_1(x) - \phi_1(y)} \leq \frac{e^{Kd(x_n,x_0)^\alpha}\phi_2(x_0) - \phi_2(x_n)}{e^{Kd(x_n,x_0)^\alpha}\phi_1(x_0) - \phi_1(x_n)} \leq \frac{\phi_2(x_n)}{\phi_1(x_n)}.$$

As n goes to infinity,

$$\inf_{d(x,y) \leq a} \frac{e^{Kd(x,y)^\alpha}\phi_2(x) - \phi_2(y)}{e^{Kd(x,y)^\alpha}\phi_1(x) - \phi_1(y)} \leq \frac{\phi_2(x_0)}{\phi_1(x_0)}.$$

We get

$$A = \inf_{d(x,y) \leq a} \frac{e^{Kd(x,y)^\alpha}\phi_2(x) - \phi_2(y)}{e^{Kd(x,y)^\alpha}\phi_1(x) - \phi_1(y)}$$

Similarly we can get

$$B = \sup_{d(z,w)\le a} \frac{e^{Kd(z,w)^\alpha}\phi_2(z) - \phi_2(w)}{e^{Kd(z,w)^\alpha}\phi_1(z) - \phi_1(w)}.$$

Thus

$$\Theta(\phi_1,\phi_2) = \log \sup_{d(x,y),d(z,w)\le a} \frac{e^{Kd(x,y)^\alpha}\phi_1(x) - \phi_1(y)}{e^{Kd(z,w)^\alpha}\phi_1(z) - \phi_1(w)} \frac{e^{Kd(z,w)^\alpha}\phi_2(z) - \phi_2(w)}{e^{Kd(x,y)^\alpha}\phi_2(x) - \phi_2(y)}.$$

□

Let

$$\Delta = \sup_{\phi_1,\phi_2\in\mathcal{C}^\alpha_{\tau K}} \Theta(\phi_1,\phi_2)$$

and

$$\Lambda = \tanh\left(\frac{\Delta}{4}\right).$$

Let k_0 be the minimal positive integer such that there are k_0 balls of radius a covering X.

Lemma 10.3.

$$\Delta \le 2\log\left(\frac{1+\tau}{1-\tau}\right) + 2(1-\tau+2k_0)Ka^\alpha.$$

Thus $0 < \Lambda < 1$.

Proof. For any $\phi_1,\phi_2\in\mathcal{C}^\alpha_{\tau K}$,

$$e^{Kd(z,w)^\alpha}\phi_2(z)-\phi_2(w) = \left(e^{Kd(z,w)^\alpha} - \frac{\phi_2(z)}{\phi_2(w)}\right)\phi_2(z) \le \left(e^{Kd(z,w)^\alpha} - e^{-\tau Kd(z,w)^\alpha}\right)\phi_2(z).$$

In the same way, we get that

$$e^{Kd(x,y)^\alpha}\phi_2(x) - \phi_2(y) \ge \left(e^{Kd(x,y)^\alpha} - e^{\tau Kd(x,y)^\alpha}\right)\phi_2(x)$$

and

$$e^{Kd(x,y)^\alpha}\phi_1(x) - \phi_1(y) \le \left(e^{Kd(x,y)^\alpha} - e^{-\tau Kd(x,y)^\alpha}\right)\phi_1(x),$$

$$e^{Kd(z,w)^\alpha}\phi_1(z) - \phi_1(w) \ge \left(e^{Kd(z,w)^\alpha} - e^{\tau Kd(z,w)^\alpha}\right)\phi_1(z).$$

This implies that

$$\begin{aligned}
&\Theta(\phi_1,\phi_2)\\
\le\ &\log \sup_{d(x,y),d(z,w)\le a} \frac{(e^{Kd(x,y)^\alpha}-e^{-\tau Kd(x,y)^\alpha})}{(e^{Kd(x,y)^\alpha}-e^{\tau Kd(x,y)^\alpha})}\frac{(e^{Kd(z,w)^\alpha}-e^{-\tau Kd(z,w)^\alpha})}{(e^{Kd(z,w)^\alpha}-e^{\tau Kd(z,w)^\alpha})}\frac{\phi_2(z)}{\phi_2(x)}\frac{\phi_1(x)}{\phi_1(z)}\\
=\ &\log \sup_{d(x,y),d(z,w)\le a} \frac{(1-e^{-(1+\tau)Kd(x,y)^\alpha})}{(1-e^{-(1-\tau)Kd(x,y)^\alpha})}\frac{(1-e^{-(1+\tau)Kd(z,w)^\alpha})}{(1-e^{-(1-\tau)Kd(z,w)^\alpha})}\frac{\phi_2(z)}{\phi_2(x)}\frac{\phi_1(x)}{\phi_1(z)}.
\end{aligned}$$

Since

$$e^{-C}t \le 1-e^{-t} = \int_{-t}^{0} e^{\xi}d\xi \le t, \quad 0\le t\le C$$

and since

$$\frac{\phi_2(z)}{\phi_2(x)}, \frac{\phi_1(x)}{\phi_1(z)} \le e^{k_0Ka^\alpha},$$

we have

$$\begin{aligned}
\Theta(\phi_1,\phi_2) &\le \frac{(1+\tau)Kd(x,y)^\alpha}{(1-\tau)Kd(x,y)^\alpha e^{-(1-\tau)Ka^\alpha}}\frac{(1+\tau)Kd(z,w)^\alpha}{(1-\tau)Kd(z,w)^\alpha e^{-(1-\tau)Ka^\alpha}}\left(e^{k_0Ka^\alpha}\right)^2\\
&= \left(\frac{1+\tau}{1-\tau}\right)^2 e^{2(1-\tau)Ka^\alpha}e^{2k_0Ka^\alpha}.
\end{aligned}$$

Therefore,

$$\Delta \le 2\log\left(\frac{1+\tau}{1-\tau}\right) + 2(1-\tau+2k_0)Ka^\alpha.$$

□

Lemma 10.4. *For any* $\phi_1,\phi_2\in\mathcal{C}_K^\alpha$,

$$\Theta(\mathcal{L}\phi_1,\mathcal{L}\phi_2)\le\Lambda\Theta(\phi_1,\phi_2).$$

Proof. By the definition,

$$\Theta(\phi_1,\phi_2)=\log\frac{B}{A}$$

and $\phi_2-A\phi_1$ and $B\phi_1-\phi_2$ are both in $\mathcal{C}_K^\alpha$. From Lemma 10.1,

$$\Theta(\mathcal{L}(\phi_2-A\phi_1),\mathcal{L}(B\phi_1(x)-\phi_2))\le\Delta.$$

Let A_0 and B_0 be two corresponding numbers in the definition of

$$\Theta(\mathcal{L}(\phi_2-A\phi_1),\mathcal{L}(B\phi_1(x)-\phi_2)).$$

Then we have

$$\frac{B_0}{A_0} \leq e^{\Delta}$$

and

$$A_0\mathcal{L}(\phi_2 - A\phi_1)) \preceq \mathcal{L}(B\phi_1 - \phi_2) \preceq B_0\mathcal{L}(\phi_2 - A\phi_1).$$

This gives us that

$$\frac{B + B_0A}{1 + B_0}\mathcal{L}\phi_1 \preceq \mathcal{L}\phi_2 \preceq \frac{B + A_0A}{1 + A_0}\mathcal{L}\phi_1.$$

So

$$\begin{aligned}
\Theta(\mathcal{L}\phi_1, \mathcal{L}\phi_2) &\leq \log\frac{(B + A_0A)(1 + B_0)}{(B + B_0A)(1 + A_0)} \\
&= \log\frac{e^{\Theta(\phi_1,\phi_2)} + A_0}{e^{\Theta(\phi_1,\phi_2)} + B_0} - \log\frac{1 + A_0}{1 + B_0} \\
&= \int_0^{\Theta(\phi_1,\phi_2)} \frac{(B_0 - A_0)e^{\xi}}{(e^{\xi} + A_0)(e^{\xi} + B_0)}\, d\xi \\
&\leq \frac{1 - \frac{A_0}{B_0}}{(1 + \sqrt{\frac{A_0}{B_0}})^2}\Theta(\phi_1, \phi_2) \\
&\leq \Lambda\Theta(\phi_1, \phi_2).
\end{aligned}$$

□

It is easy to check that the functional ρ satisfies

1. $\rho(s\phi) = s\rho(\phi)$ for any $s \geq 0$ and $\phi \in \mathcal{C}^{\alpha}$ and
2. if $\phi_1 \preceq \phi_2$, then $\rho(\phi_1) \leq \rho(\phi_2)$.

Use these properties, we have

Lemma 10.5. *For any ϕ_1 and ϕ_2 in $\mathcal{C}^{\alpha}_{K}$ satisfying $\rho(\phi_1) = \rho(\phi_2) \neq 0$,*

$$||\phi_2 - \phi_1|| \leq (e^{\Theta(\phi_1,\phi_2)} - 1)||\phi_1||.$$

Proof. From the definition,

$$\Theta(\phi_1, \phi_2) = \log\frac{B}{A}$$

where

$$A\phi_1 \preceq \phi_2 \preceq B\phi_1.$$

We get

$$A\rho(\phi_1) \leq \rho(\phi_2) \leq B\rho(\phi_1).$$

So

$$A \leq 1 \leq B.$$

Thus

$$||\phi_2 - \phi_1|| \leq ||\frac{\phi_2}{\phi_1} - 1|| ||\phi_1|| \leq (B-1)||\phi_1|| \leq (\frac{B}{A} - 1)||\phi_1|| \leq (e^{\Theta(\phi_1,\phi_2)} - 1)||\phi_1||.$$

□

Lemma 10.6. *Suppose $\phi \in \mathcal{C}^\alpha$, $\phi > 0$. Then there is an integer $N = N(\phi) > 0$ such that*

$$\mathcal{L}^n \phi \in \mathcal{C}_K^\alpha$$

for all $n \geq N$.

Proof. Let $[\log \phi]_\alpha$ be the Hölder constant of $\log \phi$. From the proof of Lemma 3.2 we see that

$$[\log \mathcal{L}\phi]_\alpha \leq ([\log \phi]_\alpha + K_0)\lambda^{-\alpha}.$$

In general,

$$[\log \mathcal{L}^n \phi]_\alpha \leq K_0(\lambda^{-\alpha} + \cdots + \lambda^{-n\alpha}) + K_\phi \lambda^{-n\alpha} \to \frac{K_0}{\lambda^\alpha - 1} \quad \text{as} \quad n \to \infty.$$

Since $K > K_0/(\lambda^\alpha - 1)$, we can find an integer $N > 0$ satisfying the lemma. □

Theorem 10.1. *Suppose $\phi \in \mathcal{C}^\alpha$. Then there is a constant $C_0 > 0$ independent of ϕ and an integer $N = N(\phi) > 0$ such that*

$$||\delta^{-n}\mathcal{L}^n \phi - \rho(\phi)h|| \leq C_0 \Lambda^{n-N} ||\phi||, \quad n \geq N.$$

Proof. Suppose $||\phi|| > 0$. Otherwise it is trivial. Take

$$b = \frac{2||\phi||}{\min_x h(x)}.$$

Then $\phi + bh > 0$. From Lemma 10.6, there is an integer $N = N(\phi + bh) > 0$ such that

$$\mathcal{L}^n(\phi + bh) \in \mathcal{C}_K^\alpha, \quad n \geq N.$$

For $n \geq 0$,

$$\begin{aligned} \rho(\delta^{-n}\mathcal{L}^n(\phi + bh)) &= \delta^{-n} \int \mathcal{L}^n(\phi + bh)\; d\nu \\ &= \delta^{-n} \int (\phi + bh)\; d\mathcal{L}^{*n}\nu \\ &= \int (\phi + bh) d\nu \\ &= \rho(\phi + bh) \\ &= \rho(\phi) + b. \end{aligned}$$

Because

$$|\rho(\phi)| = |\rho(\frac{\phi}{h}h)| \leq \frac{||\phi||}{\min_x h(x)},$$

$\rho(\phi) + b \neq 0$. From Lemma 10.4,

$$\begin{aligned} ||\delta^{-n}\mathcal{L}^n\phi - \delta^{-m}\mathcal{L}^m\phi|| &= ||\delta^{-n}\mathcal{L}^n(\phi + bh) - \delta^{-m}\mathcal{L}^m(\phi + bh)|| \\ &\leq \left(e^{\Theta\left(\mathcal{L}^n(\phi+bh), \mathcal{L}^m(\phi+bh)\right)} - 1\right) ||\delta^{-m}\mathcal{L}^m(\phi + bh)|| \end{aligned}$$

for any $m > n \geq N$. Since $\mathcal{L}^k(\phi + bh) \in \mathcal{C}^\alpha_K$ for $k \geq N$,

$$\Theta(\mathcal{L}^n(\phi+bh), \mathcal{L}^m(\phi+bh)) \leq \Lambda^{n-N}\Theta(\mathcal{L}^N(\phi+bh), \mathcal{L}^{m-n+N}(\phi+bh)) \leq \Lambda^{n-N}\Delta.$$

Thus

$$||\delta^{-n}\mathcal{L}^n\phi - \delta^{-m}\mathcal{L}^m\phi|| \leq (e^{\Lambda^{n-N}\Delta} - 1)||\delta^{-m}\mathcal{L}^m(\phi + bh)||.$$

Let $m \to \infty$. We get

$$||\delta^{-n}\mathcal{L}^n\phi - \rho(\phi)h|| \leq (e^{\Lambda^{n-N}\Delta} - 1)(\rho(\phi) + b)||h|| \leq C_0\Lambda^{n-N}||\phi||$$

for $n \geq N$, where $C_0 > 0$ is a constant independent of ϕ. □

Remember that $\mu = h\nu$ is the Gibbs measure for (f, ψ) and is f-invariant. For any $\phi \in \mathcal{C}^0$, define

$$\Phi(n) = \Big| \int \phi \cdot \phi \circ f^n\; d\mu - (\int \phi\; d\mu)^2 \Big|.$$

It is called the correlation for ϕ. Since $\mathcal{L}^{*n}\nu = \delta^n\nu$,

$$\begin{aligned} \int \phi \cdot \phi \circ f^n\; d\mu &= \delta^{-n} \int \phi \cdot \phi \circ f^n h\; d\mathcal{L}^{*n}\nu \\ &= \delta^{-n} \int \mathcal{L}^n(\phi \cdot h \cdot \phi \circ f^n)\; d\nu \\ &= \int (\delta^{-n}\mathcal{L}^n(\phi h))\phi\; d\nu. \end{aligned}$$

Thus for $n \geq N$,

$$\Phi(n) \leq \int ||\delta^{-n}\mathcal{L}^n(h\phi) - \rho(h\phi)h|| \cdot ||\phi|| \, d\nu \leq C_0||h||\Lambda^{n-N}||\phi||^2 = C\Lambda^{n-N}||\phi||^2$$

where $C = C_0||h||$ is a constant independent of ϕ. This gives us that

Corollary 10.1. *The decay of correlation is exponential. More precisely, there is a constant $C > 0$ such that for any $\phi \in \mathcal{C}^\alpha$, there is an integer $N = N(\phi) > 0$ such that*

$$\Phi(n) \leq C\Lambda^{n-N}||\phi||^2, \quad n \geq N.$$

Assume $\mathcal{L}$ is normalized, i.e., $\mathcal{L}1 = 1$. Let

$$\mathcal{C}^{\alpha\perp} = \{\phi \in \mathcal{C}^\alpha \, ; \, \rho(\phi) = 0\}.$$

Then

$$\mathcal{C}^\alpha = \mathcal{C}^{\alpha\perp} \oplus \mathbb{R}$$

since $\phi = (\phi - \rho(\phi)) + \rho(\phi)$. For $\phi \in \mathcal{C}^{\alpha\perp}$, Theorem 10.1 says that there is an integer $N = N(\phi) > 0$ such that

$$||\mathcal{L}^n\phi|| \leq C_0\Lambda^{n-N}||\phi||, \quad n \geq N.$$

This is useful in the proof of the next lemma, from which will deduce an estimate on the essential spectral radius of $\mathcal{L} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ (see Theorem 10.2. Let

$$\kappa = \max\{\lambda^{-\alpha}, \Lambda\} < 1.$$

Lemma 10.7. *There is a constant $C > 0$ independent of ϕ such that*

$$[\mathcal{L}^n\phi]_\alpha \leq C \cdot (n + 1 - N) \cdot \kappa^{n-N}||\phi||$$

for all $n \geq N$.

Proof. Suppose $x, y \in X$ with $d(x, y) \leq a$. Let $f^{-1}(x) = \{z\}$ and $f^{-1}(y) = \{w\}$. We can arrange z's and w's such that $d(z, w) \leq \lambda^{-1}d(x, y)$. Then

$$\begin{aligned} |\mathcal{L}\phi(x) - \mathcal{L}\phi(y)| &= |\sum_{z\in f^{-1}(x), w\in f^{-1}(y)} (\psi(z)\phi(z) - \psi(w)\phi(w))| \\ &\leq \sum_{z\in f^{-1}(x), w\in f^{-1}(y)} |\psi(z) - \psi(w)||\phi(z)| + \psi(w)|\phi(w) - \phi(z)|. \end{aligned}$$

So

$$[\mathcal{L}\phi]_\alpha \le \frac{[\psi]_\alpha n_0}{\lambda^\alpha}||\phi|| + \frac{1}{\lambda^\alpha}[\phi]_\alpha.$$

Let $C_2 = [\psi]_\alpha n_0 C_0 \Lambda^{-1}\lambda^{-\alpha}$. Then for $n > N$,

$$\begin{aligned} [\mathcal{L}^n\phi]_\alpha &= [\mathcal{L}(\mathcal{L}^{n-1}\phi)]_\alpha \\ &\le \frac{[\psi]_\alpha n_0}{\lambda^\alpha}||\mathcal{L}^{n-1}\phi|| + \frac{1}{\lambda^\alpha}[\mathcal{L}^{n-1}\phi]_\alpha \\ &\le \frac{[\psi]_\alpha n_0}{\lambda^\alpha}C_0\Lambda^{n-1-N}||\phi|| + \frac{1}{\lambda^\alpha}[\mathcal{L}^{n-1}\phi]_\alpha \\ &\le C_2\Lambda^{n-N}||\phi|| + \frac{1}{\lambda^\alpha}[\mathcal{L}^{n-1}\phi]_\alpha. \end{aligned}$$

Further,

$$\begin{aligned} [\mathcal{L}^n\phi]_\alpha &\le C_2\Lambda^{n-N}||\phi|| + \frac{1}{\lambda^\alpha}(C_2\Lambda^{n-1-N}||\phi|| + \frac{1}{\lambda^\alpha}[\mathcal{L}^{n-2}\phi]_\alpha) \\ &\le 2C_2\kappa^{n-N}||\phi|| + \frac{1}{\lambda^{2\alpha}}[\mathcal{L}^{n-2}\phi]_\alpha \\ &\le \cdots \le (n-N)C_2\kappa^{n-N}||\phi|| + \frac{1}{\lambda^{(n-N)\alpha}}[\mathcal{L}^N\phi]_\alpha \\ &\le C\cdot(n-N+1)\cdot\kappa^{n-N}||\phi|| \end{aligned}$$

where $C = \max\{C_2, C_1\} > 0$. □

Lemma 10.8. *For any $\phi \in \mathcal{C}^{\alpha\perp}$ and for any $\xi \in \mathbb{R}$ such that $|\xi| > \kappa$, the series*

$$\mathcal{K}\phi = \xi^{-1}\phi + \xi^{-2}\mathcal{L}\phi + \xi^{-3}\mathcal{L}^2\phi + \cdots + \xi^{-(n+1)}\mathcal{L}^n\phi + \cdots$$

converges in the supremum norm and its sum $\mathcal{K}\phi$ belongs to $\mathcal{C}^{\alpha\perp}$. Thus $\mathcal{K} : \mathcal{C}^{\alpha\perp} \to \mathcal{C}^{\alpha\perp}$.

Proof. The convergence of the series in $\mathcal{C}^0$ follows directly from Theorem 10.1. Because $\rho(\mathcal{L}^n\phi) = 0$ for all $n > 0$, $\rho(\mathcal{K}\phi) = 0$. From Lemma 10.7,

$$\begin{aligned} [\mathcal{K}\phi]_\alpha &\le \sum_{n=0}^{\infty}|\xi|^{-n-1}[\mathcal{L}^n\phi]_\alpha \\ &\le \sum_{n=0}^{N-1}|\xi|^{-n-1}[\mathcal{L}^n\phi]_\alpha + C\sum_{n=N}^{\infty}|\xi|^{-n-1}\kappa^{n-N}(n-N+1) < \infty. \end{aligned}$$

Therefore, $\mathcal{K}\phi$ is in $\mathcal{C}^{\alpha\perp}$. □

Now consider complex valued function spaces $\mathcal{C}^{\alpha}_{\mathbb{C}}$ and $\mathcal{C}^{\alpha\perp}_{\mathbb{C}}$. Both of them are Banach spaces under the norm $||\phi||_{\alpha}$. Then

$$\mathcal{L}\phi = \mathcal{L}\phi_1 + i\mathcal{L}\phi_2 \quad \text{for} \quad \phi = \phi_1 + i\phi_2,\ \phi_1, \phi_2 \in \mathcal{C}^{\alpha}.$$

Thus we have $\mathcal{L} : \mathcal{C}^{\alpha}_{\mathbb{C}} \to \mathcal{C}^{\alpha}_{\mathbb{C}}$ and $\mathcal{L} : \mathcal{C}^{\alpha\perp}_{\mathbb{C}} \to \mathcal{C}^{\alpha\perp}_{\mathbb{C}}$. Similarly, $\mathcal{K}$ is also defined on $\mathcal{C}^{\alpha\perp}_{\mathbb{C}}$ and for any $\phi \in \mathcal{C}^{\alpha\perp}_{\mathbb{C}}$ and any $\xi \in \mathbb{C}$ such that $|\xi| > \kappa$, the series

$$\mathcal{K}\phi = \xi^{-1}\phi + \xi^{-2}\mathcal{L}\phi + \xi^{-3}\mathcal{L}^2\phi + \cdots + \xi^{-(n+1)}\mathcal{L}^n\phi + \cdots$$

converges in the maximal norm and belongs to $\mathcal{C}^{\alpha\perp}_{\mathbb{C}}$. So $\mathcal{K} : \mathcal{C}^{\alpha\perp}_{\mathbb{C}} \to \mathcal{C}^{\alpha\perp}_{\mathbb{C}}$. Now we prove a stronger result than Theorem 7.1.

Theorem 10.2. *Suppose $\mathcal{L}$ is a normalized RPF operator, i.e., $\mathcal{L}1 = 1$. Then the spectrum radius of $\mathcal{L}|\mathcal{C}^{\alpha\perp}_{\mathbb{C}} : \mathcal{C}^{\alpha\perp}_{\mathbb{C}} \to \mathcal{C}^{\alpha\perp}_{\mathbb{C}}$ is less than or equal to κ.*

Proof. From Lemma 10.8, for $\xi \in \mathbb{C}$ such that $|\xi| > \kappa$,

$$\mathcal{K}(\xi I - \mathcal{L}) = (\xi I - \mathcal{L})\mathcal{K} = I \quad \text{on} \quad \mathcal{C}^{\alpha\perp}_{\mathbb{C}}.$$

So $\xi I - \mathcal{L} : \mathcal{C}^{\alpha\perp}_{\mathbb{C}} \to \mathcal{C}^{\alpha\perp}_{\mathbb{C}}$ is bijective. By the Banach theorem, it is invertible. This implies that ξ is not a spectrum point of $\mathcal{L}|\mathcal{C}^{\alpha\perp}_{\mathbb{C}}$. This proves the theorem. □

Corollary 10.2. *For any $\phi \in \mathcal{C}^{\alpha\perp}_{\mathbb{C}}$ and any $\epsilon > 0$, there is an integer $N = N(\phi, \epsilon) > 0$ such that*

$$||\mathcal{L}^n\phi||_{\alpha} \leq (\kappa + \epsilon)^n ||\phi||_{\alpha}, \quad n \geq N.$$

Proof. From the spectrum formula,

$$\limsup_{n\to\infty} ||\mathcal{L}^n|\mathcal{C}^{\alpha\perp}_{\mathbb{C}}||_{\alpha}^{\frac{1}{n}} \leq \kappa.$$

We have an integer $N = N(\phi, \epsilon) > 0$ such that

$$||\mathcal{L}^n\phi||_{\alpha} \leq (\kappa + \epsilon)^n ||\phi||_{\alpha}, \quad n \geq N.$$

□

Corollary 10.3. *For a normalized RPF operator $\mathcal{L} : \mathcal{C}^{\alpha}_{\mathbb{C}} \to \mathcal{C}^{\alpha}_{\mathbb{C}}$, the maximal eigenvalue is 1 and the rest of its spectrum is inside the closed disk centered 0 of radius κ.*

From the relation between an RPF operator and its normalization, Corollary 10.3 implies that

Corollary 10.4. *For any RPF operator $\mathcal{L} : \mathcal{C}^{\alpha}_{\mathbb{C}} \to \mathcal{C}^{\alpha}_{\mathbb{C}}$, let δ be its maximal positive simple eigenvalue. Then the rest of its spectrum is inside the closed disk centered 0 of radius $\kappa\delta$.*

11 Nuclear operators

Suppose $\mathcal{B}$ and $\mathcal{D}$ are two Banach spaces. We denote by $\mathcal{B}'$ the dual space of $\mathcal{B}$, i.e., the space of all bounded linear functionals from $\mathcal{B}$ to $\mathbb{C}$. Let $L(\mathcal{B}, \mathcal{D})$ be the space of bounded linear operators from $\mathcal{B}$ to $\mathcal{D}$. If $\mathcal{B} = \mathcal{D}$, we write $L(\mathcal{B}) = L(\mathcal{B}, \mathcal{D})$.

For $x' \in \mathcal{B}'$ and $y \in \mathcal{D}$ and $\lambda \in \mathbb{C}$, denote by $x' \otimes y$ the linear operator $x' \otimes y(z) = x'(z)y$ defined for $z \in \mathcal{B}$ and taking values in $\mathcal{D}$. It is a bounded linear operator. A bounded linear operator N is said to have *finite rank* if there are $\{x_i'\}_{i=1}^n \subset \mathcal{B}'$ and $\{y_i\}_{i=1}^n \subset \mathcal{D}$ and $\{\lambda_i\}_{i=1}^n \subset \mathbb{C}$ such that

$$N = \sum_{i=1}^n \lambda_i x_i' \otimes y_i.$$

When $\mathcal{D} = \mathcal{B}$, the operator N defined above belongs to $L(\mathcal{B})$. Then we can define its *trace* as

$$\mathrm{tr}(N) = \sum_{i=1}^n \lambda_i x_i'(y_i).$$

The finite rank operator N has different representations. However as the following lemma shows, the trace is independent of representations. First remark that if $\{y_i\}$ are independent and x_i' are such that $x_i'(y_j) = \delta_{ij}$, then

$$\mathrm{tr}(N) = \sum_{i=1}^n \lambda_i = \sum_{i=1}^n x_j'(N(y_i)).$$

Lemma 11.1. *The trace* $\mathrm{tr}(N)$ *of a linear operator of finite rank in* $L(\mathcal{B})$ *is independent of its representations.*

Proof. Let $\{e_1, \cdots, e_m\}$ be a linearly independent set in $\mathcal{B}$ such that $||e_i|| = 1$ and such that $\{y_1, \cdots, y_n\} \subset \mathrm{span}\{e_1, \cdots, e_m\}$. Then

$$y_i = a_{i1}e_1 + \cdots + a_{im}e_m, \qquad i = 1, \cdots, n.$$

Let $e_i' \in B'$ be the linear functional satisfying that $e_i'(e_j) = \delta_{ij}$. Then

$$N(e_k) = \sum_{i=1}^n \lambda_i x_i'(e_k) \sum_{j=1}^m a_{ij} e_j = \sum_{i=1}^n \sum_{j=1}^m \lambda_i a_{ij} x_i'(e_k) e_j = \sum_{j=1}^m \Big(\sum_{i=1}^n \lambda_i a_{ij} x_i'(e_k) \Big) e_j.$$

It follows that

$$e_k'\Big(N(e_k)\Big) = \sum_{i=1}^n \lambda_i a_{ik} x_i'(e_k).$$

So

$$\begin{aligned}\mathrm{tr}(N) &= \sum_{i=1}^{n} \lambda_i x_i'(y_i) = \sum_{i=1}^{n} \lambda_i x_i'(\sum_{k=1}^{m} a_{ik} e_k) \\ &= \sum_{i=1}^{n}\sum_{k=1}^{m} \lambda_i a_{ik} x_i'(e_k) = \sum_{k=1}^{m}\sum_{i=1}^{n} \lambda_i a_{ik}) x_i'(e_k) = \sum_{k=1}^{m} e_k'\Big(N(e_k)\Big).\end{aligned}$$

This implies that the trace $\mathrm{Tr}(N)$ is independent of representations. □

A sequence $\mathbf{a} = \{\lambda_n\}_{n=1}^{\infty}$ of complex numbers is in l_p, $0 < p \leq 1$ if

$$||\mathbf{a}||_p = \Big(\sum_{n=1}^{\infty} |\lambda_n|^p\Big)^{\frac{1}{p}} < \infty;$$

and it is in l_∞ if

$$||\mathbf{a}||_\infty = \sup_{n\geq 0} \max_{1\leq i\leq n} |\lambda_i| < \infty.$$

A linear operator N in $L(\mathcal{B}, \mathcal{D})$ is called a *nuclear operator* of order $0 \leq q < 1$ if

$$N = \sum_{n=1}^{\infty} \lambda_n x_n' \otimes y_n$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is in l^p for any $p > q$, $\{y_n\}_{n=1}^{\infty}$ is a sequence of elements in $\mathcal{D}$ with norms $||y_n||_D = 1$, and $\mathbf{x} = \{x_n'\}_{n=1}^{\infty}$ is a sequence of elements in $\mathcal{B}'$ with also norms $||x_n'||_{B'} = 1$. In other words, N is a nuclear operator if and only if it can be decomposed as the product $N = ABC$ where $C : \mathcal{B} \to l_\infty$, $B : l_\infty \to \cap_{q<p\leq 1} l_p$, and $A : \cap_{q<p\leq 1} l_p \to \mathcal{D}$ are bounded linear operators.

A nuclear operator of order q is compact but a compact operator may not be nuclear because all eigenvalues of a compact operator may accumulate to 0 very slowly.

Theorem 11.1. *Suppose $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$ are three Banach spaces. Suppose $N \in \mathcal{L}(\mathcal{B}, \mathcal{D})$ and $M \in \mathcal{L}(\mathcal{D}, \mathcal{E})$ are bounded linear operators. If one of N and M is nuclear of order $0 \leq q < 1$, then MN is a nuclear operator of order q.*

Proof. Suppose N is nuclear and has a representation

$$N = \sum_{n=1}^{\infty} \lambda_n x_n' \otimes y_n.$$

It is easy to see that

$$\begin{aligned} MN &= \sum_{n=1}^{\infty} \lambda_n x'_n \otimes M(y_n) \\ &= \sum_{n=1}^{\infty} \lambda_n ||M(y_n)|| x'_n \otimes \frac{M(y_n)}{||M(y_n)||} \end{aligned}$$

for all $n > 0$ such that $M(y_n) \neq 0$. Therefore, we have

$$MN = \sum_{n=1}^{\infty} \tilde{\lambda}_n x'_n \otimes \tilde{y}_n$$

where $\tilde{y}_n = M(y_n)/||M(y_n)||$ or 0 and $\tilde{\lambda}_n = \lambda_n M(y_n)$. Since

$$\sum_{n=1}^{\infty} |\tilde{\lambda}_n|^p \leq \sum_{n=1}^{\infty} |\lambda_n|^p ||M||^p < \infty$$

for any $q < p \leq 1$, we get that MN is a nuclear operator of order q. Similarly, we can prove that MN is a nuclear operator of order q when M is nuclear of order q. □

For a nuclear operator in $L(\mathcal{B})$ having its representation $N = \sum_{n=1}^{\infty} \lambda_n y_n \otimes x'_n$, we could define its *"formal" trace*

$$\text{“tr”}(N) = \sum_{n \in \mathbb{N}} \lambda_n x'_n(y_n).$$

A Banach space $\mathcal{B}$ is said to have approximation property if there is a sequence $\{e_n\}_{n=1}^{\infty}$ of independent unit elements in $\mathcal{B}$ such that any element x in $\mathcal{B}$ can be written as a linear combination of these independent elements:

$$x = \sum_{n=1}^{\infty} a_n e_n,$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers (such a system $\{e_n\}$) is called a Schauder basis).

If $\mathcal{B}$ has approximation property and N is a nuclear operator of order $0 \leq q < 1$, then it may be proved that the trace defined above is independent of representations (the same proof for finite rank operators). Not every Banach space has approximation property. However, we have

Theorem 11.2 (Grothendick [Gr]). *If $N \in \mathcal{L}(\mathcal{B})$ is a nuclear operator of order* 0*, then* "tr"(N) *is unique, this means that it is independent of representations, and we have a well-defined trace formula*

$$\mathrm{tr}(N) = \sum_{n=1}^{\infty} \lambda_n x_n'(y_n) = \sum_{\rho \in \mathrm{sp}(N)} \rho$$

and a well-defined determinant

$$\det(I - zN) = \exp\Big(- \sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(N^n)\Big) = \prod_{\tau \in \mathrm{sp}(N)} (1 - \tau z),$$

where $\mathrm{sp}(N)$ *means the spectrum of* N*. Moreover,* $\det(I - zN)$ *is an entire function of* $\mathbb{C}$*.*

Remark 11.1. *To prove that* $\det(I - zN)$ *is entire, Grothendick proved that*

$$\det(I - zN) = \sum_{n=0}^{\infty} (-z)^n \bigwedge^n N$$

where $\bigwedge^n N$ *means the* n*-fold exterior product of the linear operator* N*. Then following the Hadamard Inequality, he showed that*

$$|\mathrm{tr}(\bigwedge^n N)| \leq C n^{-\frac{n}{2}} = C e^{-\frac{n \log n}{2}}.$$

So det$(I - zN)$ *is an entire function.*

Now let us look at an example of nuclear operators.

Let U and V be two bounded Jordan domains in the complex plane such that $\overline{U}$ is compact and $U \subset \overline{U} \subset V$. Suppose $C^{\omega}(V)$ (resp. $C^{\omega}(U)$) is the space of all complex analytic functions on U (resp. V) and suppose $C^0(\overline{V})$ (resp. $C^{\omega}(\overline{U})$) is the space of all complex valued continuous functions on $\overline{U}$ (resp. $\overline{V}$). Let $A(U) = C^{\omega}(U) \cap C^0(\overline{U})$ and $A(V) = C^{\omega}(V) \cap C^0(\overline{V})$. Define the restriction operator $R_U : A(V) \to A(U)$ by

$$R_U(\phi) = \phi|_U, \qquad \forall \phi \in A(V).$$

Example 11.1. *The restriction operator* $R_U \in \mathcal{L}(A(V), A(U))$ *is a nuclear operator of order* 0*.*

Proof. Let $D_r = \{z \in \mathbb{C}; ||z|| < r\}$ for any $0 < r \leq 1$. First let us consider a special case that $U = D_r$ for some $0 < r < 1$ and $V = D_1$, we want to prove that $R_r = R_{D_r} : A(D_1) \to A(D_1)$ is nuclear of order 0.

For any ϕ in $A(D_1)$, we have

$$\phi(z) = \oint_{\partial D_1} \frac{\phi(u)}{u-z} \cdot \frac{du}{2\pi i} = \sum_{n=1}^{\infty} \oint_{\partial D_1} z^{n-1} \frac{\phi(u)}{u^n} \frac{du}{2\pi i} = \sum_{n=1}^{\infty} \lambda_n x'_n(\phi) \otimes y_n(z)$$

where $y_n(z) = (z/r)^{n-1}$, $\lambda_n = r^{n-1}$ and

$$x'_n(\phi) = \oint_{\partial D_1} \frac{\phi(u)}{u^n} \cdot \frac{du}{2\pi i}.$$

Notice that $||x'_n||_{A(D_1)} = 1$ and $y_n = (z/r)^{n-1}$ is a function defined on D_r with $||y_n(z)||_{A(D_r)} = 1$ and that $\sum_{n=1}^{\infty} r^{p(n-1)} < \infty$ for all $0 < p \leq 1$. So, the operator

$$R_r = \sum_{n=1}^{\infty} \lambda_n x'_n \otimes y_n(z)$$

is nuclear of order 0.

Return to the general case. Since U and V are Jordan domains, let $f : D_1 \to V$ be a conformal mapping (Riemann mapping). Then f can be continuously extended to $\overline{D}_1$. Let $f^*\phi = \phi \circ f$ for $\phi \in A(V)$. Let $0 < r < 1$ such that $f^{-1}(U) \subset D_r$ and let $g : D_r \to U$ be another conformal mapping (Riemann mapping). Also g can be continuously extended to $\overline{D}_r$. Let $g_*\phi = \phi \circ g^{-1}$ for $\phi \in A(D_r)$. Let $(g \circ f^{-1})^*\phi(z) = \phi(g \circ f^{-1}(z))$ for $\phi \in A(U)$. Then

$$R_U = (g \circ f^{-1})^* \circ g_* \circ R_r \circ f^*.$$

Since R_{D_r} is nuclear of order 0 and $(g \circ f^{-1})^*$, g_*, and f^* are all bounded, R_U is nuclear of order 0. □

12 Nuclear transfer operators and dynamical determinants

Consider two complex numbers c, θ, $|\theta| < 1$. Define an operator $N_{\theta,c} : A(D_1) \to A(D_1)$ by

$$N_{\theta,c}\phi(w) = c\phi(\theta w), \qquad \phi \in A(D_1).$$

Theorem 12.1. *The operator $N_{\theta,c}$ is nuclear of order* 0.

Proof. Consider the restriction operator $R_{D_{|\theta|}} : A(D_1) \to A(D_{|\theta|})$ which is nuclear of order 0 (Example 11.1), and the bounded operator $\tilde{N} : A(D_{|\theta|}) \to A(D_1)$ defined by

$$\tilde{N}\phi(w) = c\phi(\theta w), \qquad \phi \in A(D_{|\theta|}).$$

Then, by Theorem 11.1, the composition

$$N_{\theta,c} = \tilde{N} \circ R_{D_{|\theta|}}.$$

is nuclear of order 0. □

How to calculate the trace $\mathrm{Tr}(N_{\theta,c})$? Consider the bases 1, w, w^2, $\cdots$, w^n, $\cdots$ of $A(D_1)$ and the Taylor expansion

$$\phi(w) = \sum_{n=0}^{\infty} a_n w^n$$

for ϕ in $A(D_1)$. Notice that $N_{\theta,c}(w^n) = c\theta^n w^n$ for $n = 0, 1, \cdots$. Thus $\{w^n\}_{n=0}^{\infty}$ are all eigenvectors of $N_{\theta,c}$ with eigenvalues $\{c\theta^n\}_{n=0}^{\infty}$. So the trace is equal to

$$\mathrm{tr}(N_{\theta,c}) = \sum_{n=0}^{\infty} c\theta^n = \frac{c}{1-\theta}.$$

It is clear that the determinant

$$\det(1 - zN_{\theta,c}) = \prod_{n=0}^{\infty}(1 - c\theta^n z)$$

is an entire function.

Let V and U are Jordan domains in the complex plane $\mathbb{C}$ with $\overline{U} \subset V$. Suppose $g : V \to U$ is conformal. Then g has a unique fixed point z_g which is attractive (following hyperbolic geometry). Let $\psi \in A(V)$. Define $N_{g,\psi} : A(V) \mapsto A(V)$ by

$$N_{g,\psi}\phi = \psi \circ g \cdot \phi \circ g, \qquad \phi \in A(V).$$

Lemma 12.1. *The operator $N_{g,\psi}$ is nuclear of order* 0 *and*

$$\mathrm{tr}(N_{g,\psi}) = \frac{\psi(z_g)}{1 - g'(z_g)}$$

Proof. The restriction operator $R_U : A(V) \to A(U)$ is nuclear of order 0. Since

$$N_{g,\psi} = \tilde{N} \circ R_U,$$

where

$$\tilde{N}\phi = \psi \circ g \cdot \phi \circ g, \qquad \phi \in A(V).$$

is bounded, N is nuclear of order 0.

For any $\phi \in A(V)$,

$$N_{g,\psi}\phi(w) = \oint_{\partial V} \phi(z) \frac{g(z)}{z - g(w)} \frac{dz}{2\pi i}.$$

Since for $w \in \overline{U}$ and $z \in \partial V$, $z - g(w) \neq 0$,

$$\mathrm{tr}(N_{g,\psi}) = \oint_{\partial V} \frac{g(z)}{z - f(z)} \frac{dz}{2\pi i} = \frac{\psi(z_g)}{1 - g'(z_g)}.$$

□

Remark 12.1. *In general if $K(w, z)$ is a kernel function such that*

$$N\phi(w) = \oint_{\partial V} K(w, z)\phi(z)dz : A(V) \to A(V)$$

is an integral operator (called the standard Fredholm operator). The trace of N can be calculated by $\mathrm{tr}(N) = \oint_{\partial V} K(z, z)dz$ (see [DS]).

Finally let us consider some conformal dynamics and its dynamical determinants.

Suppose that V is a Jordan domain and $U_1, \cdots, U_d$ are sub-Jordan domains of V such that

$$\bigcup_{k=1}^{d} \overline{U}_k \subset V \quad \text{and} \quad U_i \bigcap U_j = \emptyset, \quad 1 \leq i \neq j \leq d.$$

Consider the dynamics $f : \bigcup_{k=1}^{d} U_k \to V$. Suppose that each $f|U_k : U_k \to V$ is conformal. Let $\psi \in A(V)$. The transfer operator $\mathcal{L}$ for f with weight ψ is defined as

$$\mathcal{L}\phi(z) = \sum_{w \in f^{-1}(z)} \psi(w)\phi(w), \quad \phi \in A(V).$$

Let $g_k = (f|U_k)^{-1}$ and z_k be the unique fixed point of g_k in U_k, $1 \leq k \leq d$. Define $N_k : A(V) \to A(V)$ by

$$N_k \phi = \psi \circ g_k \cdot \phi \circ g_k.$$

Then we have

$$\mathcal{L} = \sum_{k=1}^{d} N_k.$$

Since each N_k is nuclear of order 0, so is $\mathcal{L}$ and

$$\mathrm{tr}(\mathcal{L}) = \sum_{k=1}^{d} \mathrm{tr}(N_k) = \sum_{k=1}^{d} \frac{\psi(z_k)}{1 - g_k'(z_k)}.$$

Let $w_n = i_0 i_1 \cdots i_0$ be a sequence of 1's, $\cdots$, d's of length n. Let

$$N_{w_n} = N_{i_{n-1}} \circ \cdots \circ N_{i_0}.$$

Define

$$\psi_{w_n} = \psi \circ g_{i_0} \cdot \psi \circ g_{i_0} \circ g_{i_1} \cdots \psi \circ g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{n-1}}$$

and

$$g_{w_n} = g_{i_0} \circ g_1 \circ \cdots \circ g_{i_{n-1}}.$$

We have

$$N_{w_n} \phi = \psi_{w_n} \cdot \phi \circ f_{w_n}.$$

Furthermore,

$$\mathcal{L}^n = \sum_{w_n} N_{w_n}$$

and

$$\mathrm{tr}(\mathcal{L}^n) = \sum_{w_n} \mathrm{tr}(N_{w_n}),$$

where the summations are over all sequence of 1's, $\cdots$, d's of length n. Let z_{w_n} be the unique fixed point of g_{w_n} in $U_{w_n} = g_{w_n}(V)$. Then we have

$$\mathrm{tr}(N_{w_n}) = \frac{\psi_{w_n}(z_{w_n})}{1 - g'_{w_n}(z_{w_n})}.$$

Theorem 12.2 (Ruelle). *The operator $\mathcal{L}$ is nuclear of order* 0 *and the determinant*

$$\det(I - z\mathcal{L}) = \exp\Big(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{w_n} \frac{\psi_{w_n}(z_{w_n})}{1 - g'(z_{w_n})}\Big) = \prod_{\tau \in \mathrm{sp}(\mathcal{L})} (1 - z\tau)$$

is an entire function and the roots of $\det(I - \lambda\mathcal{L})$ *are exactly one over the eigenvalues of* $\mathcal{L}$ *(with the same multiply), where* $\mathrm{sp}(\mathcal{L})$ *means the spectrum of* $\mathcal{L}$.

Proof. Because

$$\det(I - z\mathcal{L}) = \exp(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathcal{L}^n))$$

and

$$\mathrm{tr}(\mathcal{L}^n) = \sum_{w_n} \frac{\psi_{w_n}(z_{w_n})}{1 - g'(z_{w_n})}.$$

Now from Theorem 11.2, we have that

$$\det(I - z\mathcal{L}) = \exp\Big(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{w_n} \frac{\psi_{w_n}(z_{w_n})}{1 - g'(z_{w_n})}\Big) = \prod_{\tau \in \mathrm{sp}(\mathcal{L})} (1 - \tau z)$$

is an entire function. It is clear from the last formula that the roots of $\det(I - z\mathcal{L})$ are exactly one over the eigenvalues of $\mathcal{L}$ (with the same multiply). □

13 Dynamical Zeta functions

We consider the same dynamics in the last section. Recall that V, U_1, $\cdots$, U_d are Jordan domains such that

$$\cup_{k=1}^{d} \overline{U}_k \subset V \quad \text{and} \quad U_i \cap U_j = \emptyset, \quad 1 \leq i \neq j \leq d.$$

The dynamics $f : \cup_{k=1}^{d} U_k \to V$ is defined so that and each $f|U_k : U_k \to V$ is conformal.

Let

$$\Lambda = \bigcap_{n=0}^{\infty} f^{-n}(V).$$

Let $C^0(\Lambda)$ be the space of all continuous functions $\phi : \Lambda \to \mathbb{C}$. Suppose ψ is in $C^0(\Lambda)$. For any $n > 0$, let

$$G_n(x) = \prod_{k=0}^{n-1} \psi(f^k(x)), \quad x \in \Lambda.$$

Remember that $g_k = (f|U_k)^{-1}$ and that for $w_n = i_0 i_1 \cdots i_{n-1}$, a sequence of 1's, 2's, $\cdots$, d's of length n,

$$g_{w_n} = g_{i_0} g_{i_1} \cdots g_{i_{n-1}} : V \to U_{w_n} = g_{w_n}(V).$$

The map g_{w_n} has a unique fixed point z_{w_n} in $\Lambda_{w_n} = \Lambda \cap U_{w_n}$ because it is a contracting map. For each $n > 0$, define

$$Z_n(\psi) = \sum_{w_n} G_n(z_{w_n})$$

where w_n runs over all sequences of 1's, 2's, $\cdots$, d's of length n. The functional $Z_n(\psi) : \mathcal{C}^0(\Lambda) \to \mathbb{C}$ is called a partition function. The zeta function of (f, ψ) is defined as

$$\xi(z, \psi) = \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\psi) \Big).$$

The number

$$P(\psi) = \limsup_{n \to \infty} \frac{1}{n} \log |Z_n(\psi)|$$

is called the *topological pressure* of the observable $\log \psi$. The convergence radius of $\xi(z, \psi)$ is e^P.

Remark 13.1. *If $\psi > 0$, we can define a measure μ_n on Λ as*

$$d\mu_n = \frac{G_n(z_{w_n})}{Z_n(\psi)} \quad on \quad \Lambda_{w_n}.$$

All weak limits of $\{\mu_n\}$ will be f-invariant probability measures on Λ and are called Gibbs measures for (f, ψ) (refer to §6).

Now let us see some examples to calculate zeta functions.

Example 13.1. *Suppose $\psi = 1$. Then $Z_n(1) = \#(Fix(f^n)) = d^n$. Let A be the $d \times d$ matrix with all entries 1. Then it is easy to see $Z_n(1) = \mathrm{tr}(A^n)$. So*

$$\xi(z,\psi) = \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n}\mathrm{tr}(A^n)\Big) = \frac{1}{\det(I - zA)}$$

where I means the $d \times d$ identity matrix. The pressure is

$$P = \lim_{n\to\infty} \frac{1}{n} \log Z_n(1) = \log d.$$

Example 13.2. *Suppose ψ is a function only depending on $i_0 i_1$, that is,*

$$\psi(x) = a_{i_0 i_1}, \quad x \in \Lambda_{i_0 i_1}$$

Let $A = (a_{i_0 i_1})$ be the $d \times d$-matrix whose $i_0 i_1$-entry is $a_{i_0 i_1}$. One can see easily that $Z_1(\psi) = \sum_{i=1}^{d} a_{ii} = \mathrm{tr}(L)$ and $Z_2(\psi) = \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}a_{ji} = \mathrm{tr}(A^2)$. In general, $Z_n = \mathrm{tr}(A^n)$. Therefore,

$$\xi(z,\psi) = \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n}\mathrm{tr}(A^n)\Big) = \frac{1}{\det(I - zA)}.$$

Now let us calculate the zeta function for a function $\psi \in A(V)$.

Theorem 13.1 (Ruelle). *Suppose $\psi \in A(V)$. Then the zeta function $\xi(z,\psi)$ can be extended to a meromorphic function of z and its pole are exactly the inverses of all eigenvalues of the transfer operator $\mathcal{L}_\psi$ (counted by multiplicity).*

Proof. The transfer operator $\mathcal{L} = \mathcal{L}_\psi$ can be written as

$$\mathcal{L}\phi(z) = \sum_{i=1}^{d} \psi(g_k(z))\phi(g_k(z)).$$

Furthermore, for any $n \geq 1$, we have

$$\mathcal{L}^n\phi(z) = \sum_{w_n} \psi_{w_n}(z)\phi(g_{w_n}(z)),$$

where $w_n i_0 i_1 \cdots i_{n-1}$ runs over all strings of 1's, $\cdots$, d's of length n and

$$g_{w_n} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{n-1}}$$

and

$$\psi_{w_n} = \psi \circ g_{i_0} \cdot \psi \circ g_{i_0} \circ g_{i_1} \cdot \cdots \cdot \psi \circ g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{n-1}}.$$

As we already know, $\mathcal{L}^n$ is nuclear of order 0 and its trace is

$$\mathrm{tr}(\mathcal{L}^n) = \sum_{w_n} \frac{\psi_{w_n}(z_{w_n})}{1 - g'_{w_n}(z_{w_n})}.$$

The difference between $\mathrm{tr}(\mathcal{L}^n)$ and $Z_n(\psi)$ is the factor $1 - g'_{w_n}(z_{w_n})$. However, it is quite easy to remove the factor as follows. Define $\mathcal{L}_{(0)} = \mathcal{L}$ and

$$\mathcal{L}_{(1)}\phi(z) = \sum_{i=1}^{d} \psi(g_k(z)) g'_k(z) \phi(g_k(z)).$$

Then

$$\mathcal{L}^n_{(1)}\phi(z) = \sum_{w_n} \psi_{w_n}(z) g'_{w_n}(z) \phi(g_{w_n}(z))$$

and

$$\mathrm{tr}\Big(\mathcal{L}^n_{(1)}\Big) = \sum_{w_n} \frac{\psi_{w_n}(z_{w_n}) g'_{w_n}(z_{w_n})}{1 - g'_{w_n}(z_{w_n})}.$$

So

$$Z_n(\psi) = \mathrm{tr}\Big(\mathcal{L}^n_{(0)}\Big) - \mathrm{tr}\Big(\mathcal{L}^n_{(1)}\Big).$$

This gives us that

$$\begin{aligned} \xi(z,\psi) &= \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n}\Big(\mathrm{tr}\Big(\mathcal{L}^n_{(0)}\Big) - \mathrm{tr}\Big(\mathcal{L}^n_{(1)}\Big)\Big)\Big) \\ &= \frac{\exp\Big(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\Big(\mathcal{L}^n_{(1)}\Big)\Big)}{\exp\Big(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\Big(\mathcal{L}^n_{(0)}\Big)\Big)} \\ &= \frac{\det(I - z\mathcal{L}_{(1)})}{\det(I - z\mathcal{L}_{(0)})}. \end{aligned}$$

Since $\det(I - z\mathcal{L}_{(1)})$ and $\mathrm{Det}(I - z\mathcal{L}_{(0)})$ are both holomorphic functions of z, $\xi(z,\psi)$ is a meromorphic function of z. It is clearly the poles of $\xi(z,\psi)$ are exactly one over the eigenvalues of $\mathcal{L}$ (counted by multiplicity). □

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Aihua FAN: LAMFA CNRS UMR 6140, Faculté de Mathématiques et Informatique, Université de Picardie, 39 Rue Saint Leu, 80039 Amiens, France & Department of Mathematics, Wuhan University, 420073 Wuhan, China

Email: ai-hua.fan@u-picardie.fr

Yunping JIANG: Department of Mathematics, Queens College of CUNY, Flushing, NY 11367 and Department of Mathematics, CUNY Graduate School, 365 Fifth Avenue, New York, NY 10016 and the 100 Man Program in the Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, P. R. China

Email: yunqc@forbin.qc.edu