On geometrically finite branched covering*
III. A direct proof of CJS's theorem†

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Abstract

We studied the rational realization problem for sub-hyperbolic semi-rational branched coverings. By using the shielding ring lemma, we are able to give a direct proof of CJS's Theorem following the lines of the proof of Thurston's Theorem given in the paper of Douady-Hubbard.

Contents

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1 Introduction

Thurston's Theorem in complex dynamical systems gives a topological description of a critically finite rational map. A proof of this theorem is an interesting application of the finite dimensional Teichmüller theory. The reader may refer to Douady and Hubbard's paper for this proof or Appendix to get some idea about the proof. The statement of Thurston's Theorem can be summarized as follows: A critically finite branched covering of the two-sphere associated with a hyperbolic orbifold is combinatorially equivalent to a unique rational map (up to conformal conjugation) if and only if it has no Thurston's obstruction. McMullen [Mc] further showed that no Thurston's obstruction is essentially true for any rational map with a hyperbolic orbifold–only trivial Thurston obstructions inside Siegel disks or Herman rings may occur for a rational map with a hyperbolic orbifold (see [CJS1] in this proceedings for a precise statement of McMullen's Theorem). So a basic problem is whether a branched covering having no essential Thurston obstruction is combinatorially equivalent to a rational map?

Cui, Jiang, and Sullivan studied this problem in 1994 for a geometrically finite branched covering. Local combinatorial structures, like combinatorial contraction and combinatorially invariant shrinking family of curves, have been studied in [CJS1]. Furthermore, a counter-example of a geometrically finite branched covering is constructed in [CJS1] by using this study. The example has no Thurston obstruction but is not combinatorially equivalent to a rational map. Therefore a combinatorially contracting property is introduced into the further study.

Following the study in [CJS1], a class of semi-rational branched coverings are introduced in [CJS2]. More precisely, a semi-rational branched covering is a geometrically finite branched covering whose periodic accumulation points of post-critical orbits can be combinatorially normalized into super-attracting, or attracting, or parabolic periodic points. Furthermore, they proved that a semi-rational branched covering is combinatorially equivalent to a rational map if and only if it has no Thurston obstruction. However, the uniqueness in the geometrically finite case is quite different from the critically finite case. By using combinatorially invariant shrinking family of curves, it is proved in [CJS1] that a semi-rational branched covering is always combinatorially equivalent to a sub-hyperbolic semi-rational branched covering, which is a semi-rational branched covering whose periodic accumulation points of post-critical orbits are only super-attractive or attractive. Therefore, to study the rational realization of semi-rational branched covering, we only need to study the class of sub-hyperbolic semi-rational branched coverings. In order to have a well-defined rational realization problem in the class of semi-rational sub-hyperbolic branched coverings, a CLH-equivalence is introduced in [CJS2] (see also §2). The theorem proved in [CJS2] is that a sub-hyperbolic semi-rational branched covering is CLH-equivalent to a unique rational map (up to conformal conjugation) if and only if it has no Thurston obstruction. The proof given in [CJS2] is by the study of combinatorial properties of a sub-hyperbolic semi-rational branched covering. An interesting point in the proof is to use a similar partition technique like those used in Branner-Hubbard's work [BH] in the study of cubic polynomials and in Yoccoz' work (refer to [Hu, Ji1]) in the study of quadratic polynomials.

In this paper, we only concentrate in sub-hyperbolic semi-rational branched coverings defined in [CJS2] and show a direct proof of CJS's Theorem following lines of the proof of Thurston's Theorem given in Douady and Hubbard's paper [DH].

An essential difference between the studies of critically finite case and geometrically finite case is the finite dimensional Teichmüller theory and the infinite dimensional Teichmüller theory. Later one makes the study of the problem harder. However, in both cases, following bounded geometry properties should be studied first. These bounded geometry properties enable us to have a similar study in both cases.

In the critically finite case, the base point of the Teichmüller space is the Riemann sphere minus finite points. The branched covering induces an operator on the Teichmüller space. Iterations of the operator produce a sequence of sets of finite points in the Riemann sphere. The bounded geometry in this case means that the sets of points move with a bounded distance in the Riemann sphere (refer to [DH, Pi]).

In the geometrically finite case, the base point of the Teichmüller space is the Riemann sphere minus infinite points. The branched covering induces an operator on the Teichmüller space too. Iterations of the operator produce a sequence of sets of infinite points in the Riemann sphere. In order to have a similar bounded geometry concept like that in the critically finite case, we need to find a holomorphic disk about each accumulation point. Note that we only need to find finitely many such disks. These holomorphic disks should keep a definite size under iterations of the operator. More importantly, there should have an annulus with a definite modulus around each holomorphic disk to keep away the rest of finitely many points from these holomorphic disks. To have this property, we prove our shielding ring lemma (see §2). After having this lemma, the problem essentially becomes a problem in the finite dimensional Teichmüller theory. So we can just follow main lines of the proof of Thurston's Theorem given in Douady-Hubbard paper [DH].

The article is organized as follows. In §2, we review the definition of sub-hyperbolic semi-rational maps and prove the shield ring lemma. We also state the main theorem in §2. In §3, we define the Thurston pull back operator on the Teichmüller space and push forward operator on quadratic differentials. We also define bounded geometry for Riemman surfaces, which are the Riemann sphere munius finite number of disks and points. We then give our direct proof of CJS's Theorem. To make the paper a self contained, we write down main ideas of a proof of Thurston's theorem in Appendix.

Further result and study. The shielding ring lemma is shown by the first author(YJ) and then the rest is finished by the second author(GZ) under supervision of the first author(YJ). So it is put in the second author(GZ)'s thesis [Zh] as Chapter 2. The origination of this research is started in the preparation of a suitable and workable thesis problem for the second author (GZ) by the first author (YJ). During this preparation, the first author (YJ) formulates a conjecture that a branched covering of the two sphere without essential Thurston's obstruction and linearizable at Siegel disks and Herman rings with bounded type rotation numbers and with finite number post-critical points outside Siegel disks and Herman rings is CLH (combinatorially and local holomorphically) equivalent (see §3) to a rational map and unique up to holomorphic conjugation. In this conjecture, one need assume that there are finitely many branch points on the boundary of each Siegel disks or each Herman rings. Our successful story is that the conjecture is solved partially for a subclass of simple Siegel disk type topological polynomials which forms the main body of the second author(GZ)'s thesis [Zh]. A simple Siegel disk type topological polynomials is a branch covering in the conjecture with a completely invariant point ∞ and only one linearizable Siegel disk with bounded type rotation number. Many other problems like the measure and local connectivity of the Julia set of a simple Siegel disk type polynomial are also studied from the combinatorial equivalence point of view. The further study of this conjecture is still underway and remains an interesting problem for us. During the formulation of the conjecture mentioned in the last paragraph. The first author(YJ) had some discussion with Professor Linda Keen. He would like very much to express his thanks for her valuable advises. During this research, both authors got

great help from Professors Guizhen Cui, Fred Gardiner, and Linda Keen, they would like to express their thanks to them.

2 Sub-hyperbolic semi-rational maps

Suppose S^2 is the two-sphere. Let f be an orientation preserving branched covering. Then

$$\Omega_f = \{x \in S^2 \mid \deg_x f > 1\}$$

is called the set of branched points, where $deg_x f$ means the local degree of f at x. The set

$$P_f = \cup_{n\geq 1}^{\infty} f^n(\Omega_f)$$

is called the post-critical set of f. The map f is called geometrically finite if P_f has only finitely many accumulation points. In this case, every accumulation point of P_f is periodic. The reason is the following. Let $P_f' = \{p_1, \cdots, p_m\}$ be all accumulation points. Then it is clear that $f(P_f') \subseteq P_f'$. So every point is eventually periodic, i.e, for every $p \in P_f'$, there are minimal integers $l \geq 0$ and $k \geq 1$ such that $f^{l+k}(p) = f^l(p)$. If $l = 0$, then p is periodic. So we only need to prove that $l = 0$ for every $p \in P_f'$. Suppose there is a $p \in P_f'$ such that $l > 0$. Assume $p_i = f^{i-1}(p)$, $1 \leq i \leq l+k$. Then $O = \{p_{l+1}, \cdots, p_{l+k}\}$ is a periodic cycle. So we can find a small number $\epsilon > 0$ such that $f(B_\epsilon(x)) \cap B_\epsilon(y) = \emptyset$ for all $x \in O$ and $y \in P_f' \setminus O$, where $B_\epsilon(\cdot)$ means the disk of radius ϵ and centered $\cdot$. Now let $c \in \Omega_f$ such that $f^{n_k}(c) \to p = p_1$ for a subsequence $\{n_k\}$. Then $f^{n_k+l}(c) \to p_{l+1}$. Since $\{f^i(c)\}_{i=M}^{\infty} \subset \cup_{x\in\Omega_f} B_\epsilon(x)$ for some $M > 0$, $\{f^i(c)\}_{i=N}^{\infty} \subset \cup_{x\in O} B_\epsilon(x)$ for some $N \geq M$. This contradicts with $f^{n_k}(c) \to p$.

Let $\mathbb{P}^1$ be the standard Riemann sphere. Suppose f is geometrically finite and suppose $p \in P_f'$. Let $k \geq 1$ be the period of f at p. Then we say f is combinatorially contracting at p if there exist homeomorphisms $\phi, \psi : S^2 \to \mathbb{P}^1$ such that

- $\phi(p) = 0$,
- ϕ is isotopic to ψ rel P_f, and
- $(\phi \circ f \circ \psi^{-1})^{\circ k}(z) = \lambda z$ for some $|\lambda| < 1$ if $deg_a f^{\circ p} = 1$ or $(\phi \circ f \circ \psi^{-1})^{\circ k}(z) = z^d$ if $deg_p f^{\circ k} = d > 1$.

In[CJS1, CJS2], the following theorem is proved. See Appendix for definitions of Thurston obstruction and combinatorial equivalence.

Theorem 1 (Cui-Jiang-Sullivan's Theorem). *A geometrically finite branched covering f is combinatorially equivalent to a rational map if and only if f has no Thurston obstruction and f is combinatorially contracting at every accumulation point of P_f.*

From the study in [CJS1], in the study of the "if" part of the above theorem, we can assume the combinatorially contracting condition. So a class of semi-rational maps and a class of sub-hyperbolic semi-rational maps are defined in [CJS2] as follows. A geometrically finite branched covering f is called semi-rational if

- f is holomorphic in a neighborhood of the set of accumulation points P'_f;
- each cycle in P'_f is either attractive or super-attractive or parabolic; and
- each attracting petal associated with a parabolic cycle in P'_f contains a post-critical point.

Furthermore, if all cycles in P'_f are either attractive or super-attractive, then f is called a sub-hyperbolic semi-rational map. It is shown in [CJS1] that a semi-rational branched covering is always combinatorially equivalent to a sub-hyperbolic semi-rational branched covering. So the rest proof of the "if" part of Theorem 1 is just for sub-hyperbolic semi-rational branched coverings.

Suppose f is a sub-hyperbolic semi-rational branched covering. For each point $z \in P'_f$, let $k \geq 1$ be the period of f at z, there is an open disk $D(z)$ centered z such that

$$f^k(w) = z + \lambda(w - z) + h.o.t, \quad w \in D(z)$$

for some λ with $0 < |\lambda| < 1$ or

$$f^k(w) = z + (w - z)^n + h.o.t$$

for some $n \geq 2$, where h.o.t. means the higher order terms. By making each $D(z)$ small enough, we can assume $D(z)$ and $D(z')$ are disjoint for $z \neq z' \in P'_f$. Moreover, by taking $D(z)$ smaller, we assume that $D(z)\setminus\{z\}$ contains no critical value. Let $P'_f = \{z_i\}$ and let $\{D_i\}$ be a collection of corresponding disks.

Lemma 1 (Shielding Ring Lemma). *There is a collection* $\{D_i\}$ *such that for each* D_i *there is an annulus* A_i *attaching to it which is mapped into some* D_j *by* f *and which contains no post-critical point.*

Proof. Start with a collection of disks $\{D_i\}$ satisfying $\overline{f^{k_i}(D_i)} \subset D_i$ where k_i is the period of the point in $D_i \cap P'_f$. Let $z_1 \in P'_f$ be a point with period $k > 0$. Suppose $\{z_{i+1} = f^i(z_1)\}_{i=0}^{k-1}$ is a periodic cycle. Let D be the disk in the collection containing z_1. Then $\overline{f^k(D)} \subset D$. Take an open topological disk B with

$$\overline{f^k(D)} \subset B \subset D \quad \text{and} \quad (\overline{B} \setminus f(D)) \cap P'_f = \emptyset.$$

Divide $\overline{B} \setminus f^k(D)$ into k sub-annuli

$$E_1, \cdots, E_k$$

such that E_1 attaches to $f^k(D)$ and E_{i+1} attaches to E_i for $i = 1, \cdots, k-1$.

Let $E_0 = \emptyset$. The new disk, which we still denote as D_j, about z_j is

$$D_j = f^{j-1}(f^{\circ k}(D) \cup (\cup_{0 \leq i \leq j-1} E_i)).$$

The annulus A_j attaching at D_j is

$$A_j = f^{\circ j}(E_j).$$

The above construction works for every periodic cycle. Then the new collection of disks satisfies the lemma. □

A collection of disks $\{D_i\}$ satisfying Lemma 1 are called standard normal disks and corresponding annuli $\{A_i\}$ are called shielding rings. Henceforth, we will assume that f is a sub-hyperbolic semi-rational branched covering with a fixed collection of standard normal disks $\{D_i\}$ and a fixed collection of shielding rings $\{A_i\}$. In order to have a well-understood problem for the rational realization of sub-hyperbolic semi-rational branched covering maps, the CLH-equivalent is defined in [CJS1, CJS2]. (CLH means combinatorially and locally holomorphically.) Two sub-hyperbolic semi-rational maps f and g are CLH-equivalent if there is a pair of homeomorphisms (ϕ, ψ) of the Riemann sphere $\mathbb{P}^1$ such that

- ψ is homotopic to ϕ rel P_f,
- $\phi f = g \psi$,

- $\phi|U_f = \psi|U_f$ is holomorphic on some open set $U_f \supset P'_f$.

After having the above result and notations, we are ready to give a direct proof of the following theorem in [CJS2].

Theorem 2 (Cui-Jiang-Sullivan's Theorem). *Suppose f is a sub-hyperbolic semi-rational map with P'_f non-empty. Then f is CLH-equivalent to a rational map R if and only if f has no Thurston obstruction. Moreover, the rational map R is unique up to holomorphic conjugation.*

3 Thurston pull back operator on Beltrami coefficients and push forward operator on quadratic differentials

We keep using the notations in §2. Let $P'_f = \{z_1, \cdots, z_m\}$ and $U_f = \cup_{i=1}^m D_i$. Denote

$$P_1 = P_f \setminus U_f.$$

Then P_1 is a set containing only finite number of points. Let

$$X = \partial U_f \cup P_1$$

Let $\mathcal{M}$ be the set of all the Beltrami coefficients on $\mathbb{P}^1$, which consists of all measurable functions μ defined on the Riemann sphere with $||\mu||_\infty < 1$. Consider the subset $\mathcal{M}_f \subset \mathcal{M}$,

$$\mathcal{M}_f = \{\mu \in \mathcal{M} \mid \mu|U_f = 0\}.$$

For each $\mu \in \mathcal{M}_f$, there is a unique quasi-conformal homeomorphism $\phi_\mu : \mathbb{P}^1 \to \mathbb{P}^1$ fixing $0, 1$, and ∞ and having complex dilation μ. Without loss of the generality, we can assume that $0, 1$, and ∞ belong to P_f. Two elements $\mu, \nu \in \mathcal{M}_f$ are said to be equivalent, denoted as $\mu \sim \nu$, if ϕ_μ is isotopic to ϕ_ν rel X. Then the space of $M_f/\sim = \{[\mu]\}$ is the Teichmüller space, which we denote as $\mathcal{T}_f$, with the base point $(\mathbb{P}^1 \setminus U_f, X)$. The Thurston pull back operator associate with f is the operator $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ defined as $\sigma_f([\mu]) = [f^*\mu]$, where

$$f^*(\mu)(z) = \frac{\mu_f(z) + \mu((f(z))\theta(z)}{1 + \overline{\mu_f(z)}\mu(f(z))\theta(z)}$$

where $\mu_f = \frac{f_{\bar{z}}}{f_z}$ and $\theta(z) = \frac{\overline{f_z}}{f_z}$. Note that for $\mu \sim \nu \in \mathcal{M}_f$, then ϕ_μ is isotopic to ϕ_ν rel X and $\phi_\mu = \phi_\nu$ on U_f is analytic. Since $f(\overline{U_f}) \subset U_f$ and

f is analytic on U_f, $\phi_\mu \circ f$ is isotopic to $\phi_\nu \circ f$ rel X and $\phi_\mu \circ f = \phi_\nu \circ f$ on U_f are analytic. Since $f^*\mu$ and $f^*\nu$ are the Beltrami coefficients of $\phi_\mu \circ f$ and $\phi_\nu \circ f$, so $f^*\mu \sim f^*\nu \in \mathcal{M}_f$. Indeed, $\sigma_f([\mu]) \in \mathcal{T}_f$ for any $[\mu] \in \mathcal{T}_f$. Furthermore, if $\gamma(t)$ is a smooth curve in $\mathcal{T}_f$, then $\sigma_f(\gamma)(t)$ is also a smooth curve in $\mathcal{T}_f$.

Let $\gamma : [0,1] \to \mathcal{M}_f$ be a C^1 Beltrami path and let $\phi_t = \phi_{\gamma_t}$ be the corresponding path of quasi-conformal homeomorphisms. For any $0 \le \tau \le 1$, $\phi_*(\gamma)(t) = [\mu_{\phi_t \circ \phi_\tau^{-1}}]$ denotes the Beltrami path passing $\alpha_0 = [0]$ at τ. Let $\mu = \gamma(\tau)$ and let $\phi_\mu = \phi_\tau$ be the corresponding quasi-conformal homeomorphism. Use $\mathcal{T}_\mu$ to denote the Teichmüller space with the base point $(\mathbb{P}^1 \setminus \phi_\mu(U_f), \phi_\mu(X))$. Denote $\eta = \gamma'(\tau)$ and $\xi = d\phi_{\mu*}(\eta) = \frac{d}{dt}(\phi_{\mu*}(\gamma)(\tau)$. The norm of $\xi = \xi(z)\frac{d\bar{z}}{dz}$ is defined as

$$\|\xi\| = sup\Big|\int\int_{\mathbb{P}^1 \setminus \phi_\mu(U_f)} q(z)\xi(z) dz \wedge d\bar{z}\Big|$$

where the *sup* is taken over all the quadratic differentials $q = q(z)dz^2$ on $\mathbb{P}^1 \setminus \phi_\mu(U_f)$ such that

- All the poles of q are in $\phi_\mu(P_1)$ and simple,
- $\int\int_{\mathbb{P}^1 \setminus \phi_\mu(U_f)} |q(z)| dz \wedge d\bar{z} = 1$.

Since ϕ_μ induces an isometric mapping between $\mathcal{T}_f$ and $\mathcal{T}_\mu$, so the norm of η can be calculated as $\|\eta\| = \|\xi\|$.

Similarly, let $\nu = f^*(\mu)$, $\tilde{\eta} = df^*(\eta)$, and $\tilde{\xi} = d\phi_{\nu*}(\tilde{\eta})$, then we have $\|\tilde{\eta}\| = \|\tilde{\xi}\|$. Since the following diagram is commutative,

$$\begin{array}{ccc} (\mathbb{P}^1, f^{-1}(P_f)) & \xrightarrow{\phi_\nu} & (\mathbb{P}^1, g^{-1}(P_g)) \\ \downarrow f & & \downarrow g \\ (\mathbb{P}^1, P_f) & \xrightarrow{\phi_\mu} & (\mathbb{P}^1, P_g) \end{array}$$

We have $\tilde{\xi} = dg^*(\xi)$, where g is a rational map. We can use the following diagram to show the relation among η, $\tilde{\eta}$, ξ, and $\tilde{\xi}$.

$$\begin{array}{cccc} \tilde{\eta} & \xrightarrow{d\phi_{\nu*}} & \tilde{\xi} & \tilde{q} \\ \uparrow df^* & & \uparrow dg^* & \downarrow \mathcal{L} \\ \eta & \xrightarrow{d\phi_{\mu*}} & \xi & q \end{array}$$

We conclude the above as follows. For each quadratic differential $\tilde{q}(w)dw^2$ on $\mathbb{P}^1 \setminus \phi_\nu(U_f)$, there is a unique quadratic differential $q(z)dz^2$ on $\mathbb{P}^1 \setminus$

$\phi_\mu(U_f)$ such that

$$\int\int_{\mathbb{P}^1\setminus\phi_\mu(U_f)} \xi(z)q(z)dz\wedge d\overline{z} = \int\int_{\mathbb{P}^1\setminus\phi_\nu(U_f)} \tilde{\xi}(w)\tilde{q}(w)dw\wedge d\overline{w}. \qquad (1)$$

Actually, $q = \delta g^*\tilde{q} = \mathcal{L}\tilde{q}(z)dz^2$ is given by

$$\mathcal{L}\tilde{q}(z) = \sum_{g(w)=z} \frac{\tilde{q}(w)}{[g'(w)]^2}$$

distributively. Note that $\mathcal{L}$ is called a transfer operator (see [Ji2]). Here we call δg^* the push forward operator.

4 Bounded geometry and contracting principle

Theorem 2 is equivalent to the statement that σ_f has a unique fixed point in $\mathcal{T}_f$. The reason is the following. If σ_f has a unique fixed point $[\mu]$ in $\mathcal{T}_f$, then $\nu = f^*\mu \in [\mu]$. Let ϕ_μ and ϕ_ν be the corresponding quasi-conformal homeomorphisms. Then $\phi_\mu = \phi_\nu$ on U_f is analytic, ϕ_μ is homotopic to ϕ_ν rel P_f, and $g = \phi_\mu f \phi_\nu^{-1}$ is a rational map. Therefore, our purpose now is to find a unique fixed point of σ_f.

Let $\mu_0 = 0$ and $\mu_n = f^*\mu_{n-1}$. Starting from $\alpha_0 = [\mu_0]$, define $\alpha_n = [f^*\mu_{n-1}]$ for all integers $n \geq 1$. If we can show σ_f is strictly contracting on $\{\alpha_n\}$, then it has a unique fixed point because $\mathcal{T}_f$ is complete and σ_n is weakly contracting. The derivative $d\sigma_f$ maps η to $\tilde{\eta}$. So to show σ_f is strictly contracting, we need only to prove that there is a uniform number $0 < \delta < 1$ such that

$$\|\tilde{\eta}\| = \|d\sigma_f\eta\| < \delta\|\eta\| \qquad (2)$$

for all η. As we have seen that $\|\eta\| = \|\xi\|$ and $\|\tilde{\eta}\| = \|\tilde{\xi}\|$. So let $\tilde{q} = \tilde{q}(w)dw^2$ and $q = q(z)dz^2 = \delta g^*\tilde{q} = \mathcal{L}\tilde{q}(z)dz^2$ be the corresponding quadratic differentials in Equation 1. Then

$$\|q\| = \int\int_{\mathbb{P}^1\setminus\phi_\mu(U_f)} |q(z)|dz\wedge d\overline{z} = \int\int_{\mathbb{P}^1\setminus\phi_\nu(U_f)} \Big|\sum_{g(w)=z} \frac{\tilde{q}(w)}{[g'(w)]^2}\Big| dz\wedge d\overline{z}$$

$$\leq \int\int_{\mathbb{P}^1\setminus\left(\phi_\nu(U_f)\cup_i\phi_\nu(A_i)\right)} |\tilde{q}(w)|dw\wedge d\overline{w}$$

$$= \int\int_{\mathbb{P}^1\setminus\phi_\nu(U_f)} |\tilde{q}(w)|dw\wedge d\overline{w} - \int\int_{\cup_i\phi_\nu(A_i)} |\tilde{q}(w)|dw\wedge d\overline{w}$$

$$= \|\tilde{q}\| - \int\int_{\cup_i\phi_\nu(A_i)} |\tilde{q}(w)|dw\wedge d\overline{w}.$$

Assume $\|\tilde{q}\| = 1$. Then

$$\|q\| \leq 1 - \int\int_{\cup_i\phi_\nu(A_i)} |\tilde{q}(w)|dw\wedge d\overline{w}.$$

Thus Equation 2 holds if we can find a uniform $\epsilon > 0$ such that

$$\int\int_{\phi_\nu(\cup_i A_i)} |\tilde{q}(w)|dw\wedge d\overline{w} > \epsilon.$$

Let $\phi_n = \phi_{\mu_n}$ be the sequence of corresponding quasi-conformal homeomorphisms. Let $m_0 = \#(P_1)$.

Lemma 2 (Bounded Geometry). *Let $\Lambda = P_1 \cup \{D_i, A_i, 1 \leq i \leq m\}$. We say Λ is of bounded geometry if there exists a uniform $\delta > 0$ such that*

- *$d(\phi_n(z_i), \phi_n(A_j)), d(\phi_n(A_i), \phi_n(A_j)), d(\phi_n(z_i), \phi_n(z_j)) > \delta$ for $z_i \neq z_j \in P_1$ and $A_i \neq A_j \in \Lambda$ and*
- *$\phi_n(D_i)$ contains a disk with radius δ and mod$(\phi_n(A_i)) > \delta$ for all $A_i \in \Lambda$.*

We have that

Lemma 3 (Bounded Geometry Implies Contracting Principle). *Suppose Λ has bounded geometry. Then there exists a uniform $\epsilon > 0$, dependent only on m, m_0, δ, such that for any quadratic differential $q(z)dz^2$ on $\mathbb{P}^1\setminus\phi_n(U_f)$, with possible simple poles at some $\phi_n(z_i) \in \phi_n(P_1)$, satisfying*

$$\int\int_{\mathbb{P}^1\setminus\phi_n(U_f)} |q(z)|dz\wedge d\overline{z} = 1,$$

we have

$$\int\int_{\cup\phi_n(A_i)} |q(z)|dz\wedge d\overline{z} > \epsilon.$$

Proof. Without loss of generality, we assume $\infty \in D_1$. We prove the lemma by the contradiction. Suppose there is a sequence of quadratic differentials $\{q_n = q_n(z)dz^2\}_{n=1}^{\infty}$ with

$$q_n(z) = \sum_{1 \leqslant i \leqslant m_0} \frac{a_{i,n}}{z - \phi_n(z_i)} + g_n(z)$$

such that

- $\int\int_{\mathbb{P}^1 \setminus \phi_n(U_f)} |q_n(z)| dz \wedge d\overline{z} = 1$ for all $n \geq 1$,
- $\int\int_{\cup_j \phi_n(A_j)} |q_n(z)| dz \wedge d\overline{z} \to 0$ as $n \to \infty$, and
- $g_n(z)$ is holomorphic on $\mathbb{P}^1 \setminus \phi_n(U_f)$.

Since

$$\int\int_{\cup_i \phi_n(A_j)} |q_n(z)| dz \wedge d\overline{z} \to 0 \quad \text{as} \quad n \to \infty,$$

we can take $\Gamma_n = \{\gamma_{1,n}, \cdots, \gamma_{m,n}\}$ where $\gamma_{j,n} \subset \phi_n(A_j)$ is a simple closed curve homotopic to the boundary of $\phi_n(A_j)$ such that

$$\int_{\Gamma_n} |q_n(z)| dz \to 0 \quad \text{as} \quad n \to \infty.$$

Since ϕ_n on every A_j is holomorphic, the modulus of $\phi_n(A_j)$ is unchanged with n. From the bounded geometry, each D_j contains a disk with a fixed radius and A_j attaches at D_j, A_j has a fixed thickness. By taking a subsequence if necessary, we can assume $\phi_n(D_j)$ and $\phi_n(A_j)$ converges to a disk E_j and an annulus B_j in the Carathéodory topology such that every E_j contains a disk with a fixed radius and $\mathrm{mod}(B_j) = \mathrm{mod}(A_j)$. Also we can assume every $\gamma_{j,n}$ converges to γ_j in the Carathéodory topology. Let $\Gamma = \{\gamma_j\}$.

From the bounded geometry we can assume by take a subsequence $z_{i,n} = \phi_n(z_i) \in \phi_n(P_1)$ converges to $z_{i,\infty}$ for every i. So we get a limiting Riemann surface $\mathbb{P}^1 \setminus (U_{f,\infty} \cup P_{1,\infty})$ where $U_{f,\infty} = \cup_j E_j$ and $P_{1,\infty} = \{z_i\}$.

For any $z \in \mathbb{P}^1 \setminus \phi_n(U_f \cup P_1)$,

$$g_n(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{g_n(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{q_n(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \sum_{1 \leq i \leq m_0} \int_{\Gamma_n} \frac{a_{i,n}}{(\xi - z_{i,n})(\xi - z)} d\xi$$

$$= \frac{1}{2\pi i} \int_{\Gamma_n} \frac{q_n(\xi)}{\xi - z} d\xi$$

since $\int_{\Gamma_n} \frac{a_{i,n}}{(\xi - z_{i,n})(\xi - z)} d\xi = 0$. Therefore, $g_n(z) \to 0$ as $n \to \infty$ on any compact set in $\mathbb{P}^1 \setminus (U_{f,\infty} \cup P_{1,\infty})$. This implies that

$$\int_{\Gamma_n} \| \sum_{1 \leqslant i \leqslant m_0} \frac{a_{i,n}}{z - z_{i,n}} \| dz \to 0, \quad \text{as} \quad n \to \infty.$$

We claim that $a_{i,n} \to 0$ as $n \to \infty$. Let $a_n = \max_i \{|a_{i,n}|\}$. If there is a constant $b > 0$ such that $a_n \geq b$ for at least a subsequence of n. Then for this subsequence we take $b_{i,n} = a_{i,n}/a_n$. Then $\max_i \{|b_{i,n}|\} = 1$ and

$$\int_{\Gamma_n} \| \sum_{1 \leqslant i \leqslant m_0} \frac{b_{i,n}}{z - z_{i,n}} \| dz \to 0 \quad \text{as} \quad n \to \infty.$$

Every $b_{i,n}$ converges to a number b_i as n goes to infinity and $\max_i \{|b_i|\} = 1$. This implies that

$$\int_{\Gamma} \| \sum_{1 \leqslant i \leqslant m_0} \frac{b_i}{z - z_{i,\infty}} \| dz = 0.$$

Thus, $\sum_{1 \leqslant i \leqslant m_0} \frac{b_i}{z - z_{i,\infty}} = 0$ on Γ. This implies each $b_i = 0$. This contradicts to $\max\{|b_i|\} = 1$. We proved the claim.

Since $g_n(z) \to 0$ and every $a_{i,n} \to 0$ as $n \to \infty$, this contradicts to $\|q_n\| = \int_{\mathbb{P}^1 \setminus \phi_n(U_f \cup P_1)} |q_n(z)| dz \wedge d\overline{z} = 1$. (For fix n, $g_n(z)$ may go to infinity as z tends to the boundary of $\phi_n(U_f)$. But note that each B_i has a fixed thickness and we assume that $\int_{\cup_i \phi_n(A_i)} |q_n(z)| dz \wedge d\overline{z} \to 0$ as $n \to \infty$.) $\square$

Let $R = \mathbb{P}^1 \setminus (U_f \cup P_1)$, we need to prove that $\phi_n(R)$ has bounded geometry for all $n = 1, 2, \cdots$. This is equivalent to prove that the length of the simple closed geodesics in $\phi_n(R \setminus P_1)$ has a uniform positive low bound for all n. Note that since f is holomorphic on U_f, for each disk D_i, if its center is attractive but not super-attractive, then critical orbits are attracted to it in a bounded geometry fashion. So any non-peripheral close curve in $\mathbb{P} \setminus P_f$ inside D_i should have a certain length. From this fact, Koebe Theorem, and Theorem 6.3 in [DH], any short simple closed geodesic in $\phi_n(\mathbb{P}^1 \setminus P_f)$ is either lies in $\phi_n(R)$ or lies deeply in a super attracting disk, which is a disk of $\phi_n(U_f)$.

For each disk D of U_f, let a be the center of D and b be a point in $D \cap P_f$ such that $d(a, b) > \alpha \mathrm{diam}(D)$ for some $\alpha > 0$. Let $P_2 = P_1 \cup \{a, b \in D \mid D \in \Lambda\}$.

For each simple closed curve $\gamma \in R$, $\phi(\gamma)$ is a closed curve in the Riemann surface $\mathbb{P}^1 \setminus \phi(P_2)$. We use $\|\gamma\|_{\phi, P_2}$ to denote the hyperbolic length of the unique closed geodesic homotopic to $\phi(\gamma)$ in $\mathbb{P}^1 \setminus \phi(P_2)$.

Lemma 4. *There exists a uniform $\delta > 0$ such that $\|\gamma\|_{\phi_n,P_2} > \delta$ for all simple closed curves in $\mathbb{P}^1 \setminus P_2$ and all $n = 1, 2, \cdots$.*

Denote
$$\mathcal{F}_\delta = \{[\gamma_i] \mid \gamma_i \subset R, \|\gamma_i\|_{\phi_{n_0},P_2} < \delta\}$$
Before we prove Lemma 4, we first prove that

Lemma 5. *For δ small enough, if there is a $n_0 > 0$ such that the family $\mathcal{F}_\delta$ is not empty, then $\mathcal{F}_\delta$ must be a f-stable family.*

Proof. Let $\gamma \in \mathcal{F}_\delta$ and γ' be a non-peripheral component of $f^{-1}(\gamma)$. Then γ' must be in R. Let $P_3 = P_2 \cup f(P_2)$. Since ϕ_n is univalent in U_f, there exists a number $K > 1$ independent of n and γ such that

$$\|\gamma\|_{\phi_n,P_3} < K\|\gamma\|_{\phi_n,P_2}$$

On the other hand, we have

$$\|\gamma'\|_{\phi_{n+1},P_2} < d\|\gamma\|_{\phi_n,P_3},$$

where d is the degree of f. So we get

$$\|\gamma'\|_{\phi_{n+1},P_2} < dK\|\gamma\|_{\phi_n,P_2}$$

If δ is small enough, by Lemma A9 in Appendix (note that ϕ_n is holomorphic on U_f), we know $\gamma' \in \mathcal{F}_\delta$. Therefore $\mathcal{F}_\delta$ is a f-stable family. □

Proof of Lemma 4. By lemma A6 in Appendix, any two simple closed geodesics in $\phi_n(\mathbb{P}^1 \setminus P_1)$ whose spectra $> A = -\log\log(\sqrt{2}+1)$ are disjoint. Therefore there exists a k dependent only on f such that for all $n = 1, 2, \cdots$, there are at most k simple closed geodesics whose spectra greater than A.

Similar to Appendix, for each f-stable family $\mathcal{F}$, we can define the Thurston linear transformation and it's associated linear transformation matrix A. Since the size of A is bounded by k and each entry of A can only be one of the finite values, so there are only finite many such linear transformation matrixes. Since f has no Thurston obstruction, so we have a uniform number m such that

$$\|A^m\| \le \frac{1}{2}$$

for any Thurston transformation A associated to a f-stable family.

When δ is small enough, By Lemma 5, we know $\mathcal{F}_\delta$ is a f-stable family. Let $\mathcal{F}_n$ be the family consists of the non-peripheral components of the pre-image of $\mathcal{F}_\delta$ by $f^{\circ n}$ in $\mathbb{P}^1 \setminus P_2$.

Define

$$\|\mathcal{F}_n\| = \sum_{\gamma \in \mathcal{F}_\delta} \frac{1}{\|\gamma\|_{\phi_n, P_2}}.$$

We will prove that $\|\mathcal{F}_n\|$ has a positive lower bound for all $n = 1, 2, \cdots$. This will imply Lemma 4. Note $\mathcal{F}_m$ be the family consisting of all the non-peripheral components of the pre-image of $\mathcal{F}_\delta$ by $f^{\circ m}$ in $\mathbb{P}^1 \setminus P_2$. Denote $Y = P_2 \cup f^{\circ m}(P_2)$ and $Y_m = f^{-m}(Y)$. Then $P_2 \subset Y_m$. By Lemma A7 in Appendix,

$$\mathcal{F}_m \leq \sum_{\gamma' \in \mathcal{F}_m} \|\gamma'\|^{-1}_{\phi_{m+n_0}, Y_m} + M$$

where M is a constant dependent only on f and m.

On the other hand, from the following diagram,

$$\begin{array}{ccc} (S^2, Y_m) & \stackrel{\phi_{m+n_0}}{\longrightarrow} & (\mathbb{P}^1, \phi_{m+n_0}(Y_m)) \\ \downarrow f^{\circ m} & & \downarrow g_m \\ (S^2, Y) & \stackrel{\phi_{n_0}}{\longrightarrow} & (\mathbb{P}^1, \phi_{n_0}(Y)) \end{array}$$

where $g_m : (\mathbb{P}^1, \phi_{m+n_0}(Y_m)) \to (\mathbb{P}^1, \phi_{n_0}(Y))$ is a holomorphic covering. We have

$$\sum_{\gamma' \in \mathcal{F}_m} \|\gamma'\|^{-1}_{\phi_{m+n_0}, Y_m} = \sum_{i,j} b_{ij} \|\gamma_i\|^{-1}_{\phi_{n_0}, Y} \leq \sum_j b_{ij} \frac{1}{\|\gamma_i\|_{\phi_{n_0}, P_2}}$$

where b_{ij} is exactly the (i, j) entry in A^m. Since $\|A^m\|_{\max} \leq 1/2$, $\sum_j b_{ij} \leq \frac{1}{2}$. So

$$\sum_{\gamma' \in \mathcal{F}_m} \|\gamma'\|^{-1}_{\phi_{m+n_0}, Y_m} \leq \frac{1}{2} \|\mathcal{F}_0\|.$$

Therefore,

$$\|\mathcal{F}_{m+n_0}\| \leq \frac{1}{2} \|\mathcal{F}_0\| + M.$$

Now for any $n \geq n_0$, write $n = km + j$, where $0 \leq j < l - 1$ and $k \geq 1$. Then we have

$$\|\mathcal{F}_0\| \leq \frac{1}{2^k} (\|\mathcal{F}_j\| - 2M) + 2M.$$

This implies that $\|\mathcal{F}_n\|$ has a uniform upper bound for all $n \geq 1$. Therefore there is a positive lower bound for $\|\gamma\|_{\phi_n, P_2}$ for all $\gamma \in \mathcal{F}_n$ and all $n \geq 1$. This completes the proof of Lemma 4. □

Proof of Theorem 2. The Thurston pull back operator α_f is strictly contraction on $\{[\mu_n]\}$. So $[\mu_n]$ converges to a unique fixed point of α_f. Thus, there is a unique rational map combinatorially equivalent to f. □

5 Appendix. Thurston's Theorem

An orientation preserving branched covering f with degree greater than one is said to be critically finite if P_f is finite.

Two branched coverings f and g are equivalent if there exist homeomorphisms $\phi, \psi : (S^2, P_f) \to (S^2, P_g)$ such that the diagram

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\psi} & (S^2, P_g) \\ \downarrow f & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\phi} & (S^2, P_g) \end{array}$$

commutes and ϕ is isotopic to ψ rel P_f.

If γ is a simple closed curve on $S^2 \setminus P_f$, then the set $f^{-1}(\gamma)$ is a union of disjoint simple closed curves. If γ moves continuously, so does each component of $f^{-1}(\gamma)$. A simple closed curve γ is non-peripheral if each component of $S^2 \setminus \gamma$ contains at least two points of P_f. Consider a multi-curve

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\}$$

of simple, closed, disjoint, non-homotopic, and non-peripheral curves on $S^2 \setminus P_f$. It is said to be f-stable if for any $\gamma \in \Gamma$, all the non-peripheral components of $f^{-1}(\gamma)$ are homotopic in $S^2 \setminus P_f$ to elements of Γ. For each f-stable multi-curve Γ, define a linear transformation,

$$f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma,$$

as follows: Let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $S^2 \setminus P_f$ and $d_{i,j,\alpha} = deg f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j$.

Then

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} \gamma_i.$$

Since the matrix of f_Γ is non-negative, there exist a largest eigenvalue $\lambda(\Gamma, f) \in \mathbb{R}_+$ and a correspondent non-negative eigenvector. A multi-curve Γ is called a Thurston obstruction if $\lambda(\Gamma, f) \geq 1$.

Lemma A1. *The linear transformation f_Γ commutes with the iteration, i.e.,*

$$(f^n)_\Gamma = (f_\Gamma)^n.$$

Let $v_f : S^2 \to Z^+ \cup \{0\}$ be the minimal positive function such that

- $v_f(x) = 1$ for all $x \notin P_f$.
- $v_f(x)$ is the integer multiple of $v_f(y) \cdot deg_y f$ for all $y \in f^{-1}(x)$.

Then $O_f = (S^2, v_f)$ is called an orbifold . It is called hyperbolic if

$$\chi(Q_f) = 2 - \sum_{x \in S^2} (1 - \frac{1}{v_f(x)}) < 0.$$

Thurston proved the following remarkable theorem.

Theorem A1 (Thurston's Theorem). *A critically finite orientation-preserving branched covering f with hyperbolic O_f is combinatorially equivalent to a unique rational map (up to conformal equivalence) if and only if f has no Thurston obstruction.*

We use $\mathbb{P}^1$ to denote the two-sphere S^2 with the standard complex structure. The Teichmüller space T_f is the Teichmüller space with the base point $\mathbb{P}^1 \setminus P_f$. It can be defined as

1. the space of smooth complex structures on S^2 module the equivalent relation that $\mu \sim \nu$ if $\mu = h^*\nu$ for some diffeomorphism $h : S^2 \to S^2$ with $h|_{P_f} = id$ and h isotopic to the identity rel P_f or
2. the space of diffeomorphisms $\phi : (S^2, P_f) \to (\mathbb{P}^1, \phi(P_f))$ module the equivalent relation $\phi \sim \psi$ if there exists an analytic isomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(P_f)) \\ \downarrow id & & \downarrow h \\ (S^2, P_f) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(P_f)) \end{array}$$

Let $\mathcal{M}$ be the space of all measurable functions μ defined on $\mathbb{P}^1$ with $\|\mu\|_\infty < 1$. Here μ is called a Beltrami coefficient. By the measurable Riemann mapping theorem, for each $\mu \in \mathcal{M}$, there exists a unique quasiconformal homeomorphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ which fixes $0, 1, \infty$ such that

$\mu(z) = \phi_{\overline{z}}/\phi_z$. A Beltrami path is a piecewise smooth map $\gamma : [0,1] \to \mathcal{M}$. We use ϕ_t to denote the correspondent path of quasiconformal homeomorphisms which fixes $0, 1$ and ∞. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be any quasiconformal homeomorphism, the pull forward of ϕ_t by ϕ is defined as

$$\phi_*(\phi_t) = \phi_t \circ \phi^{-1}.$$

We use $\phi_*\gamma(t)$ to denote the Beltrami coefficient of $\phi_*(\phi_t)$. Let $\xi = \gamma'(t)$ be the tangent vector of $\gamma(t)$ at t, we use $d\phi_*\xi$ to denote the tangent vector of $\phi_*\gamma(t)$ at t. Now we take $\phi = \phi_{t_0}$ and define the Teichmüller norm of ξ (with respect to T_f) as

$$\|\xi\| = sup \int\int_{\mathbb{P}^1} q(z) d\phi_*\xi d\overline{z}dz$$

where sup is taken over all quadratic differentials $q(z)dz^2$ on $(\mathbb{P}^1, \phi(P_f))$ with

$$\int\int_{\mathbb{P}^1} |q(z)| d\overline{z}dz = 1$$

and $\phi(P_f)$ the set of poles. Since

$$\int\int_{\mathbb{P}^1} |q(z)| d\overline{z}dz = 1,$$

any pole of $q(z)$ must be simple. For any two elements μ and ν in T_f, the Teichmüller distance between μ and ν is given by

$$d_T(\mu, \nu) = inf\{\int \|\gamma'(t)\| dt\},$$

where inf is taken over all the piecewise smooth Beltrami path connecting μ and ν. The reader may refer to [Ga, §7.1] for the above.

Let f be a critically finite orientation-preserving branched covering. Then there exists a C^1 branched covering g such that

(i) f is combinatorially equivalent to g and

(ii) $g_{\overline{z}}/g_z$ is defined almost everywhere and $\|g_{\overline{z}}/g_z\| \leq \delta < 1$ for some $\delta > 0$.

Therefore, we can always assume that $\mu_f = f_{\overline{z}}/f_z$ is defined almost everywhere and $\|\mu_f(z)\| \leq k < 1$ for some $k > 0$.

For each Beltrami differential μ on $\mathbb{P}^1$, we define the pull back of μ by f as

$$f^*(\mu)(z) = \frac{h(z) + \mu((f(z))\theta(z)}{1 + \overline{h(z)}\mu(f(z))\theta(z)}$$

where $\theta(z) = \overline{f_z}/f_z$. Clearly, the pull back of each piecewise C^1 Beltrami path is still a piecewise C^1 Beltrami path. Now let μ be a Beltrami differential on $\mathbb{P}^1$ and $\nu = f^*(\mu)$. Let ϕ_μ and ϕ_ν be the Beltrami solutions with μ and ν fixing 0, 1, and ∞. The following diagram holds,

$$\begin{array}{ccc} (S^2, f^{-1}(P_f)) & \xrightarrow{\phi_\nu} & (\mathbb{P}^1, \phi_\nu(f^{-1}(P_f))) \\ \downarrow f & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\phi_\mu} & (\mathbb{P}^1, \phi_\mu(P_g)) \end{array}$$

where g is a rational function. Here we call ϕ_ν the pull back of ϕ_μ by f. Let $\gamma(t)$ be a smooth Beltrami path passing though μ, then the pull back $f^*\gamma(t)$ is a smooth Beltrami path which passes though ν. Let η be the tangent vector of $\gamma(t)$ at μ, then $\tilde{\eta} = df^*\eta$ is the tangent vector of $f^*\gamma(t)$ at ν. Let $\xi = d\phi_{\mu*}\eta$ and $\tilde{\xi} = d\phi_{\nu*}\tilde{\eta}$. We have the following diagram,

$$\begin{array}{ccc} \tilde{\eta} & \xrightarrow{d\phi_{\nu*}} & \tilde{\xi} \\ \uparrow df^* & & \uparrow dg^* \\ \eta & \xrightarrow{d\phi_{\mu*}} & \xi \end{array}$$

Let $\tilde{q} = \tilde{\phi}(w)dw^2$ be the quadratic differential on $(\mathbb{P}^1, \phi_\nu(P_f))$. The push forward of $\tilde{q}$ by g is the quadratic differential $q = \mathcal{L}_g\tilde{\phi}(z)dz^2$ on $(\mathbb{P}^1, \phi_\mu(P_f))$ where

$$\mathcal{L}\tilde{\phi}(z) = \sum_{g(w)=z} \frac{\tilde{\phi}(w)}{[g'(w)]^2}.$$

We use $q = \delta g^*(\tilde{q})$ to denote the push forward quadratic differential. Note that $\mathcal{L}_g\tilde{\phi}(z)$ is a transfer operator (see [Ji2]). By the above formula, we have

$$< \tilde{q}, \tilde{\xi} >=< q, \xi > .$$

Therefore $\|\tilde{\xi}\| \leqslant \|\xi\|$ and the pull back will not increase the Teichmüller length of the Beltrami path.

Starting from the standard complex structure $\mu_0 = 0$, we can define a sequence of pull back complex structures $\mu_n = f^*(\mu_{n-1})$ for $n > 1$. Consequently, we have ϕ_n, $\eta_n = df^*\eta_{n-1}$, $\xi_n = d\phi_{\mu_{n-1}*}\eta_{n-1}$, and $q_{n-1} = \delta g^*_{n-1}(q_n)$ for $n > 1$. We call $\{\phi_n\}$ the pull back sequence.

Consider any composition of two consecutive pull backs in the following diagram,

$$\begin{array}{ccc} \eta_{n+2} & \xrightarrow{d\phi_{n+2*}} & \xi_{n+2} \\ \uparrow df^* & & \uparrow dg^*_{n+1} \\ \eta_{n+1} & \xrightarrow{d\phi_{n+1*}} & \xi_{n+1} \\ \uparrow df^* & & \uparrow dg^*_{n} \\ \eta_{n} & \xrightarrow{d\phi_{n*}} & \xi_{n} \end{array}$$

Lemma A2. $\|\delta g_n^* \delta g_{n+1}^* q_{n+2}\| < \|q_{n+2}\|$. *Therefore,* $\|\xi_{n+2}\| < \|\xi_n\|$.

For the proof of this lemma, see Proposition 3.3 of [DH].

Lemma A3. *Let $X \subset \mathbb{P}^1$ be a finite subset such that $0, 1, \infty \in X$. Let m be the cardinality of X and suppose $m \geq 4$. For any $a > 0$, if all the simple closed geodesic in $\mathbb{P}^1 \setminus X$ has length greater than a then the spherical distance between any two distinct points in X has a positive lower bound which depends only on a and m.*

Proof. First we will show that the spherical distance between ∞ and any finite point in X has a positive low bound which is dependent only on a and m. Let $X = \{x_1, \cdots, x_{m-1}\}$ and $x_m = \infty$. Suppose $|x_1| \leq \cdots \leq |x_{m-1}|$. Let $M = |x_{m-1}|$. Since $x_2 \leq 1$, we have

$$\prod_{2 \leq i \leq m-2} \frac{|x_{i+1}|}{|x_i|} = \frac{|x_{m-1}|}{|x_2|} \geq M.$$

So

$$max\{\frac{|x_{i+1}|}{|x_i|}\} \geq M^{\frac{1}{m-3}}.$$

Let

$$A_i = \{z \in \mathbb{P}^1 \ \Big| \ |x_i| < z < |x_{i+1}|\}.$$

Then we have an integer $i_0 > 0$ such that

$$mod(A_{i_0}) \geq \frac{logM}{2\pi(m-3)}.$$

Therefore the unique simple closed geodesic γ in A_{i_0} has hyperbolic length

$$\|\gamma\|_{A_{i_0}} = \frac{\pi}{mod(A_{i_0})} \leq \frac{2\pi^2(m-3)}{logM}.$$

Since

$$\|\gamma\|_{A_{i_0}} \geq \|\gamma\|_{\mathbb{P}^1 \setminus X} \geq a,$$

This implies that

$$M \leq e^{\frac{2\pi^2(m-3)}{a}}.$$

So the spherical distance between ∞ and any finite point in X has a positive low bound which depends only on a and m.

Now we will show that the spherical distance between any two finite points in X has a positive low bound dependent only on a and m. Without loss of generality, we assume

$$d(x_1, x_2) = \min_{x_i \neq x_j}\{d(x_i, x_j)\} \quad \text{and} \quad d(x_1, x_i) \leq d(x_1, x_{i+1})$$

for $i = 1, \cdots, m-1$. Then

$$\prod_{2 \leq i \leq m-2} \frac{d(x_{i+1}, x_1)}{d(x_i, x_1)} = \frac{d(x_{m-1}, x_1)}{d(x_2, x_1)} \geq max\{\frac{d(x_1, 0)}{d(x_1, x_2)}, \frac{d(x_1, 1)}{d(x_1, x_2)}\} \geq \frac{1}{2d(x_1, x_2)}.$$

Thus we have an integer $i_0 > 0$ such that

$$mod(B_{i_0}) \geq \frac{log\frac{1}{2d(x_1,x_2)}}{2\pi(m-3)}$$

where

$$B_{i_0} = \{z \in \mathbb{P}^1 \mid d(x_1, x_{i_0}) < d(x_1, z) < d(x_1, x_{i_0+1})\}.$$

Let η be the unique simple closed geodesic in B_{i_0}. Then we have

$$\|\eta\|_{B_{i_0}} = \frac{\pi}{mod(B_{i_0})} \leq \frac{2\pi^2(m-3)}{log\frac{1}{2d(x_1,x_2)}}.$$

Since

$$\|\eta\|_{B_{i_0}} \geq \|\eta\|_{\mathbb{P}^1 \setminus X} \geq a,$$

again we have

$$d(x_1, x_2) \geq \frac{1}{2}e^{-\frac{2\pi^2(m-3)}{a}}.$$

This implies the spherical distance between any two finite points in X has a positive low bound dependent only on a and m. □

Let $\mathcal{R}^d$ be the set of all the rational functions with degree $d > 1$. For a sequence $\{f_n\} \subset \mathcal{R}^d$ and $f \in \mathcal{R}^d$, we say $f_n \to f$ if f_n is convergent to f uniformly in the spherical metric. Therefore we have a topological space $\mathcal{R}^d$. We define an equivalent relation in $\mathcal{R}^d$ as follows: for any f and g in $\mathcal{R}^d$, we say f is equivalent to g if $f = h_1 \circ g \circ h_2$ for some linear transformation $h_i, i = 1, 2$. We denote the quotient space by $\mathcal{S}^d$. For $a > 0$, let

$$\mathcal{F}_{d,a} = \{[f] \in \mathcal{S}^d \mid \text{there is a } f \in [f] \text{ such that } d(x,y) \geq a \text{ for all distinct critical values } x \text{ and } y \text{ of } f\}$$

Lemma A4. *$\mathcal{F}_{d,a}$ is compact.*

Proof. For $f \in \mathcal{F}_{d,a}$, we use X to denote the set of its critical values. If $|X| = 1$, the number of the pre-images of the critical value would be $\sum d_i \geq \sum (d_i - 1) + 1 = 2d - 1$. So we get $d \geq 2d - 1$ and therefore $d \leq 1$ which is a contradiction with $d > 1$. If $|X| = 2$, f is equivalent to z^d. So $F_{d,a}$ consists of one element and must be compact. So we need only consider the case when $|X| \geqslant 3$. By a composition with a linear transformation on the left and right, we can assume both X and $f^{-1}(X)$ contain $0, 1, \infty$.

Any simple closed geodesic in $\mathbb{P}^1 \setminus X$ has length bigger than a fixed constant only depending on a. (The fixed constant is independent of $f \in \mathcal{F}_{d,a}$, refer to [Le, Chapter 1].) Since $f : \mathbb{P}^1 \setminus f^{-1}(X) \to \mathbb{P}^1 \setminus X$ is a covering, any simple closed short geodesic in $\mathbb{P}^1 \setminus f^{-1}(X)$ will be mapped to a shorter closed geodesic in $\mathbb{P}^1 \setminus X$. So the length of the closed geodesic in $\mathbb{P}^1 \setminus f^{-1}(X)$ has a positive low bound. Since $f^{-1}(X)$ contains $0, 1, \infty$, so by Lemma A3, we have a constant $b > 0$ (only depends on a) such that $d(w, v) \geqslant b$ for any $w \neq v \in f^{-1}(X)$. For any sequence f_n in $\mathcal{F}_{d,a}$, by taking a subsequence we can assume $X_n \to X$ and $f_n^{-1}(X_n) \to Y$. Since X_n contains $0, 1, \infty$, so $\{f_n\}$ is normal in $\mathbb{P}^1 \setminus Y$. Let g be the limit point of any convergent subsequence f_{n_k} of $\{f_n\}$. Then g is holomorphic in $\mathbb{P}^1 \setminus Y$. Clearly any point in Y can not be an essential pole of g, so g must be a rational function. Now we show that $f_{n_k} \to g$ uniformly on the sphere under the spherical metric. For each $x \in X$, there is a small open disk centered at x with boundary circle γ such that the closure of the disk does not contain any other points in X. We denote this disk by $D(\gamma)$. We can take $D(\gamma)$ small enough such that any component of $g^{-1}(D(\gamma))$ must be an open topological disk contains only one point y in Y with a smooth boundary curve η. We denote such a component by

$D(\eta)$. We have $g : \eta \to \gamma$ a covering with degree $\deg g_y$. Since $f_{n_k} \to g$ uniformly in any compact subset of $\mathbb{P}^1 \setminus Y$, therefore, as k is large enough, we have η_{n_k} near η such that $f_{n_k} : \eta_{n_k} \to \gamma$ is a covering with the same degree as $g : \eta \to \gamma$. Since when k is large enough, $D(\eta_{n_k})$ contains only one point in $f_{n_k}^{-1}(X_{n_k})$, so $f_{n_k} : D(\eta_{n_k}) \to D_\gamma$ is a branched covering with the same degree as $g : D(\eta) \to D(\gamma)$. Therefore $f_{n_k} \to g$ uniformly on the whole sphere. So g is rational map of degree d and $dist(x, y) \geq a$ for any two distinct critical values of g. Therefore $g \in \mathcal{F}_{d,a}$. This completes the proof that $\mathcal{F}_{d,a}$ is compact. □

Now we can have that

Lemma A5. *For the pull back sequence $\{\phi_n\}$, if there exists a constant $a > 0$ independent of n such that the hyperbolic length of any simple closed geodesic in $\mathbb{P}^1 \setminus \phi_n(P_f)$ is greater than a, then the pull back uniformly contracts the Teichmüller metric on T_f, and therefore, $\{\phi_n\}$ is exponentially convergent in the Teichmüller space T_f.*

Proof. Note that we have

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi_{n+2}} & (\mathbb{P}^1, \phi_{n+2}(P_f)) \\ \downarrow f & & \downarrow g_{n+1} \\ (S^2, P_f) & \xrightarrow{\phi_{n+1}} & (\mathbb{P}^1, \phi_{n+1}(P_f)) \\ \downarrow f & & \downarrow g_n \\ (S^2, P_f) & \xrightarrow{\phi_n} & (\mathbb{P}^1, \phi_n(P_f)) \end{array}$$

By Lemma A2,

$$\|\delta g_n^* \delta g_{n+1}^* q_{n+2}\| < \|q_{n+2}\|$$

for any quadratic differential q_{n+2} on $\mathbb{P}^1 \setminus \phi_{n+2}(P_f)$. Since the set of poles of q_n is contained in $\phi_n(P_f)$, the space the space of such quadratic differentials satisfying $\|q\| = 1$ is compact. We have the biggest $0 < b_n < 1$ such that

$$\|\delta g_n^* \delta g_{n+1}^* q_{n+2}\| \leq (1 - b_n)\|q_{n+2}\|.$$

Let

$$b_0 = inf\{b_n\} \geq 0.$$

Al quadratic differentials q_{n+2} are contained in a compact set by the assumption and Lemma A3, and the family of $\{g_n\}$ is a compact family by Lemma A4 and the assumption. Therefore we can assume $q_{n+2} \to q$, $g_n \to h_1$ and $g_{n+1} \to h_2$ such that

$$b_0 = 1 - \frac{\|\delta h_1^* \delta h_2^* q\|}{\|q\|}$$

Now by Lemma A2, we get $b_0 > 0$. □

We list two lemmas in hyperbolic Riemann surfaces. Let S be a hyperbolic Riemann surface. A spectrum of S is defined as $w(\gamma) = -\log l(\gamma)$, where γ is a closed geodesic of S. Let $A = -\log\log(\sqrt{2}+1)$.

Lemma A6. *If the spectrum $w(\gamma) > A$, then γ must be a simple closed curve. And any two such geodesics are disjoint.*

For the proof, see Corollary 6.6 and Proposition 6.7 in [DH]. From now on, we say a closed geodesic γ is short if $w(\gamma) > A$.

Lemma A7. *Let $X \subset S$ be a finite set with $|X| = p$. Let γ be a short closed geodesic in S. Let $S' = S \setminus X$. Let $\gamma_i, i = 1, \cdots, n$, be all short closed geodesics on S' which are homotopic to γ in S. Then*

$$\Big| \sum_{1 \leqslant i \leqslant n} \frac{1}{l(\gamma_i)} - \frac{1}{l(\gamma)} \Big| < C(p),$$

where $C(p)$ is a constant only dependent on p.

For the proof, see Theorem 7.1 in [DH]. Let $\gamma \subset \mathbb{S}^2 \setminus P_f$ be a simple closed curve. Let $\phi : \mathbb{S}^2 \to \mathbb{P}^1$ be a quasi-conformal homeomorphism. Consider the hyperbolic Riemann surface $\mathbb{P}^1 \setminus \phi(P_f)$. Let ξ be the unique simple closed geodesic in the homotopy class of $\phi(\gamma)$. We define the ϕ-norm $\|\gamma\|_\phi$ of γ to be the hyperbolic length of ξ in $\mathbb{P}^1 \setminus \phi(P_f)$. Furthermore, for a subset $X \in \mathbb{S}^2$ and a simple closed curve $\gamma \in \mathbb{S}^2 \setminus X$, we use $\|\gamma\|_{\phi,X}$ to denote the hyperbolic length of the unique simple closed geodesic homotopic $\phi(\gamma)$ in $\mathbb{P}^1 \setminus \phi(X)$.

Lemma A8. *For any simple closed curve $\gamma \subset \mathbb{S}^2 \setminus P_f$, the map $\tau : T(\mathbb{S}^2, P_f) \to R$ defined by $[\phi] \to log\|\gamma\|_{\phi,P_f}$ is a Lipshitz map with Lipshitz constant* 2.

For the proof of this lemma, see Proposition 7.2 in [DH]. Now we mention the gap lemma for the set of spectra of a Riemann surface. Suppose P is a finite set in S^2. Let $p = \#(P) > 3$. For any quasiconformal map $\phi : (\mathbb{S}^2, P) \to (\mathbb{P}^1, \phi(P))$. Let $\mathrm{Sp}(\phi) = \{w(\gamma)\}$ be the set of all spectra of S. Then $\mathrm{Sp}(\phi) \cap [A, \infty)$ is a discrete set and $(A, \infty) \setminus \mathrm{Sp}(\phi)$ consists of intervals. Each interval is called a gap.

Lemma A9. *There is a sequence of positive numbers $J_k = J_k(p) \to \infty$ as $k \to \infty$ such that for any quasi-conformal map $\phi : (\mathbb{S}^2, P) \to (\mathbb{P}^1, \phi(P))$, there is a smallest gap $I_k = I_k(\phi)$ in (A, ∞) with length greater than J_k.*

Now consider the pull back sequences $\{\phi_n\}$ starting from $\phi_0 = id$.

Lemma A10. *Suppose f has no Thurston obstruction. Then there exists a positive number a such that*

$$\|\gamma\|_{\phi_n} > a$$

for for all simple closed curves $\gamma \subset \mathbb{S}^2 \setminus P_f$ and all $n = 1, 2, \cdots$.

Proof. Without lost of generality, we assume f is a C^1 (or quasi-regular) covering. Since there can only be finitely many distinct Thurston linear transformations, so by Lemma A1, we have a universal number m such that

$$(f_\Gamma)^m = A^m = (b_{ij})$$

with $\|A^m\| < \frac{1}{2}$ for any f-stable multi-curve Γ.

For $\epsilon > 0$, if there is a $n_0 > 0$ such that

$$\{\gamma \subset S^2 \setminus P_f \mid \|\gamma\|_{\phi_{n_0}} < \epsilon\} \neq \emptyset,$$

By Lemma A10, as long as ϵ small enough, there is an $a(\epsilon) > 0$ such that

$$\mathcal{F} = \{\gamma \subset S^2 \setminus P_f \mid \|\gamma\|_{\phi_{n_0}} < a(\epsilon)\} \neq \emptyset$$

is a $f^{\circ m}-$ stable family.

Let $\mathcal{F}_n$ be the family consists of the non-peripheral components of the pre-image of $\mathcal{F}$ by f^n. Define

$$||\mathcal{F}_n|| = \sum_{\gamma \in \mathcal{F}_n} \frac{1}{\|\gamma\|_{\phi_{n+n_0}}}.$$

Let $P_m = f^{-m}(P_f)$. We have

$$\begin{array}{ccc} (S^2, P_m) & \overset{\phi_{m+n_0}}{\longrightarrow} & (\mathbb{P}^1, \phi_{m+n_0}(P_m)) \\ \downarrow f^{\circ m} & & \downarrow g_m \\ (S^2, P_f) & \overset{\phi_{n_0}}{\longrightarrow} & (\mathbb{P}^1, \phi_{n_0}(P_f)) \end{array}$$

Since

$$g_m : (\mathbb{P}^1, \phi_{m+n_0}(P_m) \to (\mathbb{P}^1, \phi_{n_0}(P_f))$$

is a holomorphic covering, so

$$\sum_{\xi\in\mathcal{F}_m}\frac{1}{\|\xi\|_{\phi_{m+n_0},P_m}}=\sum_{i,j}b_{ij}\frac{1}{\|\gamma_i\|_{\phi_0}}\leqslant\frac{1}{2}||\mathcal{F}_{n_0}||.$$

By Lemma A7, we have

$$||\mathcal{F}_m||\leqslant\sum_{\xi\in\mathcal{F}_m}\frac{1}{\|\xi\|_{\phi_{m+n_0},P_m}}+M$$

where the constant M is dependent only on f and m. So

$$||\mathcal{F}_m||\leqslant\frac{1}{2}||\mathcal{F}_0||+M.$$

Similar arguments, for any $j=0,1,\cdots,m-1$ and $k=1,2,\cdots$, we get that

$$||\mathcal{F}_{km+j}||\leq\frac{1}{2^k}(||\mathcal{F}_j||-2M)+2M.$$

So $\mathcal{F}_n$ has a uniform upper bound for all $n\ \geq n_0$. Therefore there is a positive lower bound for all $\|\gamma\|_{\phi_n}$ for all $n\geq 0$ and all simple closed curves $\gamma\subset\mathbb{S}^2\setminus P_f$. □

Proof of Theorem A1. For the necessary part, see Section 4 of [DH]. The sufficient part follows now from Lemma A10, Lemma A4, and Lemma A5. □

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