Local connectivity of the Mandelbrot set at certain infinitely renormalizable points*†

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Abstract

We construct a subset consisting of infinitely renormalizable points in the Mandelbrot set. We show that Mandelbrot set is locally connected at this subset and for every point in this subset, corresponding infinitely renormalizable quadratic Julia set is locally connected. Since the set of Misiurewicz points is in the closure of the subset we construct, therefore, the subset is dense in the boundary of the Mandelbrot set.

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1 Introduction, review, and statements of new results

1.1 Quadratic polynomials

Let $\mathbb{C}$ and $\overline{\mathbb{C}}$ be the complex plane and the extended complex plane (the Riemann sphere). Suppose f is a holomorphic map from a domain $\Omega \subset \overline{\mathbb{C}}$ into itself. A point z in Ω is called a periodic point of period $k \geq 1$ if $f^{\circ k}(z) = z$ but $f^{\circ i}(z) \neq z$ for all $1 \leq i < k$. The number $\lambda = (f^{\circ k})'(z)$ is called the multiplier of f at z. A periodic point of period 1 is also called a fixed point. A periodic point z of f is said to be super-attracting, attracting, neutral, or repelling if $|\lambda| = 0$, $0 < |\lambda| < 1$, $|\lambda| = 1$, or $|\lambda| > 1$. The proof of the following theorem can be found in Milnor's book [22, pp. 86-88].

Theorem 1.1 (Theorem of Böttcher). *Suppose p is a super-attracting periodic point of period k of a holomorphic map f from a domain Ω into itself. There is a neighborhood W of p, a holomorphic diffeomorphism $h : W \to h(W)$ with $h(p) = 0$, and a unique integer $n > 1$ such that*

$$h \circ f^{\circ k} \circ h^{-1}(w) = w^n$$

for $w \in h(W)$. Furthermore, h is unique up to multiplication by an $(n-1)$-st root of unity.

Consider a quadratic polynomial $f(z) = az^2 + bz + d$. Conjugating f by an appropriate linear map $h(z) = ez + s$, we get $h \circ f \circ h^{-1}(z) = z^2 + c$. So from dynamical systems point of view, quadratic polynomials form a one-parameter family. We call $q_c(z) = z^2 + c$ the Douady-Hubbard family of quadratic polynomials. When we study this family, we always deal with two kinds of sets, one is the class of sets in the phase space (the z-plane) and the other is the class of sets in the parameter space (the c-plane). In this paper, we use regular letters to denote sets in the phase space and curly letters to denote sets in the parameter space.

Denote $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ the disk of radius r centered 0. Let $q_c(z) = z^2 + c$ be a quadratic polynomial. For r large enough, $U = q_c^{-1}(D_r)$ is a simply connected domain and its closure is relatively compact in D_r. Then $q_c : U \to V = D_r$ is a holomorphic, proper, degree two branched covering map. This is a model of quadratic-like maps defined by Douady and Hubbard [2] as follows: A quadratic-like map is a triple (f, U, V) such that U and V are simply connected domains isomorphic to a disc with $\overline{U} \subset V$ and such that $f : U \to V$ is a holomorphic, proper, degree two branched covering mapping. For a quadratic-like map (f, U, V),

$$K_f = \cap_{n=0}^{\infty} f^{-n}(U)$$

is called the filled-in Julia set. The Julia set J_f is the boundary of K_f. Both K_f and J_f are compact. A quadratic-like map (f, U, V) has only one critical point which we always denote as 0. Refer to [22, pp. 91-92] for a proof of the following theorem.

Theorem 1.2. *The set K_f (as well as J_f) is connected if and only if 0 is in K_f. Moreover, if 0 is not in K_f, $K_f = J_f$ is a Cantor set.*

Since $q_c : U \to V = D_r$ is a quadratic-like map, the filled-in Julia set of q_c is the set of all points not going to infinity under forward iterates of q_c. We use K_c and J_c to mean its filled-in Julia set and Julia set. For $c = 0$ and every $r > 1$, (q_0, D_r, D_{r^2}) is a quadratic-like map whose filled-in Julia set is $K_0 = \overline{D}_1$. For any c, ∞ is a super-attracting fixed point of q_c, applying Theorem 1.1, there is a unique holomorphic diffeomorphism h_c (called the Böttcher coordinate) defined on a neighborhood B_0 about ∞ with $h_c(\infty) = \infty$, $h_c'(\infty) = 1$ such that

$$h_c \circ q_c \circ h_c^{-1}(z) = z^2, \quad z \in B_0.$$

Assume $h_c(B_0) = \overline{\mathbb{C}} \setminus \overline{D}_r$ (for a fixed large number $r > 1$). Let $B_n = q_c^{-n}(B_0)$. If $0 \notin B_n$, then

$$q_c : B_n \cap \mathbb{C} \to B_{n-1} \cap \mathbb{C}, \quad n \geq 1,$$

is unramified covering map of degree two. So we can inductively extend

$$h_c : B_n \to \overline{\mathbb{C}} \setminus \overline{D}_{r^{\frac{1}{2^n}}}$$

such that

$$h_c \circ q_c \circ h_c^{-1}(z) = z^2 \quad z \in B_n.$$

Let

$$B_c(\infty) = \{z \in \overline{\mathbb{C}} \mid q_c^{\circ n}(z) \to \infty \quad \text{as} \quad n \to \infty\}$$

be the basin of ∞. Then $K_c = \mathbb{C} \setminus B_c(\infty)$. Let

$$G_c(z) = \lim_{n \to \infty} \frac{\log^+ |q_c^{\circ n}(z)|}{2^n},$$

where $\log^+ x = \sup\{0, \log x\}$, be the Green function of K_c. It is a proper harmonic function whose zero set is K_c and whose critical points are bounded by $G_c(0)$. So the Böttcher coordinate can be extended to

$$h_c : U_c = \{z \in \mathbb{C} \mid G_c(z) > G_c(0)\} \to \{z \in \mathbb{C} \mid |z| > \exp G_c(0)\}.$$

such that

$$h_c \circ q_c \circ h_c^{-1}(z) = z^2.$$

Moreover, $G_c = \log |h_c|$. The set $G_c^{-1}(\log r)$, $r > 1$, is called an equipotential curve.

If K_c is connected (equivalent to say $0 \notin B_c(\infty)$), then $U_c = B_c(\infty) \setminus \{\infty\}$; if K_c is a Cantor set (equivalent to say $0 \in B_c(\infty)$), then U_c is a punctured disk bounded by an ∞ shape curve passing 0. By the definition, the Mandelbrot set $\mathcal{M}$ is defined as the set of parameters c such that K_c is connected.

Suppose $c \in \mathcal{M}$. Let $S_r = \{z \in \mathbb{C} \mid |z| = r\}$ be a circle of radius $r > 1$. Then

$$s_r = h_c^{-1}(S_r) = G_c^{-1}(\log r)$$

is an equipotential curve. Note that

$$q_c(s_r) = s_{r^2}.$$

For every $r > 1$, let $U_r = h_c^{-1}(D_r)$. Then U_r is a domain bounded by the equipotential curve s_r. Furthermore, (q_c, U_r, U_{r^2}) is a quadratic-like map whose filled-in Julia set is K_c. In the rest of this paper, when we talk about a quadratic polynomial $q(z) = z^2 + c$, it also mean a quadratic-like map (q, U_r, U_{r^2}) for any $r > 1$.

Take $E_\theta = \{z \in \mathbb{C} \mid |z| > 1, \arg(z) = 2\pi\theta\}$ for $0 \le \theta < 1$. Let

$$e_\theta = h_c^{-1}(E_\theta).$$

Then e_θ is called an external ray of angle θ and

$$q_c(e_\theta) = e_{2\theta \pmod 1}.$$

An external ray is an integral curve of the gradient vector field ∇G on $B_c(\infty)$. An external ray e_θ is said to land at J_c if e_θ has only one limiting point at J_c. It is periodic with period m if $q_c^{\circ i}(e_\theta) \cap e_\theta = \emptyset$ for $1 \le i < m$ and if $q_c^{\circ m}(e_\theta) = e_\theta$. Refer to [9, 23] for the following theorem (the reader can also find a proof in [10, pp. 199].

Theorem 1.3 (Douady and Yoccoz). *Suppose $q_c(z) = z^2 + c$ is a quadratic polynomial with a connected filled-in Julia set K_c. Every repelling periodic point of q_c is a landing point of finitely many periodic external rays with the same period.*

Two quadratic-like maps (f, U, V) and (g, U', V') are said to be topologically conjugate if there is a homeomorphism h from a neighborhood $K_f \subset X \subset U$ to a neighborhood $K_g \subset Y \subset U'$ such that

$$h \circ f(z) = g \circ h(z), \quad z \in X.$$

If h is quasiconformal (see [1]) (resp. holomorphic), then they are quasiconformally (resp. holomorphically) conjugate. If h can be chosen such that $h_{\overline{z}} = 0$ a.e. on K_f, then they are hybrid equivalent. The reader can find the proof of the following theorem in [2].

Theorem 1.4 (Douady and Hubbard). *Suppose (f, U, V) is a quadratic-like map whose Julia set J_f is connected. Then there is a unique quadratic polynomial $q_c(z) = z^2 + c$ which is hybrid equivalent to f.*

The following basic facts about the Julia set of a quadratic-like map (f, U, V) will be used in the paper but the reader can check them by himself by referring to [22]. The Julia set J_f is completely invariant, i.e., $f(J_f) = J_f$ and $f^{-1}(J_f) = J_f$. The Julia set J_f is perfect, i.e., $J_f' = J_f$, where J_f' means the set of limit points of J_f. The set of all repelling periodic points E_f is dense in J_f, i.e., $\overline{E_f} = J_f$. The limit set of $\{f^{-n}(z)\}_{n=0}^{\infty}$ is J_f for every z in V. The Julia set J_f has no interior point. If f has no attracting, super-attracting, and neutral periodic points in V, then $K_f = J_f$.

1.2 Local connectivity for quadratic Julia sets.

Consider a quadratic polynomial $q_c(z) = z^2 + c$ whose Julia set J_c is connected. The external ray e_0 is the only one fixed by q_c. It lands either at a repelling or parabolic fixed point β of q_c. Suppose β is repelling. From Theorem 1.3, e_0 is the only external ray landing at β. So $J_c \setminus \{\beta\}$ is still connected. We call β the non-separating fixed point. Let $\alpha \neq \beta$ be the other fixed point of q_c. If α is attracting or super-attracting, then $J_c = K_c \setminus B(\alpha)$ where $B(\alpha) = \{z \in \overline{\mathbb{C}} \mid q_c^{\circ n}(z) \to \alpha \text{ as } n \to \infty\}$ is the basin of α. In this case, q_c is hyperbolic and J_c is a Jordan curve. If α is repelling. Then there are at least two periodic external rays landing at

α. All of them are in one cycle of period $k \geq 2$. We use Λ to denote the union of this cycle of external rays. Then Λ cuts $\mathbb{C}$ into finitely many simply connected domains

$$\Omega_0, \Omega_1, \ldots, \Omega_{k-1}.$$

Each domain contains points in J_c. This implies that $J_c \setminus \{\alpha\}$ is disconnected. We call α the separating fixed point.

The point 0 is the only finite critical point of q_c. Let $c_i = q_c^{\circ i}(0)$, $i \geq 1$, be the i^{th} critical value. We use CO to denote the critical orbit $\{c_i\}_{i=0}^{\infty}$. The critical point 0 is said to be recurrent if for any neighborhood W of 0, there is a critical value $c_i \neq 0$ in W. We will only consider those quadratic polynomial whose critical point is recurrent and which has no neutral periodic points in this paper. These are assumed for a quadratic-like map too. Under these assumptions, q_c has only repelling periodic points and $J_c = K_c$.

Let $U_r \subset \mathbb{C}$ be the open domain bounded by an equipotential curve s_r. For a fixed $r > 1$, $(q, U_{\sqrt{r}}, U_r)$ is a quadratic-like map. The set Λ cuts U_r into finitely many Jordan domains. Let C_0 be the closure of the one containing 0, and let $B_{0,i}$ be the closure of the domain containing c_i for $1 \leq i < k$. We call the collection

$$\eta_0 = \{C_0, B_{0,1}, \ldots, B_{0,k-1}\}$$

the original partition about J_c.

Since Λ is forward invariant under q_c, the image $q_c(C_0 \cap J_c)$ (resp. $q(B_{0,i} \cap J_c)$ for every $1 \leq i < k$) is the union of some sets from

$$\eta_0 \cap J_c = \{C_0 \cap J_c, B_{0,1} \cap J_c, \ldots, B_{0,k-1} \cap J_c\}.$$

Each $q_c|B_{0,i}$ is degree one, proper, and holomorphic but $q_c|C_0$ is degree two, proper, and holomorphic.

For each $n > 0$, let $\alpha_n = q_c^{-n}(\alpha)$ and $\Lambda_n = q_c^{-n}(\Lambda)$ (denote $\alpha_0 = \{\alpha\}$ and $\Lambda_0 = \Lambda$). The set Λ_n is the union of external rays landing at points in α_n. It cuts the closure of the domain $U_{r^{\frac{1}{2^n}}}$ into a finite number of closed Jordan domains,

$$\eta_n = \{C_n, B_{n,1}, \ldots, B_{n,k_n}\}.$$

Here we use C_n to denote the one containing 0 and $B_{n,1}$, ..., B_{n,q_n} to denote others. Since $q_c(\Lambda_n) = \Lambda_{n-1}$, $q_c(C_n)$ (resp. $q_c(B_{n,i})$) is in η_{n-1}.

Each $q_c|B_{n,i}$ is degree one, holomorphic, proper but $q_c|C_n$ is degree two, holomorphic, and proper. We call η_n the n^{th}-partition about J_c and within it, C_n is called the critical piece.

Thus we get a sequence

$$\xi = \{\eta_n\}_{n=0}^{\infty}$$

of nested partitions, which we call the Yoccoz partition about J_c, and a sequence of critical pieces,

$$0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_2 \subseteq C_1 \subseteq C_0.$$

From Theorem 1.4, the sequence ξ can be also constructed similarly for any quadratic-like map whose Julia set is connected.

Definition 1.1. *Let (f, U, V) be a quadratic-like map whose Julia set J_f is connected. We say it is (once) renormalizable if there is an integer $n' \geq 2$ and a domain $0 \in U' \subset U$ such that*

$$f_1 = f^{\circ n'} : U' \to V' = f_1(U') \subset V$$

is a quadratic-like map with connected Julia set $J_{f_1} = J(n', U', V')$. In this situation, we call (f_1, U', V') a renormalization of (f, U, V). Otherwise, we call f non-renormalizable.

If f is renormalizable and f_1 is also renormalizable, then we call f twice renormalizable. So on one can define a finitely renormalizable and infinitely renormalizable f. The relation between ξ and the renormalizability of a quadratic polynomial is proved by Yoccoz (refer to [9, 23, 10]).

Theorem 1.5 (Yoccoz). *The polynomial $q_c(z) = z^2+c$ is non-renormalizable if and only if $\cap_{n=0}^{\infty} C_n$ contains one point. Moreover, if q_c is non-renormalizable, then the Julia set J_c is locally connected.*

Furthermore, we have that

Theorem 1.6 (Yoccoz). *If q_c is finitely renormalizable, then the Julia set J_c is locally connected.*

We have studied the extension of Yoccoz partitions (the first three-dimensional partition in [11]) (see §1.4) for an infinitely renormalizable quadratic polynomial firstly and used it in the study of the local connectivity of the Julia set of an infinitely renormalizable quadratic polynomial. More precisely, we first proved the following theorem.

Theorem 1.7 (Modulus Inequality, Jiang). *Suppose $f : U \to V$ is a renormalizable quadratic-like map and 0 is not periodic. For any n'-renormalization*

$$f_1 = f^{\circ n'} : U' \to V', \quad n' \geq 2,$$

we have

$$mod(U \setminus \overline{U'}) \geq \frac{1}{2} mod(V \setminus \overline{U}).$$

In the theorem, mod$(\cdot)$ means the modulus of an annulus. Using this theorem, we showed the following corollary. The reader may refer to [11] for the concept, complex bounds.

Corollary 1.1 (Jiang). *Suppose q_c is an infinitely renormalizable quadratic polynomial having complex bounds. Then the Julia set J_c is locally connected at 0.*

Furthermore, we constructed the second three-dimensional partition for an infinitely renormalizable quadratic polynomial in [11] (see §1.4), we proved the following theorem. Again, the reader may refer to [11] for the concept, unbranched.

Theorem 1.8 (Jiang). *Suppose q_c is a unbranched infinitely renormalizable quadratic polynomial having complex bounds. Then the Julia set J_c is locally connected.*

The unbranched condition and the complex bounds condition are important in the study of local connectivity. A real infinitely renormalizable quadratic polynomial is unbranched. Many people have worked out some results about complex bounds for real infinitely renormalizable quadratic polynomials. The Feigenbaum polynomial is a real infinitely renormalizable quadratic polynomial $q_\infty(z) = z^2 + t_\infty$, t_∞ real, such that

$$f_i(z) = q_\infty^{\circ 2^i}(z) : U_i \to V_i$$

is a sequence of simple renormalizations (see [20]). Sullivan [Su] (see also [MS,Ji1]) proved that q_∞ has the complex bounds. In the study of Sullivan's result, we concluded in [15] (see also [8, Introduction] for some historic remark).

Corollary 1.2 (Jiang-Hu). *The Julia set of the Feigenbaum polynomial is locally connected.*

Furthermore, Levins and van Strien [17], and later, Lyubich and Yampolsky [19] showed that any real infinitely renormalizable quadratic polynomial has complex bounds. Therefore,

Corollary 1.3 (Levins-van Strien and Lyubich-Yampolsky). *The Julia set of a real infinitely renormalizable quadratic polynomial is locally connected.*

1.3 Some basic facts about the Mandelbrot set from Douady-Hubbard

Consider the Mandelbrot set $\mathcal{M}$ which is the compact set of all parameters c in $\mathbb{C}$ such that the Julia set J_c of q_c is connected. Equivalently, $\mathcal{M}$ consists of all parameters c such that 0 has bounded orbit (see Theorem 1.2). We will often identify c with the corresponding polynomial q_c. A point $c \in \mathcal{M}$ (really means q_c) is called hyperbolic if and only if it has a unique attracting periodic orbit. Let $\{z_0, \cdots, z_{p-1}\}$ be the attracting periodic orbit for a hyperbolic c and let $\lambda(c) = q_c^{\circ p}(z_0)$. The hyperbolic maps form an open subset in $\mathbb{C}$. When c changes in one connected component $\mathcal{H}$ of this open set, the period p and the combinatorial type of the attracting periodic orbit are fixed. Here p is called the period of $\mathcal{H}$. Moreover, $\lambda_{\mathcal{H}}(c) = q_c^{\circ p}(z_0)$ maps $\mathcal{H}$ to $D_1 = \{z \in \mathbb{C} \mid |z| < 1\}$ conformally and can be extends uniquely to a homeomorphism

$$\lambda_{\mathcal{H}}(c) : \overline{\mathcal{H}} \to \overline{D_1}.$$

Note that $\mathcal{H}$ has a unique center $c_{\mathcal{H}} = \lambda_{\mathcal{H}}^{-1}(0)$. The point $r_{\mathcal{H}} = \lambda_{\mathcal{H}}^{-1}(1)$ is called the root of $\mathcal{H}$. For example, the component $\mathcal{H}_0$ bounded by the cardioid is the hyperbolic component of c such that q_c has an attracting fixed point. Then $\lambda_{\mathcal{H}_0}(c) = 1 - \sqrt{1-4c}$. The center is 0, the root is $1/4$, and $\lambda_{\mathcal{H}_0}^{-1}(e^{2\pi i\theta}) = e^{2\pi i\theta}(2 - e^{2\pi i\theta})/4$.

Consider a hyperbolic component $\mathcal{H}$ of period $p \geq 1$ and the corresponding $\lambda_{\mathcal{H}}(c)$. For each $r \in \partial\mathcal{H}$ such that $\lambda_{\mathcal{H}}(r) = e^{2\pi i \frac{m}{p'}}$, $(m, p') = 1$, it is the root of a hyperbolic component $\mathcal{H}'$ of period pp'. (Note that in the case $m = 0$ and $p' = 1$, $\mathcal{H}' = \mathcal{H}$ and in all other cases, $\mathcal{H}' \neq \mathcal{H}$.) Here $\mathcal{H}'$ is called the satellite of $\mathcal{H}$ with an internal angle m/p' and denoted as $\mathcal{H}' = \mathcal{H} * \mathcal{H}(m/p')$.

For each $c \in \mathbb{C}$, let

$$h_c : U_c = \{z \in \mathbb{C} \mid G_c(z) > G_c(0)\} \to \{z \in \mathbb{C} \mid |z| > \exp{(G_c(0))}\}$$

be the Böttcher coordinate from §1.1, where G_c is the Green function for K_c. Then

$$h_c \circ q_c \circ h_c^{-1}(z) = z^2.$$

and $G_c = \log |h_c|$. For $c \in \mathbb{C} \setminus \mathcal{M}$, $c \in U_c$ and thus we have $h_c(c)$.

Theorem 1.9 (Douady-Hubbard). *The map*

$$\Phi_{\mathcal{M}}(c) = h_c(c) : \mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \overline{D}_1$$

is a conformal map. Thus $\mathcal{M}$ is connected.

The equipotential curve of radius $r > 1$ of $\mathcal{M}$ is

$$\mathcal{S}_r = \Phi_M^{-1}(\{c \in \mathbb{C} \mid |c| = r\})$$

and the external ray of angle $0 \le \theta < 1$ of $\mathcal{M}$ is

$$\mathcal{E}_\theta = \Phi_M^{-1}(\{c \in \mathbb{C} \mid |c| > 1 \text{ and } \arg(c) = 2\pi\theta\}).$$

For example, $\mathcal{E}_0 = (1/4, \infty)$ and $\mathcal{E}_{1/2} = (-\infty, -2)$. An external ray $\mathcal{E}_\theta$ is said to land at $\mathcal{M}$ if it has only one limiting point at $\mathcal{M}$. Both of $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ land at $\mathcal{M}$ ($\mathcal{E}_0$ lands at 0 and $\mathcal{E}_{1/2}$ lands at -2). Furthermore,

1) every external ray $\mathcal{E}_\theta$ of rational angle $\theta = m/p$ lands at a point in $\mathcal{M}$;

2) if $\theta = m/p$ with $(m,p) = 1$ and $p = 2p'$, then $\mathcal{E}_\theta$ lands at a point $c \in \mathcal{M}$ such that the critical orbit CO of q_c is preperiodic (such a c is called a Misiurewicz point). Conversely, every Misiurewicz point is a landing point of an external ray $\mathcal{E}_\theta$ of angle $\theta = m/p$ with $(m,p) = 1$ and $p = 2p'$;

3) For $\theta = m/p$ with $(m,p) = 1$ and $p = 2p'$, let c be the landing (Misiurewicz) point of $\mathcal{E}_\theta$, the external ray e_θ in the z-plane for q_c lands at its critical value of c (see [28] for some similarity between $\mathcal{M}$ around a Misiurewicz c and the corresponding J_c around c);

4) let $\mathcal{H}$ be a hyperbolic component of period $p > 1$, there are exactly two external rays $\mathcal{E}_{\theta^-}$ and $\mathcal{E}_{\theta^+}$ land at the root $r_{\mathcal{H}}$ where $\theta^- = m^-/(2^p - 1)$ and $\theta^+ = m^+/(2^p - 1)$ for $1 \le m^- < m^+ < 2^p - 1$.

Consider the component $\mathcal{H}_0$ bounded by the cardioid and a satellite $\mathcal{H}(m/p) = \mathcal{H} * \mathcal{H}_0(m/p)$ of angle m/p, $(m,p) = 1$. Let $\mathcal{E}_{\theta^-}$ and $\mathcal{E}_{\theta^+}$ be two external rays landing at the root $r_{\mathcal{H}(m/p)}$. Then the closure of $\mathcal{E}_{\theta^-} \cup \mathcal{E}_\theta^+$ cuts $\mathbb{C}$ into two connected components. Let $\mathcal{W}_{m/p}$ be the one containing $\mathcal{H}(m/p)$. See [7] for the following theorem.

Theorem 1.10 (Goldberg-Milnor). *For any $c \in \mathcal{W}_{m/p}$, consider q_c. There are exactly p external rays in the z-plane that land at the separating fixed point α_c of q_c. These p external rays have angles $2^i\theta^+$, $i = 0, \cdots, p-1$. Moreover, c is in the domain bounded by two external rays of angle θ^- and θ^+.*

The main conjecture in this direction is that the Mandelbrot set $\mathcal{M}$ is locally connected. From the Carathéodory theorem, if $\mathcal{M}$ is locally connected, then $\Phi_{\mathcal{M}}^{-1}$ can be extended to a comtinuous map from $\mathbb{C} \setminus D_1$ to $\mathbb{C} \setminus \mathcal{M}$, thus every external ray lands at a unique point in $\mathcal{M}$.

By applying the Douady-Hubbard map $\Phi_{\mathcal{M}}$, one can transfer Yoccoz partition $\xi = \{\eta_n\}_{n=1}^{\infty}$ to a partition $\Xi = \{\Theta_n\}_{n=0}^{\infty}$ around all finitely renormalizable points in the parameter space. Using this para- partition, Yoccoz further proved the following theorem (refer to [9, 24]).

Theorem 1.11 (Yoocoz). *The Mandelbrot set $\mathcal{M}$ is locally connected at every finitely renormalizable point.*

For indifferent points c (i.e., q_c has a neutral periodic point), Yoccoz proved that the Mandelbrot set is locally connected at these points. The key step in his proof is the Yoccoz inequality as follows (refer to [9]). (For rational indiferent points, one needs to argue more, see [25] or [29] for a clarification.)

Consider a monic polynomial $P(z)$ of degree d whose Julia set is connected. Let p be a repelling fixed point of P and let $\lambda = P'(p)$. Suppose there are totally m' external rays of P landing at p. Label these external rays in cyclical order. Suppose the i^{th} external ray is mapped to the $[(i + k') \pmod{m'}]^{th}$ external ray. Let $r = \gcd(m', k')$. Then there are r cycles of external rays landing at p. Let $m' = rm$ and $k' = rk$, $(m, k) = 1$. The Yoccoz inequality says that there is a branch τ of $\log \lambda$ satisfying

$$\frac{\Re\tau}{|\tau - 2\pi i \frac{k}{m}|^2} \geq \frac{rm}{2\log d}.$$

In other words, τ belongs to the closed disc of radius $(\log d)/(rm)$ tangent to the imaginary axis at $2\pi ik/m$. The Yoccoz inequality reveals a relation between the analytic derivative τ and the combinatorial derivative $2\pi ik/m$ of P at a fixed point.

Concluding from Yoccoz' results, the Mandelbrot set $\mathcal{M}$ is locally connected at those points c such that q_c are not infinitely renormalizable. Therefore, the study of the local connectivity of the Mandelbrot set $\mathcal{M}$ is concentrated at all infinitely renormalizable points.

By applying the Yoccoz inequality, Douady-Hubbard constructed a generic subset of infinitely renormalizable quadratic polynomials whose Julia set is non-locally connected but $\mathcal{M}$ is locally connected at every point in this subset. They used a method called tuning (or called unrenormalization) in this construction. The tuning can be described roughly as follows.

Consider a hyperbolic component $\mathcal{H}$ of period p. There are two external rays $\mathcal{E}_{\theta^-(\mathcal{H})}$ and $\mathcal{E}_{\theta^+(\mathcal{H})}$ land at its root $r_{\mathcal{H}}$. The set $\mathcal{E}_{\theta^-(\mathcal{H})} \cup \mathcal{E}_{\theta^+(\mathcal{H})} \cup \{r_{\mathcal{H}}\}$ cuts $\mathbb{C}$ into two domains. Denote the one containing $\mathcal{H}$ as $\mathcal{W}(\mathcal{H})$. Then $\mathcal{W}(\mathcal{H})$ contains a small copy $\mathcal{M}(\mathcal{H})$ of the Mandelbrot set $\mathcal{M}$ such that $\mathcal{H}$ is the image of $\mathcal{H}_0$. For every $c \in \mathcal{M}(\mathcal{H})$, q_c is renormalizable and, more precisely, there is a simple renormalization

$$q_c^{\circ m(\mathcal{H})} : U[c] \to V[c].$$

Let $\mathcal{H}'$ be a satellite of $\mathcal{H}$ of period p'. We can similarly construct $\mathcal{W}(\mathcal{H}')$ and $\mathcal{M}(\mathcal{H}')$ for $\mathcal{H}'$ but we will denote them as $\mathcal{W}(\mathcal{H} * \mathcal{H}')$ and $\mathcal{M}(\mathcal{H} * \mathcal{H}')$. (Without having confusion, we also denote them as $\mathcal{W}(p * p')$ and $\mathcal{M}(p * p')$). For every $c \in \mathcal{W}(p * p')$,

$$q_c^{\circ m(p)} : U[c] \to V[c]$$

is renormalizable and has a simple renormalization

$$q_c^{\circ m(p)p'} : U'[c] \subset U[c] \to V'[c] \subset V[c].$$

This processing is called tuning. Starting from the component $\mathcal{H}_0$ bounded by the cardioid and a sequence of positive integers, $p_0, p_1, \cdots$, Douady and Hubbard constructed a sequence of nested tuning sets

$$\tilde{\mathcal{W}}_{p_0 * p_1 * \cdots * p_n} = \mathcal{W}(p_0 * p_1 * \cdots * p_n) \cap \mathcal{M}.$$

Using the Yoccoz inequality, they showed that if p_n tends to ∞ fast enough (i.e., $p_n >> p_{n-1}$), the diameter $d(\tilde{\mathcal{W}}_{p_0 * p_1 * \cdots * p_n})$ tends to zero as n goes to infinity, therefore, $\mathcal{M}$ is locally connected at the intersection point

$$c \in \cap_{n=0}^{\infty} \tilde{\mathcal{W}}_{p_0 * p_1 * \cdots * p_n}.$$

Furthermore, q_c is infinitely renormalizable whose Julia set is not locally connected (see [23]) due to the fast growth of p_n.

In this paper, we will study those infinitely renormalizable points whose Julia sets are locally connected.

1.4 Constructions of three-dimensional partitions.

Take an infinitely renormalizable quadratic polynomial $q_c(z) = z^2 + c$. Let U and V be domains bounded by equipotential curves such that $f_0 = q_c : U \to V$ is a quadratic-like map. Let $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\eta_n^0 = \eta_n$, $\xi^0 = \xi$, $C_n^0 = C_n$, $\Lambda_n^0 = \Lambda_n$, and $\Lambda_\infty^0 = \cup_{n=0}^\infty \Lambda_n^0$ be the same as §1.2. Let $n_1 \geq 0$ and $k_1 \geq 2$ be two integers such that

$$f_1 = f_0^{\circ k_1} : C_{k_1+n_1}^0 \to C_{n_1}^0 \subset N(J_1, 1) \quad \text{where} \quad J_1 = \cap_{i=0}^\infty C_i^0.$$

Suppose β_1 and α_1 are the non-separating and separating fixed points of f_1, i.e., $J_1 \setminus \{\beta_1\}$ is still connected and $J_1 \setminus \{\alpha_1\}$ is not. The points β_1 and α_1 are also repelling periodic points of q_c. There are at least two, but a finite number, external rays of q_c landing at α_1. Let Λ_0^1 be the union of external rays landing at α_1. Then Λ_0^1 cuts $V_0^1 = C_{n_1}^0$ into a finite number of closed domains. Let η_0^1 be the collection of these domains. Let $\Lambda_n^1 = f_1^{-n}(\Lambda_0^1)$ for any $n > 0$. Then Λ_n^1 cuts $V_n^1 = f_1^{-n}(V_0^1)$ into a finite number of closed domains. Let η_n^1 be the collection of these domains. The sequence $\xi^1 = \{\eta_n^1\}_{n=0}^\infty$ is a sequence of nested partitions about J_1. We call it the first partition. (We also call ξ^0 the 0^{th} partition.) Let $\Lambda_\infty^1 = \cup_{n=0}^\infty \Lambda_n^1$.

The domain $C_n^1 \in \eta_n^1$ containing 0 is called the critical piece. It is clear the restriction f_1 to C_n^1 is degree two branched covering map but to all other domains in η_n^1 are degree one. Let

$$J_2 = \cap_{n=0}^\infty C_n^1.$$

There are two integers $n_2 \geq 0$ and $k_2 \geq 2$ such that

$$f_2 = f_1^{\circ k_2} : C_{n_2+k_2}^1 \to C_{n_2}^1$$

is a degree two branched cover map and such that $C_{n_2}^1 \subset N(J_2, 1/2)$.

Inductively, for every $i \geq 2$, suppose we have already constructed

$$f_i = f_{i-1}^{\circ k_i} : C_{n_i+k_i}^{i-1} \to C_{n_i}^{i-1}.$$

Let β_i and α_i be the non-separating and separating fixed points of f_i; i.e., $J_i \setminus \{\beta_i\}$ is still connected and $J_i \setminus \{\alpha_i\}$ is not. The points β_i and α_i are also repelling periodic points of q_c. There are at least two, but a finite number, external rays of q_c landing at α_i. Let Λ_0^i be the union of external rays landing at α_i. Then Λ_0^i cuts $V_0^i = C_{n_i}^{i-1}$ into finitely many closed domains. Let η_0^i be the collection of these domains. Let $\Lambda_n^i = f_i^{-n}(\Lambda_0^i)$ for any $n > 0$. Then Λ_n^i cuts $V_n^i = f_i^{-n}(V_0^i)$ into finitely many closed domains. Let η_n^i be the collection of these domains. The domain C_n^i in η_n^i containing 0 is called the critical piece in η_n^i. It is clear that f_i restricted to all domains but C_n^i are bijective and $f_i|C_n^i$ is a degree two branched covering map. Let

$$J_{i+1} = \cap_{n=0}^{\infty} C_n^i.$$

There are two integers $n_{i+1} \geq 0$, $k_{i+1} \geq 2$ such that

$$f_{i+1} = f_i^{\circ k_{i+1}} : C_{n_{i+1}+k_{i+1}}^i \to C_{n_{i+1}}^i$$

is a degree two branched cover map and such that $C_{n_{i+1}}^i \subset N(J_{i+1}, 1/(i+1))$. Let $\xi^i = \{\eta_n^i\}_{n=0}^{\infty}$. It is the sequence of nested partitions about J_i. We call it the i^{th} partition.

Remark 1.1. *For any k_{i+1}-renormalization $f_{i+1} = f_i^{\circ k_{i+1}} : U' \to V'$ of $f_i : U_i \to V_i$, we have an integer $n > 0$ such that $C_n^i \subset V' \cap N(J_{i+1}, 1/(i+1))$ and $f_{i+1} = f_i^{\circ k_{i+1}} : C_{n+k_{i+1}}^i \to C_n^i$ is a degree two branched covering map. We will still use ξ^i to mean $\xi^i \cap C_{n+k_{i+1}}^i$. Therefore, (U_{i+1}, V_{i+1}) can be an arbitrary domains such that $f_{i+1} = f_i^{\circ k_{i+1}} : U_{i+1} \to V_{i+1}$ is a k_{i+1}-renormalization of $f_i : U_i \to V_i$.*

Let $m_i = \prod_{j=1}^{i} k_i$, $1 \leq i < \infty$. We have thus constructed a most natural infinite sequence of simple renormalizations,

$$\{f_i = q_c^{\circ m_i} : U_i \to V_i\}_{i=1}^{\infty},$$

and the nested-nested sequence $\{\xi^i\}_{i=0}^{\infty}$ of partitions about $\{J_i\}_{i=0}^{\infty}$ (where $J_0 = J_c$). Then $\Xi = \{\xi^i\}_{i=0}^{\infty}$ is our first three-dimensional partition about J_c.

Now we construct our second three-dimensional partition Υ about J_c. Denote by κ_1 the first partition which will be constructed as follows: Consider $\xi^0 = \{\eta_n^0\}_{n=0}^{\infty}$ in Ξ. Take $C_{k(0)}^0 \in \eta_{k(0)}^0 \in \xi^0$ where $k(0) = n_1 + k_1$. Put all domains in $\eta_{k(0)+1}^0$ which are the preimages of $C_{k(0)}^0$ under q_c into κ_1 and let $\eta_{k(0)+1}^{0c}$ be the rest of the domains. Consider $\eta_{k(0)+2}^0 \cap \eta_{k(0)+1}^{0c}$ consisting of all domains in $\eta_{k(0)+2}^0$ which are subdomains of the domains in $\eta_{k(0)+1}^{0c}$. Put all domains in $\eta_{k(0)+2}^0 \cap \eta_{k(0)+1}^{0c}$ which are the preimages of $C_{k(0)}^0$ under $q_c^{\circ 2}$ into κ_1 and let $\eta_{k(0)+2}^{0c}$ be the rest of the domains. Suppose we already have $\eta_{k(0)+s}^{0c}$ for $s \geq 2$. Consider $\eta_{k(0)+s+1}^0 \cap \eta_{k_0+s}^{0c}$ consisting of all domains in $\eta^0_{k(0)+s+1}$ which are subdomains of the domains in $\eta^{0c}_{k(0)+s}$. Put all domains in $\eta^0_{k(0)+s+1} \cap \eta^{0c}_{k(0)+s}$ which are the preimages of $C^0_{k(0)}$ under $q_c^{\circ(s+1)}$ into κ_1 and let $\eta^{0c}_{k(0)+s+1}$ be the rest of the domains. Thus we can construct the partition κ_1 inductively. This partition covers points in J_c minus all points not entering the interior of $C^0_{k(0)}$ under all forward iterations of q_c.

Next consider the $\xi^1 = \{\eta^1_n\}_{n=0}^{\infty}$ in Ξ. Take $C^1_{k(1)} \in \eta^1_{k(1)} \in \xi^1$ where $k(1) = n_2 + k_2$. We can use similar arguments to those in the previous paragraph by considering $f_1 : C^0_{k(0)} \to C^0_{k(0)-k_1}$ (to replacing $q_c : U \to V$) to get a partition $\kappa_{1,1}$ in $C^0_{k(0)}$. Then we use all iterations of q_c to pull back this partition following κ_1 to get a partition κ_2. It is a sub-partition of κ_1 and covers points in J_c minus all points not entering the interior of $C^1_{k(1)}$ under iterations of q_c.

Suppose we have already constructed the $(j-1)^{th}$ partition κ_{j-1} for $j \geq 2$. Consider the partition $\xi^j = \{\eta^j_n\}_{n=0}^{\infty}$ in Ξ. Take $C^j_{k(j)} \in \eta^j_{k(j)} \in \xi^j$ where $k(j) = n_{j+1} + k_{j+1}$. Similarly, by considering $f_j : C^{j-1}_{k(j-1)} \to C^{j-1}_{k(j-1)-k_j}$, we get a partition $\kappa_{j,1}$ in $C^{j-1}_{k(j-1)}$. Then we use all backward iterations of f_{j-1} to pull back this partition following κ_{j-1} to get a partition $\kappa_{j,2}$ in $C^{j-2}_{k(j-2)}$ and all backward iterations of f_{j-2} to pull back this partition following κ_{j-1} to get a partition $\kappa_{j,3}$ in $C^{j-3}_{k(j-3)}$, and so on to obtain a partition $\kappa_j = \kappa_{j,j}$ in V. It is a sub-partition of κ_{j-1} and covers points in the Julia set minus all points not entering the interior of $C^j_{k(j)}$ under forward iterations of q_c. By the induction, we have a sequence of nested partitions

$$\Upsilon = \{\kappa_j\}_{j=1}^{\infty}$$

which covers points in $J_c \setminus \Gamma$. Then $\Upsilon = \{\kappa_j\}_{j=1}^{\infty}$ is our second three-dimensional partition about J_c.

1.5 Statements of new results.

Yoccoz partitions can be transferred completely to the parameter space naturally by using the Douady-Hubbard map $\Phi_{\mathcal{M}}$. But it is more difficult to transfer three-dimensional partitions to the parameter space completely. The difficulty is around those Feigenbaum-like points. In this paper, we will partially transfer three-dimensional partitions to the parameter space. We will construct a subset of the Mandelbrot set such that (1) this subset consists of infinitely renormalizable points, (2) this subset is dense on the boundary of the Mandelbrot set, (3) we can transfer three-dimensional partitions for infinitely renormalizable quadratic polynomials in this subset to a partition around this subset set in the parameter space, and (4) the partition is good enough to study the local connectivity of this subset in the Mandelbrot set. More precisely, we prove in this paper that

Theorem 1.12 (Main Theorem). *Suppose c is a Misiurewicz point. Then there is a subset $\mathcal{A}(c) \subset \mathcal{M}$ such that*

(i) c is a limit point of $\mathcal{A}(c)$,

(ii) for every $c' \in \mathcal{A}(c)$, $q_{c'}(z) = z^2 + c'$ is an unbranched infinitely renormalizable quadratic polynomial having complex bounds and the Julia set $J_{c'}$ is locally connected, and

(iii) the Mandelbrot set $\mathcal{M}$ is locally connected at every point $c' \in \mathcal{A}(c)$.

Since the set $\mathcal{B}$ of Misiurewicz points is dense on the boundary of $\mathcal{M}$, we have

Corollary 1.4. *The set $\mathcal{A} = \cup_{c\in\mathcal{B}}\mathcal{A}(c)$ is dense on the boundary $\partial\mathcal{M}$.*

In the proof of Theorem 1.12, we will use the following classical theorem in complex analysis (refer to [16]).

Theorem 1.13 (Rouché's Theorem). *Suppose D is a domain in the complex plane $\mathbb{C}$. Let $\gamma \subset D$ be a closed path homologous to 0 and assume that γ has an interior. Let f and g be analytic on D, and*

$$|f(z) - g(z)| < |f(z)|, \quad z \in \gamma.$$

Then f and g have the same number of zeros in the interior of γ.

The rest of the paper is arranged as follows. To have a better explanation to our idea, we first study a special Misiurewicz point -2 in the Mandelbrot set and prove in §2 the following theorem.

Theorem 1.14 (Special Case). *There is a subset $\mathcal{A}(-2) \subset \mathcal{M}$ consisting of infinitely renormalizable points such that -2 is a limit point of $\mathcal{A}(-2)$ and the Mandelbrot set $\mathcal{M}$ is locally connected at $\mathcal{A}(-2)$, Moreover, for every $c \in \mathcal{A}(-2)$, the Julia set J_c of q_c is locally connected.*

Then by combining the proof of Theorem 1.14, we give a complete proof of Theorem 1.12.

2 Basic idea in our construction

Let $q_{-2}(z) = z^2 - 2$. Its Julia set J_2 is $[-2, 2]$. It has the non-separating fixed point $\beta(-2) = 2$ and the separating fixed point $\alpha(-2)$, i.e., $J_{-2} \setminus \{\beta(-2)\}$ is still connected and $J_{-2} \setminus \{\alpha(2)\}$ is disconnected. To use notations clearly, we use the letter G to denote a set in the phase space and the Letter Λ to denote the same set but in the Böttcher coordinate.

Let $G_{0,l}(-2)$ be the closure of the union of two external rays landing at $\alpha(-2)$. Let h_{-2} be the Bötteche coordinate for q_{-2}, i.e.,

$$h_{-2} \circ q_{-2} \circ h_{-2}^{-1}(z) = z^2.$$

Then h_{-2} maps $G_{0,l}(-2)$ to two straight rays in $\mathbb{C} \setminus \overline{D}_1$ (which have angles $1/3$ and $2/3$). We use $\Lambda_{0,l}$ to denote the closure of the union of these two straight rays.

Let

$$\mathbb{L}H = \{z = x + yi \in \mathbb{C} \mid x < 0\} \quad \text{and} \quad \mathbb{R}H = \{z = x + yi \in \mathbb{C} \mid x > 0\}$$

be the left and right half planes. Consider $q_0(z) = z^2$. The restriction $q_0|(\mathbb{C} \setminus (-\infty, 0])$ has two inverse branches

$$g_{l,0} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{L}H \qquad \text{and} \qquad g_{r,0} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{R}H.$$

Let $\Lambda_{n,r} = g_{r,0}^{n+1}(\Lambda_{0,l})$ and $\Lambda_{n+1,l} = g_{l,0}^{n+1}(\Lambda_{n,r})$ for $n = 0, 1, \cdots$. It is easy to see that $\Lambda_{n,r}$ tends to the ray of angle 0 and $\Lambda_{n,l}$ tends to the ray of angle $1/2$. Let

$$\Lambda = [\cup_{n=0}^{\infty}(\Lambda_{n,r} \cup \Lambda_{n,l})] \cup ((-\infty, -1] \cup [1, \infty)).$$

Let $G(-2) = h_{-2}^{-1}(\Lambda)$. Then it is the union of external rays for q_{-2} landing at the set

$$A(-2) = [\cup_{n=0}^{\infty}(g_{r,-2}^{n}(\alpha(-2)) \cup g_{l,-2}^{n}(\alpha(-2)))] \cup \{-2, 2\}.$$

We also denote $G_{n,r}(-2) = h_{-2}^{-1}(\Lambda_{n,r})$ and $G_{n,l}(-2) = h_{-2}^{-1}(\Lambda_{n,l})$.

For $c \in \mathcal{W}_{1/2}$, let $G_{0,l}(c)$ be the closure of union of two external rays landing at its separating fixed point $\alpha(c)$. Let $\overline{\alpha}(c)$ be another preimage of $\alpha(c)$. Let $G_{0,r}(c)$ be the closure of union of two external rays landing at $\overline{\alpha}(c)$. Then $G_{0,l}(c) \cup C_{0,r}(c)$ cuts $\mathbb{C}$ into three domains. One, which we call $E_r(c)$, contains the non-separating fixed point $\beta(c)$ of q_c. The other,

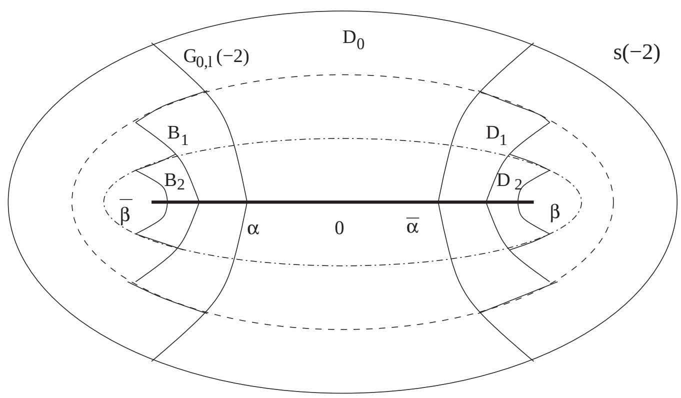

Figure 1: Basic idea in our construction in the dynamical plane.

which we call $E_l(c)$, contains the other preimage $\overline{\beta}(c)$ of $\beta(c)$ under q_c. The restriction $q_c|(E_{l,c}(c)\cup E_{r,c})(c)$ has two inverse branches

$$g_{l,c} : E \to E_l(c) \quad \text{and} \quad g_{r,c} : E \to E_r(c).$$

Let h_c be the Böttcher coordinate for $q_c(z) = z^2 + c$, i.e.,

$$h_c \circ q_c \circ h_c^{-1} = z^2.$$

Let $G(c) = h_c^{-1}(\Lambda)$. Then it is the union of external rays for q_c landing at the set

$$A(c) = [\cup_{n=0}^{\infty}(g_{r,c}^n(\alpha(c)) \cup g_{l,c}^n(\alpha(c)))] \cup \{\beta(c), \overline{\beta}(c)\}.$$

We also denote $G_{n,r}(c) = h_c^{-1}(\Lambda_{n,r})$ and $G_{n,l}(c) = h_c^{-1}(\Lambda_{n,l})$.

Let

$$\phi_c = h_c^{-1} \circ h_{-2}.$$

When restricted to $\mathcal{G}(-2)$, ϕ_c conjugates $q_{-2}|\mathcal{G}(-2)$ to $q_c|\mathcal{G}(c)$.

Let $U(-2)$ be a fixed domain bounded by an equipotential curve $s(-2)$ for q_{-2}. Then $q_{-2} : U(-2) \to V(-2)$ is a quadratic-like map where $V(-2) = q_{-2}(U(-2))$. The set $G_{0,l}(-2)\cup G_{0,r}(-2)$ cuts $U(-2)$ into three disjoint domains. Denote $D_0(-2)$ the closure of the one containing 0 (see Figure 1).

Let $D_n(-2) = g_{r,-2}^{\circ n}(D_0(-2))$ and let $B_n(-2) = g_{l,-2}^{\circ n}(D_{n-1}(-2))$ for $n \geq 1$ (see Figure 1). Since 2 is an expanding fixed point of q_{-2} and $q(-2) = 2$, the diameter $\mathrm{diam}(B_n)$ of B_n tends to zero exponentially as n goes to infinity. Let $0 \notin \mathcal{U}_0$ be a small neighborhood about -2 in the parameter space such that $\mathrm{diam}(\mathcal{U}_0) \leq 1$ and such that the corresponding graph $G(c)$ for q_c exists for c in $\mathcal{U}_0$. For c in $\mathcal{U}_0$, let $\phi_c : G(-2) \to G(c)$ be the conjugacy from q_{-2} to q_c. Let $s_c = \phi_c(s)$ be the corresponding equipotential curve for q_c. Let $U(c)$ be the closure of the domain bounded by s_c. Similarly, $G_{0,l}(c) \cup G_{0,r}(c)$ cuts $U(c)$ into three disjoint domains. Denote $D_0(c)$ be the closure of the one containing 0. Let $D_n(c) = g_{r,c}^{\circ n}(D_0(c))$ and let $B_n(c) = g_{l,c}(D_{n-1}(c))$ for $n \geq 1$. Note that $\beta(c)$ and $\alpha(c)$ are the non-separating and separating fixed points of q_c and $\overline{\beta}(c)$ is the other inverse image of $\beta(c)$ under q_c. Then $B_n(c)$ and $D_n(c)$ tend to $\overline{\beta}(c)$ and $\beta(c)$, respectively, as n goes to infinity. Since $\beta(c)$ is an expanding fixed point of q_c and since there is a constant $\mu > 1$ such that $|q_c'(\beta(c))| \geq \mu$ for all c in $\mathcal{U}_0$, the diameter $\mathrm{diam}(B_n(c))$ tends to 0 uniformly on $\mathcal{U}_0$ and the set $B_n(c)$ approaches to $\overline{\beta}(c)$ uniformly on $\mathcal{U}_0$ as n goes to infinity. Let

$$\mathcal{W}_n = \{c \mid c \in B_n(c)\}.$$

Then $\mathcal{W}_n$, for $n = 1, 2, \cdots$, give a partition in the parameter space.

Let $f(c) = q_c(0) - \overline{\beta}(c)$ and $g(c) = q_c(0) - x$ for $x \in B_n(c)$. Suppose $\gamma = \partial\mathcal{U}_0$ is a closed path homologous to 0 in $\mathcal{U}_0$ such that

$$m = \min_{c \in \partial\mathcal{U}_0} = \min_{c \in \partial\mathcal{U}_0} |q_c(0) - \overline{\beta}(c)| > 0.$$

Because $B_n(c)$ approaches $\overline{\beta}(c)$ uniformly on $\mathcal{U}_0$ as n goes to infinity, there is an integer $N_0 > 0$ such that for $n \geq N_0$,

$$|f(c) - g(c)| = |\overline{\beta}(c) - x| < m \leq |f(c)|, \quad c \in \partial\mathcal{U}_0.$$

Since the equation $f(c) = q_c(0) - \overline{\beta}(c) = 0$ has a unique solution -2 in $\mathcal{U}_0$, Theorem 1.13 (Rouché's Theorem) implies that $g(c) = q_c(0) - x = 0$ has a unique solution, which is in $\mathcal{U}_0$, for any x in $B_n(c)$ for $n \geq N_0$. Therefore $\mathcal{W}_n \subseteq \mathcal{U}_0$ for $n \geq N_0$. So $\mathrm{diam}(\mathcal{W}_n) \leq 1$ for $n \geq N_0$ (see Figure 2).

Lemma 2.1. *The intersection $\tilde{\mathcal{M}}_n = \mathcal{W}_n \cap \mathcal{M}$ for $n \geq N_0$ is connected.*

Proof. The boundary $B_n(c)$ consists of four curves $G_{n,l}(c)$, $G_{n-1,l}(c)$, and some pieces of the equipotential curve $s_n(c) = q_c^{-n}(s(c))$. Note that $q_c^{n+1}(G_{n,l}(c)) = G_{0,l}(c)$ and $q_c^n(G_{n-1,r}(c)) = G_{0,l}(c)$. The domain $\mathcal{W}_n$ is bounded by the closure of two external rays

$$\mathcal{G}_n = \{c \mid c \in G_{n,l}(c)\}, \quad \mathcal{G}_{n-1} = \{c \mid c \in G_{n-1}(c)\}$$

and some pieces of the equipotential curve

$$\mathcal{S}_n = \{c \mid c \in s_n(c)\}.$$

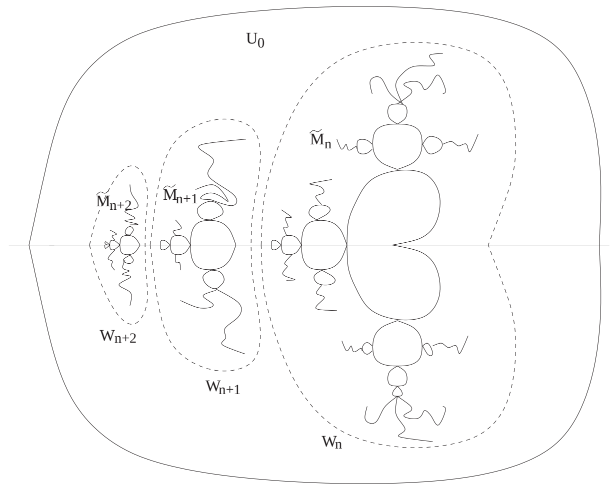

Figure 2: Basic idea in our construction in the parameter space.

Each of intersections $\mathcal{G}_n \cap \mathcal{M}$ and $\mathcal{G}_{n-1} \cap \mathcal{M}$ consists only one point, denoted as a_n and a_{n-1}. The closure of $\mathcal{G}_n$ (or $\mathcal{G}_{n-1}$) in the extended complex plane is a topological curve isomorphic to a circle and it intersects $\mathcal{M}$ only at a_n (or a_{n-1}). Let us still denote these extended curves as $\mathcal{G}_n$ and $\mathcal{G}_{n-1}$. Then $\mathcal{G}_n$ and $\mathcal{G}_{n-1}$ cut $\mathbb{C}$ into three domains. The one on a_n side is called $\mathcal{X}_n$ and the one on a_{n-1} side is called $\mathcal{X}_{n-1}$.

If $\tilde{\mathcal{M}}_n$ is disconnected. Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty domains such that $\mathcal{U} \cap \mathcal{V} = \emptyset$ and such that

$$(\mathcal{U} \cap \tilde{\mathcal{M}}_n) \cup (\mathcal{V} \cap \tilde{\mathcal{M}}_n) = \tilde{\mathcal{M}}_n.$$

Assume $a_n \in \mathcal{U}$ and $a_{n-1} \in \mathcal{V}$ (other cases can be proved similarly). Let $\tilde{\mathcal{U}} = \mathcal{U} \cup \mathcal{X}_n$ and $\tilde{\mathcal{V}} = \mathcal{V} \cup \mathcal{X}_{n-1}$. Then $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ are two non-empty domains such that $\tilde{\mathcal{U}} \cap \tilde{\mathcal{V}} = \emptyset$ and such that

$$(\tilde{\mathcal{U}} \cap \mathcal{M}) \cup (\tilde{\mathcal{V}} \cap \mathcal{M}) = \mathcal{M}.$$

This would say that $\mathcal{M}$ is disconnected. This contradiction implies that $\tilde{\mathcal{M}}_n$ must be connected. □

Proof of Theorem 1.14 (Special Case). For each $\mathcal{W}_n$ where $n \geq N_0$, since $c \in B_n(c)$, $C_n(c) = q_c^{-1}(B_n(c))$ is the closure of a connected and simply connected domain which contains 0 and which is a sub-domain of $D_0(c)$. Let

$$f_{n,c} = q_c^{\circ(n+1)} : \mathring{C}_n(c) \to \mathring{D}_0(c).$$

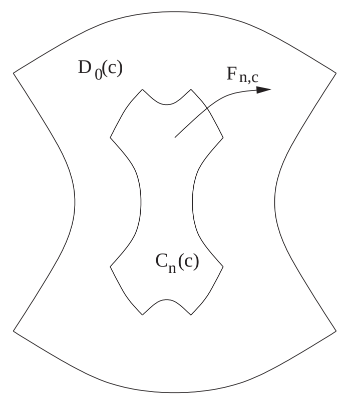

Figure 3: Construction of a quadratic-like map by renormalization.

Then it is a quadratic-like map (see Figure 3). Furthermore, since $\mathrm{diam}(B_n(c))$ tends to zero as n goes to infinity uniformly on $\mathcal{U}_0$, for N_0 large enough we can have that the modulus $\mathrm{mod}(A_n(c)) \geq 1$ where $A_n(c) = \mathring{D}_0(c) \setminus C_n(c)$. Moreover, the family

$$\{f_{n,c} : \mathring{C}_n(c) \to \mathring{D}_0(c) \ ; \ c \in \mathcal{W}_n\}$$

is a full family, so $\mathcal{W}_n$ contains a copy $\mathcal{M}_n$ of the Mandelbrot set $\mathcal{M}$ (refer to [23, pp. 102-106],). For $c \in \mathcal{M}_n$, the Julia set $J_{f_{n,c}}$ of $f_{n,c} : \mathring{C}_n(c) \to \mathring{D}_0(c)$ is connected. Therefore, for $c \in \mathcal{A}_1 = \cup_{n \geq N_0}^{\infty} \mathcal{M}_n$, q_c is once renormalizable.

For a fixed integer $i_0 \geq N_0$, consider $\mathcal{W}_{i_0}$ and $\mathcal{M}_{i_0}$; there is a parameter $c_{i_0} \in \mathcal{M}_{i_0}$ such that $f_{i_0} = f_{i_0, c_{i_0}} : \mathring{C}_{i_0} = \mathring{C}_{i_0}(c_{i_0}) \to \mathring{D}_{i_0} = \mathring{D}_0(c_{i_0})$ is hybrid equivalent to $q_{-2}(z) = z^2 - 2$. The quadratic-like map $f_{i_0} : \mathring{C}_{i_0} \to \mathring{D}_{i_0}$ has the non-separate fixed point and the separate fixed point, which we simple denote as β and α. Let $\overline{\beta}$ be another pre-image of β under f_{i_0}.

Let G be the closure of the union of two external rays for $q_{c_{i_0}}$ landing at α. Then $f_{i_0}(G) = G$ and $G \cap J_{f_{i_0}} = \{\alpha\}$. Let $\tilde{G} = f_{i_0}^{-1}(G)$. Then $\tilde{G}$ cuts C_{i_0} into three domains. Denote $D_{i_0 0}$ be the one containing 0. Let $\overline{\beta} \in E_{i_0 0}$ and $\beta \in E_{i_0 1}$ be the components of the closure of $C_{i_0} \setminus D_{i_0 0}$. Let $g_{i_0 0}$ and $g_{i_0 1}$ be the inverses of $f_{i_0}|E_{i_0 0}$ and $f_{i_0}|E_{i_0 1}$. Let

$$D_{i_0 n} = g_{i_0 1}^{\circ n}(D_{i_0 0}) \qquad \text{and} \qquad B_{i_0 n} = g_{i_0 0}(D_{i_0 (n-1)})$$

for $n \geq 1$. Again by the Böttcher coordinates $h_{c_{i_0}}$ and h_c, we can similarly prove that β, $\overline{\beta}$, α, and G, $\tilde{G}$, and ∂C_{i_0} are preserved for c close to c_{i_0}. Similar to the argument in the above, we can find a small neighborhood

$\mathcal{U}_{i_0}$ about c_{i_0} with $\mathrm{diam}(\mathcal{U}_{i_0}) \leq 1/2$ such that the corresponding domains $B_{i_0}(c)$ and $D_{i_0}(c)$ can be constructed for q_c for $c \in \mathcal{U}_{i_0}$. Let

$$\mathcal{W}_{i_0 n} = \{c \in \mathbb{C} \mid f_{i_0,c}(0) \in B_{i_0 n}(c)\}.$$

Then $\{\mathcal{W}_{i_0 n}\}_{i_0 \geq 1, n \geq 1}$ give a sub-partition of $\{\mathcal{W}_n\}_{n \geq 1}$.

The diameter $\mathrm{diam}(B_{i_0 n}(c))$ tends to zero uniformly on $\mathcal{U}_{i_0}$ and the set $B_{i_0 n}(c)$ approaches to $\overline{\beta}(c)$ uniformly on $\mathcal{U}_{i_0}$ as n goes to infinity. Suppose $\mathcal{U}_{i_0}$ is simply connected and the boundary curve $\gamma = \partial \mathcal{U}_{i_0}$ is a closed path homologous to 0 in $\mathcal{U}_{i_0}$. Since the equation $f_{i_0,c}(0) - \overline{\beta}_{i_0}(c) = 0$ has a unique solution c_{i_0} (following Thurston's theorem for critically finite rational maps (see [5]) and also refer to [4]), $m = \min_{c \in \partial U_{i_0}} |f_{i_0,c}(0) - \overline{\beta}_{i_0}(c)| > 0$. Let $f(c) = f_{i_0,c}(0) - \overline{\beta}_{i_0}(c)$ and $g(c) = f_{i_0,c}(0) - x$ for $x \in B_{i_0 n}(c)$. There is an integer $N_{i_0} > 0$ such that for $n \geq N_{i_0}$,

$$|f(c) - g(c)| = |\overline{\beta}_{i_0}(c) - x| < m \leq |f(c)|, \quad c \in \partial \mathcal{U}_{i_0}.$$

Theorem 1.13 (Rouché's Theorem) now implies that $g(c) = f_{i_0,c}(0) - x = 0$ has a unique solution in $\mathcal{U}_{i_0}$ for any x in $B_{i_0 n}(c)$ for $n \geq N_{i_0}$. Therefore $\mathcal{W}_{i_0 n} \subseteq \mathcal{U}_{i_0}$ for $n \geq N_{i_0}$. So $\mathrm{diam}(\mathcal{W}_{i_0 n}) \leq 1/2$ for $n \geq N_{i_0}$.

Similar to the proof of Lemma 2.1, we have $\tilde{\mathcal{M}}_{i_0 n} = \mathcal{W}_{i_0} \cap \mathcal{M}$ is connected for $n \geq N_{i_0}$. For each c in $\mathcal{W}_{i_0 n}$, $n \geq N_{i_0}$, let $C_{i_0 n}(c) = f_{i_0,c}^{-1}(B_{i_0 n}(c))$. Then

$$f_{i_0 n,c} = f_{i_0,c}^{\circ(n+1)} : \mathring{C}_{i_0 n}(c) \to \mathring{D}_{i_0 0}(c)$$

is a quadratic-like map. Furthermore, since $\mathrm{diam}(B_{i_0 n}(c))$ tends to zero as n goes to infinity uniformly on $\mathcal{U}_{i_0}$, for N_{i_0} large enough we can have the modulus $\mathrm{mod}(A_{i_0 n}(c)) \geq 1$ where $A_{i_0 n}(c) = \mathring{D}_{i_0 0}(c) \setminus C_{i_0 n}(c)$. Moreover,

$$\{f_{i_0 n,c} : \mathring{C}_{i_0 n}(c) \to \mathring{D}_{i_0 0}(c) \mid c \in \mathcal{W}_{i_0 n}\}$$

is a full family. Thus $\mathcal{W}_{i_0 n}$ contains a copy $\mathcal{M}_{i_0 n}$ of the Mandelbrot set $\mathcal{M}$ (refer to [23, pp. 102-106]). For $c \in \mathcal{A}_2 = \cup_{i_0 \geq N_0} \cup_{i_1 \geq N_{i_0}} \mathcal{M}_{i_0 i_1}$, the Julia set of $f_{i_0 n,c} : \mathring{C}_{i_0 n}(c) \to \mathring{D}_{i_0 0}(c)$ is connected, so q_c is twice renormalizable.

We use the induction to complete the construction of our subset $\mathcal{A}(-2)$ around -2. Suppose we have constructed $\mathcal{W}_w$ where $w = i_0 i_1 \ldots i_{k-1}$ and $i_0 \geq N_0$, $i_1 \geq N_{i_1}$, …, $i_{k-1} \geq N_{i_0 i_1 \ldots i_{k-2}}$. Let $v = i_0 \ldots i_{k-2}$. There is a parameter $c_w \in \mathcal{M}_w$ such that $f_w = f_{w,c_w} : \mathring{C}_w = \mathring{C}_w(c_w) \to \mathring{D}_w = \mathring{D}_{v0}(c_w)$ is hybrid equivalent to $q_{-2}(z) = z^2 - 2$. The quadratic-like map $f_w : \mathring{C}_w \to \mathring{D}_w$ has the non-separate fixed point β_w and the separate fixed point α_w. Let G_w be the closure of the union of two external rays for q_{c_w} which land at α_w. Let $\tilde{G}_w = f_w^{-1}(G_w)$. Then $\tilde{G}_w$ cuts C_w into three domains. Let D_{w0} be the one containing 0. Denote $\overline{\beta}_w$ be another preimage of β_w under f_w. Let $\overline{\beta}_w \in E_{w0}$ and $\beta_w \in E_{w1}$ be the components of the closure of $C_w \setminus D_{w0}$. Let g_{w0} and g_{w1} be the inverses of $f_w|E_{w0}$ and $f_w|E_{w1}$. Let

$$D_{wn} = g_{w1}^{\circ n}(D_{w0}) \qquad \text{and} \qquad B_{wn} = g_{w0}(D_{w(n-1)})$$

for $n \geq 1$. By the Böttcher coordinates h_{c_w} and h_c, β_w, $\overline{\beta}_w$, α_w, and G_w, $\tilde{G}_w$, and ∂C_w are preserved for c close to c_w. We can find a small neighborhood $\mathcal{U}_w$ about c_w with $\mathrm{diam}(\mathcal{U}_w) \leq 1/2^k$ such that the corresponding domains $D_{wn}(c)$ and $B_{wn}(c)$ can be constructed for q_c, $c \in \mathcal{U}_w$. Let

$$\mathcal{W}_{wn} = \{c \in \mathbb{C} \mid f_{w,c}(0) \in B_{wn}(c)\}, \quad n = 1, 2, \cdots.$$

They give a sub-partition of $\{\mathcal{W}_w\}$.

The diameter $\mathrm{diam}(B_{wn}(c))$ tends to zero uniformly on $\mathcal{U}_w$ and the set $B_{wn}(c)$ approaches to $\overline{\beta}_w(c)$ uniformly on $\mathcal{U}_w$ as n goes to infinity. Suppose $\mathcal{U}_w$ is simply connected and the boundary curve $\partial\mathcal{U}_w$ is a closed path homologous to 0. Since the equation $f_{w,c}(0) - \overline{\beta}_w(c) = 0$ has a unique solution c_w (following from Thurston's theorem for critically finite rational maps (see [5]) and also refer to [4]),

$$m = \min_{c \in \gamma} |f_{w,c}(0) - \overline{\beta}_w(c)| > 0.$$

Let $f(c) = f_{w,c}(0) - \overline{\beta}_w(c)$ and $g(c) = f_{w,c}(0) - x$ for $x \in B_{wn}(c)$. There is an integer $N_w > 0$ such that for $n \geq N_w$,

$$|f(c) - g(c)| = |\overline{\beta}_w(c) - x| < m \leq |f(c)|, \quad c \in \partial\mathcal{U}_w.$$

Now Theorem 1.13 (Rouché's Theorem) implies that $g(c) = f_{w,c}(0) - x = 0$ has a unique solution in $\mathcal{U}_w$ for any x in $B_{wn}(c)$ and $n \geq N_w$. Therefore $\mathcal{W}_{wn} \subseteq \mathcal{U}_w$ for $n \geq N_w$. So $\mathrm{diam}(\mathcal{W}_{wn}) \leq 1/2^k$ for $n \geq N_w$.

Similar to the proof of Lemma 2.1, we have $\tilde{\mathcal{M}}_{wn} = \mathcal{W}_{wn} \cap \mathcal{M}$ for $n \geq N_w$ is connected. For each c in $\mathcal{W}_{wn}$ where $n \geq N_w$, let $C_{wn}(c) = f_{w,c}^{-1}(B_{wn}(c))$. Then

$$f_{wn,c} = f_{w,c}^{\circ(n+1)} : \mathring{C}_{wn}(c) \to \mathring{D}_{w0}(c)$$

is a quadratic-like map. Furthermore, since $\mathrm{diam}(B_{wn}(c))$ tends to zero as n goes to infinity uniformly on $\mathcal{U}_w$, for N_w large enough we can have that the modulus $\mathrm{mod}(A_{wn}(c)) \geq 1$ where $A_{wn}(c) = \mathring{D}_{w0}(c) \setminus C_{wn}(c)$. Moreover,

$$\{f_{wn,c} : \mathring{C}_{wn}(c) \to \mathring{D}_{w0}(c) \mid c \in \mathcal{W}_{wn}\}$$

is a full family. Therefore, $\mathcal{W}_{wn}$ contains a copy $\mathcal{M}_{wn} \subseteq \tilde{\mathcal{M}}_{wn}$ of $\mathcal{M}$ (refer to [23, pp. 102-106]). For $c \in \mathcal{A}_{k+1} = \cup_w \cup_{i_k \geq N_w} \mathcal{M}_{wi_k}$, q_c is $(k+1)$-*times* renormalizable where $w = i_0 i_1 \ldots i_{k-1}$ runs over all sequences of integers of length k. We thus construct a three-dimensional partition

$$\{\mathcal{W}_{i_0 \ldots i_k}\}_{k=0}^{\infty}$$

about the subset $\mathcal{A}(-2) = \cap_{k=1}^{\infty} \mathcal{A}_k$.

For each $c \in \mathcal{A}(-2)$, q_c is infinitely renormalizable. Furthermore, -2 is a limit point of $\mathcal{A}(-2)$. For each $c \in \mathcal{A}(-2)$, there is a corresponding sequence $w_\infty = i_0 i_1 \ldots i_k \ldots$ of integers such that $\{c\} = \cap_{k=0}^{\infty} \mathcal{W}_{i_0 \ldots i_k}$. Since $\tilde{\mathcal{M}}_{i_0 \ldots i_k} = \mathcal{W}_{i_0 \ldots i_k} \cap \mathcal{M}$ is connected, $\{\mathcal{W}_{i_0 \ldots i_k}\}_{k=0}^{\infty}$ is a basis of connected neighborhoods of $\mathcal{M}$ at c. In other words, $\mathcal{M}$ is locally connected at c. Moreover, from our construction, q_c is unbranched and has complex bounds. Therefore the Julia set J_c is locally connected from Theorem 1.8. □

3 The proof of our main theorem

Proof of Theorem 1.12 (Main Theorem). Let $c_0 \in \mathcal{M}$ be a Misiurewicz point and J_{c_0} be its Julia set. Then there is the smallest integer $m \geq 1$ such that $p = q_{c_0}^{\circ m}(0)$ is a repelling periodic point of q_{c_0} of period $k \geq 1$. We start the construction of our subset $\mathcal{A}(c_0)$ and a three-dimensional partition $\{\mathcal{W}_w \mid w = i_0 i_1 \cdots i_n, n = 0, 1, \cdots\}$ around it as follows.

Let α be the separating fixed point of q_{c_0}. Let G be the closure of the union of external rays landing at α. Let $\xi = \{\eta_n\}_{n=0}^{\infty}$ be the Yoccoz partition about J_{c_0} (see §1.2). Let

$$p \in \cdots \subseteq D_n(p) \subseteq D_{n-1}(p) \subseteq \cdots \subseteq D_1(p) \subseteq D_0(p)$$

be a p-end, that means that $p \in D_n(p) \subseteq D_{n-1}(p)$ and $D_n(p) \in \eta_n$. Let

$$c_0 \in \cdots \subseteq E_n(c_0) \subseteq E_{n-1}(c_0) \subseteq \cdots \subseteq E_1(c_0) \subseteq E_0(c_0)$$

be a c_0-end, this means that $c_0 \in E_n(p) \subseteq E_{n-1}(p)$ and $E_n(p) \in \eta_n$. We have $q_{c_0}^{\circ(m-1)}(E_{n+m-1}(c_0)) = D_n(p)$. Since the diameter $\mathrm{diam}(D_n(p))$ tends to zero as $n \to \infty$ and since p is a repelling periodic point, we can find an integer $l \geq m$ such that $|(q_{c_0}^{\circ k})'(x)| \geq \lambda > 1$ for all $x \in D_l(p)$ and such that $q_{c_0}^{\circ(m-1)} : E_{l+m-1}(c_0) \to D_l(p)$ is a homeomorphism. Let $t \geq 0$ be the integer such that $f = q_{c_0}^{\circ t} : D_l(p) \to C_{r_0}$ is a homeomorphism, where C_{r_0} is the domain containing 0 in η_{r_0}, $r_0 \geq 0$. There is an integer $r > r_0$ such that $r + t > l$ and such that $B_0 = f^{-1}(C_r \cap D_l(p))$ does not contain p. Thus $q_{c_0}^{\circ t} : B_0 \to C_r$ is a homeomorphism. Define

$$B_n = \left(q_{c_0}^{\circ nk}|D_{l+nk}(p)\right)^{-1}(B_0) \subseteq D_{l+nk}(p)$$

for $n \geq 1$. Note that B_n is in η_{r+q+nk}. Then $q_{c_0}^{\circ(q+nk)} : B_n \to C_r$ is a homeomorphism. By the Böttcher coordinates h_{c_0} and h_c, α, G, p, C_r, D_n, and B_n, for $n \geq 0$, are all preserved for c close to c_0. Therefore they can be constructed for q_c as long as c close to c_0 as we did for -2. Let $\mathcal{U}_0$ be a neighborhood about c_0 with $\mathrm{diam}(\mathcal{U}_0) \leq 1$ such that the corresponding $\alpha(c)$, $G_0(c)$, $p(c)$, $C_r(c)$, $D_n(c)$, and $B_n(c)$, for $n \geq 0$, are all preserved for $c \in \mathcal{U}_0$. As n goes to infinity, the diameter $\mathrm{diam}(B_n(c))$ tends to zero uniformly on $\mathcal{U}_0$ and the set $B_n(c)$ approaches to $p(c)$ uniformly on $\mathcal{U}_0$. Let

$$\mathcal{W}_n = \mathcal{W}_n(c_0) = \{c \in \mathbb{C} \mid q_c^m(0) \in B_n(c)\}, \quad n \geq 1.$$

They give a partition in the parameter space.

Suppose $\mathcal{U}_0$ is simply connected and the boundary γ is a closed path homologous to 0 in $\mathcal{U}_0$. Since the equation $q_c^m(0) - p(c) = 0$ has a unique solution c_0 in $\mathcal{U}_0$ (following Thurston's Theorem for critically finite rational maps (see [5]) and also refer to [4]), $m = \min_{c \in \gamma} |q_c^{\circ m}(0) - p(c)| > 0$. Let $f(c) = q_c^{\circ m}(0) - p(c)$ and $g(c) = q_c^{\circ m}(0) - x$ for $x \in B_n(c)$. Since $B_n(c)$ approaches $p(c)$ uniformly on $\mathcal{U}_0$ as n goes to infinity, there is an integer $N_0 = N_0(c_0) > 0$ such that for $n \geq N_0$,

$$|f(c) - g(c)| = |p(c) - x| < m \leq |f(c)|, \quad x \in \partial \mathcal{U}_0.$$

Theorem 1.13 (Rouché's Theorem) now implies that $g(c) = q_c^{\circ m}(0) - x = 0$ has a unique solution, which is in $\mathcal{U}_0$, for any x in $B_n(c)$ and $n \geq N_0$. Therefore $\mathcal{W}_n \subseteq \mathcal{U}_0$ for $n \geq N_0$. So $\mathrm{diam}(\mathcal{W}_n) \leq 1$ for $n \geq N_0$. A similar argument to the proof of Lemma 2.1 implies that, for $n \geq N_0$, $\tilde{\mathcal{M}}_n = \mathcal{M} \cap \mathcal{W}_n$ is connected (see Figure 4).

For any $c \in \mathcal{W}_n$, $n \geq N_0$, let $R_n(c)$ be the pre-image of $B_n(c)$ under the map

$$q_{c_0}^{\circ(m-1)} : E_{l+m-1}(c) \to D_l(p, c)$$

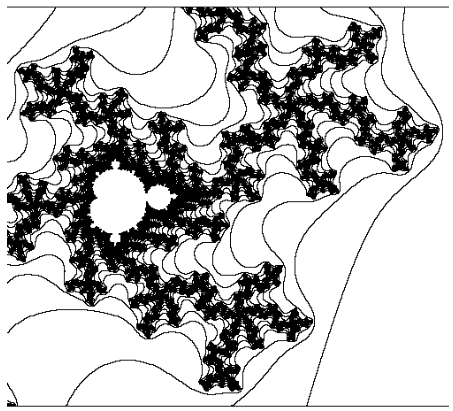

Figure 4: A small copy of the Mandelbrot set and hairs around it.

and let $C_{m+r+q+nk}(c) = q_c^{-1}(R_n(c))$. Then $C_{m+r+q+nk}(c)$ is the closure of a domain which contains 0 and which is in $\eta_{m+r+q+nk}$. Hence

$$f_{n,c} = q_c^{\circ(q+nk+m)} : \mathring{C}_{m+r+q+nk}(c) \to \mathring{C}_r(c)$$

is a quadratic-like map. Furthermore, since $\mathrm{diam}(B_n(c))$ tends to zero as n goes to infinity uniformly on $\mathcal{U}_0$, for N_0 large enough we can have the modulus $\mathrm{mod}(A_n(c)) \geq 1$ where $A_n(c) = \mathring{C}_r(c) \backslash C_{m+r+q+nk}(c)$. Moreover,

$$\{f_{n,c} : \mathring{C}_{m+r+q+nk}(c) \to \mathring{C}_r(c) \mid c \in \mathcal{W}_n\}$$

is a full family. Thus $\mathcal{W}_n$ contains a copy $\mathcal{M}_n = \mathcal{M}_n(c_0)$ of the Mandelbrot set $\mathcal{M}$ (refer to [23, pp. 102-106]). Note that $\mathcal{M}_n \subseteq \tilde{\mathcal{M}}_n$ and $\tilde{\mathcal{M}}_n \setminus \mathcal{M}_n$ is usually not empty (see Figure 4). For any $c \in \mathcal{A}_1(c_0) = \cup_{n \geq N_0} \mathcal{M}_n$, q_c is once renormalizable.

Now following almost the same arguments in the proof of Theorem 1.14, We can construct a subset $\mathcal{A}(c_0)$ and a three-dimensional partition

$$\{\mathcal{W}_w(c_0) \mid w = i_0 \cdots i_n, n = 0, 1, \cdots\}$$

about it such that c_0 is a limit point of $\mathcal{A}(c_0)$ and such that every $c \in \mathcal{A}(c_0)$ is infinitely renormalizable at which $\mathcal{M}$ is locally connected. Furthermore, q_c is unbranched and has complex bounds from our construction. Therefore the Julia set J_c is locally connected following Theorem 1.8. It completes the proof. □

Remark 3.1. *Eckmann and Epstein [6] and Douady-Hubbard [4] have estimated the size of* $\mathcal{M}_n$. *Since* $\tilde{\mathcal{M}}_n \setminus \mathcal{M}_n$ *contains hairs (see Figure 4)* *which may destroy the local connectivity of $\mathcal{M}$, we must estimate the size of $\tilde{\mathcal{M}}_n$. This is the key point in the proof.*

Remark 3.2. *Lyubich [18] constructed another subset consisting of infinitely renormalizable points such that $\mathcal{M}$ is locally connected at every point in this subset. A point in his subset must satisfies several complicate conditions. His idea is more close to Douady and Hubbard's idea. A point in our susbet may not satisfy those conditions. Our idea is different and simple and originated in [11, 12, 13, 14].*

Remark 3.3. *In our proof of Theorem 1.12, the use of Rouché's Theorem is interesting. Actually, A three-dimensional partition in the parameter space can be constructed around more general infinitely renormalizable points (those points are not eventually $(2, 2, 2, \cdots)$-renormalizable) just like we did in the proof. However finding a tool to replace Rouché's Theorem in the proof is an interesting problem. A candidate is Slodkowski's Theorem [26] for holomorphic motions. We would like to explore this in the further research.*

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