# On conformal measures for infinitely renormalizable quadratic polynomials

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Received October 15, 2004; revised June 30, 2005

**Abstract** We study a conformal measure for an infinitely renormalizable quadratic polynomial. We prove that the conformal measure is ergodic if the polynomial is unbranched and has complex bounds. The main technique we use in the proof is the three-dimensional puzzle for an infinitely renormalizable quadratic polynomial.

**Keywords: Julia set, conformal measure, three-dimensional puzzle, infinitely renormalizable quadratic polynomial**

**DOI: 10.1360/03ys173**

## 1 Introduction

Consider $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the extended complex plane and $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational function. Then $\{R^n\}_{n=0}^{\infty}$ forms a complex dynamical system, where $R^n = R \circ \cdots \circ R$ means the $n$-composition of $R$. We simply use $R$ to denote the complex dynamical system. Let $F = F(R)$ and $J = J(R)$ mean the Fatou set and the Julia set of $R$. Then $F$ is an open set representing the stable part of the complex dynamical system and $J$ is a compact set representing the chaotic part of the complex dynamical system. The dynamics of $R$ has been studied by Fatou and Julia as early as the 19th century. In 1980's, Sullivan solved the non-wandering domain problem which says that every component $W$ of $F$ is eventually periodic, i.e. there are integers $n > m > 0$ such that $R^n(W) = R^m(W)$. Therefore, the study of the dynamics of $R$ on $J$ becomes more interesting and difficult. For the fundamental theory of complex dynamical systems, there are recently several very good textbooks, see, for examples, refs. [1-3]. The readers who are interested in this could go to these books. In this paper, we are interested in the measure-theoretical properties of $R$ on $J$.

Let us start to define the Hausdorff measure and the conformal measure. Let $d$ be the Euclid spherical metric on $\hat{\mathbb{C}}$ and

$$B(z, r) = \{w \in \hat{\mathbb{C}} \mid d(w, z) < r\}$$

be the disk centered at $z$ of radius $r > 0$. We use

$$|A| = \sup\{d(w, w') \mid w, w' \in A\}$$

to denote the diameter of a subset $A$ of $\hat{\mathbb{C}}$.

Consider a set $E$ in $\hat{\mathbb{C}}$. For each positive number $\delta > 0$, we consider the possible coverings $\{A_i\}$ of $E$ by sets of diameters less than $\delta$, and define

$$m_s^\delta(E) = \inf\left\{\sum_i |A_i|^s \mid |A_i| < \delta, E \subset \cup_i A_i\right\}.$$

As $\delta$ decreasing, $m_s^\delta(E)$ is increasing. So we define the $s$-dimensional measure $m_s(E)$ by

$$m_s(E) = \lim_{\delta\to 0} m_s^\delta(E) = \sup_{\delta>0} m_s^\delta(E).$$

This limit always exists (could be $+\infty$). Then $m_s(E)$ is called the $s$-dimensional Hausdorff measure of $E$. In fact, $m_s$ is an outer measure on the class of all subsets of $\hat{\mathbb{C}}$. From the inequality

$$m_s^\delta(E) \leqslant \delta^{s-t} m_t^\delta, \quad s > t,$$

we have a unique non-negative number $HD(E)$ such that

$$m_s(E) = \begin{cases} +\infty, & s < HD(E); \\ 0, & s > HD(E). \end{cases}$$

The number $HD(E)$ is called the Hausdorff dimension of $E$. In general, the calculation of the Hausdorff dimension and measure for a given subset is difficult. However, the density distribution principle can be helpful.

**Density distribution principle.** Suppose $\mu$ is a finite positive measure supported on $E$, i.e. $0 < \mu(E) < \infty$. Let $s > 0$. Then
if there is a constant $C > 0$ such that for any small $r > 0$,

$$\mu(B(z,r)) \leqslant Cr^s \quad \forall z \in E,$$

then $m_s(E) > 0$ and $HD(E) \geqslant s$;
if there is a constant $C > 0$ such that for any small $r > 0$,

$$\mu(B(z,r)) \geqslant Cr^s \quad \forall z \in E,$$

then $m_s(E) < \infty$ and $HD(E) \leqslant s$;
if there are constants $C_1 \geqslant C_2 > 0$ such that for any small $r > 0$,

$$C_2 r^s \leqslant \mu(B(z,r)) \leqslant C_1 r^s \quad \forall z \in E,$$

then

$$0 < m_s(E) < \infty \quad \text{and} \quad HD(E) = s.$$

In view of the density distribution principle, the following conformal measure is reasonable and plays an important role in the study of measure-theoretic problems for the complex dynamical system $R$.

**Definition 1.** A finite positive probability measure $\mu$ is called a $t$-conformal measure if
$$\mu(R(E)) = \int_E |R'|^t d\mu$$
holds for any Borel set $E \subset J$ such that $R|_E$ is injective.

Sullivan[4](see also refs. [5,6]) proved that there are a minimal critical number,
$$0 < t_0 = t_0(R) \leqslant 2,$$
and the corresponding $t_0$-conformal measure $\mu$. Here the minimal critical number means that if there is a $t$-conformal measure $\nu$ for $R$, then $t \geqslant t_0$.

The complex dynamical system $R$ is said to be hyperbolic if there is a constant $C > 0$ and $\lambda > 1$ such that
$$|(R^n)'(z)| \geqslant C\lambda^n, \quad \forall z \in J, \ \ \forall n \in \mathbb{N}.$$
Equivalently, $R$ is hyperbolic if and only if every critical orbit tends to an attractive or super-attractive periodic orbit. If $R$ is hyperbolic, then the unique $t_0$-conformal measure is ergodic and
$$0 < t_0 = HD(J) < 2 \quad \text{and} \quad 0 < m_{t_0}(J) < \infty.$$
For a general complex dynamical system, the study of the Hausdorff dimension, the Hausdorff measure, and a $t$-conformal measure for $J$ is still an interesting research problem.

In this paper, we will study a conformal measure for an infinitely renormalizable quadratic polynomial. The main tool we use in the study is the three-dimensional partition constructed in ref. [7] (see also ref. [8]). Our main theorem in this paper is

**Theorem 1.** Suppose $P(z) = z^2 + c$ is an unbranched infinitely renormalizable quadratic polynomial having complex bounds. Then the $t$-conformal measure $\mu_t$ for $P$ is ergodic.

By the proof of this theorem, we will also discuss some other properties related to conformal measures. This kind of properties was also studied by Prado in ref. [9] for real unimodal polynomials (a polynomial with just one critical point) and complex unimodal polynomials with one parabolic periodic point or a quadratic polynomial in the $\mathcal{SL}$ class as defined in ref. [10]. Our purpose is to apply the three-dimensional puzzles in refs. [7,8] to the study of measure-theoretic problems about the Julia set of an infinitely renormalizable quadratic polynomial. In this direction, a much challenging problem is to study the Lebesgue measure of the Julia set of an infinitely renormalizable quadratic polynomial. We would like to use the three-dimensional puzzles further into this study.

The paper is organized as follows. In Section 2, we will give a brief review of the construction of the three-dimensional puzzle for an infinitely renormalizable quadratic polynomial. In Section 3, we will give a proof of Theorem 1.

## 2 A brief review of the three-dimensional puzzles

Let $P(z) = z^2 + c$ be a quadratic polynomial where $z$ is a complex variable and $c$ is a complex parameter. The filled Julia set $K$ of $P$ is, by definition, the set of points $z$ which remain bounded under the iterations of $P$. The Julia set $J$ of $P$ is the boundary of $K$. In this paper, we always assume that $P$ has no super-attractive, attractive, and neutral periodic points.

In this section, we give a brief review of the three-dimensional puzzle for an infinitely renormalizable quadratic polynomial constructed in ref. [7]. The readers may go to refs. [7,8] for more detailed information and additional properties about the three-dimensional puzzle. Let $CO = \{P^i(0)\}_{i=0}^{\infty}$ be the critical orbit of 0.

Let us first give the definition of renormalizability. A quadratic-like map $F : U \to V$ is a holomorphic, proper, degree-two branched cover map, where $U$ and $V$ are two domains isomorphic to a disc and $\overline{U} \subset V$. Then $K_F = \cap_{n=0}^{\infty} F^{-n}(U)$ and $J_F = \partial K_F$ are the filled Julia set and the Julia set of $F$, respectively. We only consider those quadratic-like maps whose Julia sets are connected. Let us assume that the only branch point of $F$ is 0. A quadratic-like map $F : U \to V$ is said to be (once) renormalizable if there are an integer $n' > 1$ and an open subdomain $U'$ containing 0 such that $U' \subset U$ and such that $F_1 = F^{\circ n'} : U' \to V' \subset V$ is a quadratic-like map with a connected Julia set $J_{F_1} = J(n', U', V')$. The choice of $(U', V')$ is called an $n'$-renormalization of $(U, V)$. A quadratic-like map $F : U \to V$ is said to be twice renormalizable if $F$ is once renormalizable, and there is an $m_1$-renormalization $(U', V')$ of $(U, V)$ such that $F_1 = F^{\circ m_1} : U' \to V'$ is once renormalizable. Consequently, we have renormalizations $F_1 = F^{\circ m_1} : U_1 \to V_1$ and $F_2 = F^{\circ m_2} : U_2 \to V_2$ and their Julia sets $(J_{F_1}, J_{F_2})$. Similarly, we can definite a $k$-times renormalizable quadratic-like map $F : U \to V$ and renormalizations $\{F_i = F^{\circ m_i} : U_i \to V_i\}_{i=1}^{k}$ where $m_1 < m_2 < \cdots < m_k$. A quadratic-like map $F : U \to V$ is infinitely renormalizable if it is $k$-times renormalizable for every $k > 0$.

For a quadratic polynomial $P(z) = z^2 + c$, let $U$ be a fixed domain bounded by an equipotential curve of $P$ and let $V = P(U)$. (See, for examples, refs. [1, 2, 8, 11, 12] for the definitions of an equipotential curve and an external ray.) Then $P : U \to V$ is a quadratic-like map whose Julia set is always $J$. We say $P$ is infinitely renormalizable if $P : U \to V$ is infinitely renormalizable.

Take a quadratic polynomial $P(z) = z^2 + c$. Assume that 0 is not eventually periodic. Let us fix an equipotential curve $S_t$ and the domain $U = U_t$ bounded by $S_t$. Then we have a quadratic-like map $F = P : U \to V = P(U)$ whose Julia set is $J$. Then $F$ has two fixed points. One of them, say $\beta$, is non-separating and the other, say $\alpha$, is separating, i.e, $J \setminus \{\beta\}$ is still connected and $J \setminus \{\alpha\}$ is not. There are at least two, but a finite number, external rays of $P$ landing at $\alpha$. Let $\Gamma_0^0$ be the union of a cycle of external rays landing at $\alpha$. Then $\Gamma_0^0$ cuts $U_0^0 = U$ into finitely many domains. Let $\eta_0$

be the collection of the closure of these domains. Let $\Gamma_n^0 = F^{-n}(\Gamma_0^0)$ for any $n > 0$. Then $\Gamma_n^0$ cuts $U_n^0 = F^{-n}(U_0^0)$ into finitely many domains. Let $\eta_n$ be the collection of the closures of these domains. The sequence $\xi^0 = \{\eta_n\}_{n=0}^{\infty}$ is called the Yoccoz puzzle for $J$. The domain $C_n$ in $\eta_n$ containing 0 is called the critical piece in $\eta_n$. It is clear that $P$ is restricted to all domains but $C_n$ is bijective to domains in $\eta_{n-1}$, and $P|C_n$ is a degree-two branched cover map onto a domain in $\eta_{n-1}$. Let

$$J_1 = \cap_{n=0}^{\infty} C_n.$$

The following result follows directly from the result of Yoccoz about the local connectivity of nonrenormalizable quadratic polynomials (refer to refs. [8,12,13]) and gives an equivalent definition of renormalizability:

**Theorem A.** Suppose $P(z) = z^2 + c$ has the recurrent critical orbit. Then $P$ is renormalizable if and only if $J_1$ consists of more than one point.

We will use $N(X, \epsilon) = \{x \in \mathbb{C} \mid d(x, X) < \epsilon\}$ to denote the $\epsilon$-neighborhood of $X$ in the complex plane in this paper. Suppose $P$ is renormalizable. We have two integers $n_1 \geqslant 0, k_1 > 1$ such that $F_1 = F^{k_1} : C_{k_1+n_1} \to C_{n_1}$ is a degree-two branched cover map and such that $C_{k_1+n_1} \subset N(J_1, 1)$. We can further take the domains $C_{n_1+k_1} \subseteq U_1 \subset U$ and $C_{n_1} \subseteq V_1 \subset V$ such that

$$F_1 = F^{\circ k_1} : U_1 \to V_1$$

is a quadratic-like map. Then its Julia set is $J_1$. (Note that $F_1 = F^{\circ k_1} : U_1 \to V_1$ is a simple renormalization of $F : U \to V$.) The Julia set $J_1$ is unique in the meaning that for any $k_1$-renormalization of $G = P^{\circ k_1} : U' \to V'$ of $P$, its filled-in Julia set is always $J_1$ (see Theorem 2 in ref. [7]). Therefore, for any $k_1$-renormalization $(U', V')$ of $F : U \to V$, there is $C_{k_1+n} \subset U'$ such that $F_1 = F^{k_1} : C_{k_1+n} \to C_n \subset V'$ is a degree-two branched cover map.

Suppose $\beta_2$ and $\alpha_2$ are the non-separating and separating fixed points of $F_1$, i.e. $J_1 \setminus \{\beta_2\}$ is still connected and $J_1 \setminus \{\alpha_2\}$ is not. The points $\beta_2$ and $\alpha_2$ are also the repelling periodic points of $P$. There are at least two, but a finite number, external rays of $P$ landing at $\alpha_2$. Let $\Gamma_0^1$ be the union of a cycle of external rays landing at $\alpha_2$. Then $\Gamma_0^1$ cuts $U_0^1 = C_{n_1+k_1}^0$ into finitely many domains. Let $\eta_0^1$ be the collection of the closures of these domains. Let $\Gamma_n^1 = F_1^{-n}(\Gamma_0^1)$ for any $n > 0$. Then $\Gamma_n^1$ cuts $U_n^1 = F_1^{-n}(U_0^1)$ into finitely many domains. Let $\eta_n^1$ be the collection of the closures of these domains. The sequence $\xi^1 = \{\eta_n^1\}_{n=0}^{\infty}$ is the two-dimensional puzzle for $J_1$. We call it the first puzzle. (We also call $\xi^0$ the $0^{\text{th}}$ puzzle.)

The domain $C_n^1$ in $\eta_n^1$ containing 0 is called the critical piece in $\eta_n^1$. It is clear that $F_1$ is restricted to all domains but $C_n^1$ is bijective to domains in $\eta_{n-1}^1$, and $F_1|C_n^1$ is a degree two branched cover map onto a domain in $\eta_{n-1}^1$. Let

$$J_2 = \cap_{n=0}^{\infty} C_n^1.$$

There are two integers $n_2 \geqslant 0$, $k_2 > 1$ such that

$$F_2 = F_1^{\circ k_2} : C_{n_2+k_2}^1 \to C_{n_2}^1$$

is a degree-two branched cover map and such that $C^1_{n_2+k_2} \subset N(J_2, 1/2)$. We take domains $C^1_{n_2+k_2} \subseteq U_2 \subset U_1$ and $C^1_{n_2} \subseteq V_2 \subset V_1$ such that

$$F_2 = F_1^{\circ k_2} : U_2 \to V_2$$

is a quadratic-like map. Then its Julia set is $J_2$.

Inductively, for every $i \geqslant 2$, suppose we have constructed

$$F_i = F_{i-1}^{\circ k_i} : C^{i-1}_{n_i+k_i} \to C^{i-1}_{n_i} \qquad \text{and} \qquad F_i = F_{i-1}^{\circ k_i} : U_i \to V_i$$

whose Julia set is $J_i$. Let $\beta_{i+1}$ and $\alpha_{i+1}$ be the non-separating and separating fixed points of $F_i$; i.e. $J_i \setminus \{\beta_{i+1}\}$ is still connected and $J_i \setminus \{\alpha_{i+1}\}$ is not. The points $\beta_{i+1}$ and $\alpha_{i+1}$ are also the repelling periodic points of $P$. There are at least two, but a finite number, external rays of $P$ landing at $\alpha_{i+1}$. Let $\Gamma^i_0$ be the union of a cycle of external rays landing at $\alpha_{i+1}$. Then $\Gamma^i_0$ cuts $U^i_0 = C^{i-1}_{n_i+k_i}$ into finitely many domains. Let $\eta^i_0$ be the collection of the closures of these domains. Let $\Gamma^i_n = F_i^{-n}(\Gamma^i_0)$ for any $n > 0$. Then $\Gamma^i_n$ cuts $U^i_n = F_i^{-n}(U^i_0)$ into finitely many domains. Let $\eta^i_n$ be the collection of the closures of these domains. The domain $C^i_n$ in $\eta^i_n$ containing $0$ is called the critical piece in $\eta^i_n$. It is clear that $F_i$ is restricted to all domains but $C^i_n$ is bijective to domains in $\eta^i_{n-1}$, and $F_1|C^i_n$ is a degree-two branched cover map onto a domain in $\eta^i_{n-1}$. Let

$$J_{i+1} = \cap_{n=0}^{\infty} C^i_n.$$

There are two integers $n_{i+1} \geqslant 0$, $k_{i+1} > 1$ such that

$$F_{i+1} = F_i^{\circ k_{i+1}} : C^i_{n_{i+1}+k_{i+1}} \to C^i_{n_{i+1}}$$

is a degree-two branched cover map and such that $C^i_{n_{i+1}+k_{i+1}} \subset N(J_{i+1}, 1/(i+1))$. We take domains $C^i_{n_{i+1}+k_{i+1}} \subseteq U_{i+1} \subset U_i$ and $C^i_{n_{i+1}} \subseteq V_{i+1} \subset V_i$ such that

$$F_{i+1} = F_i^{\circ k_{i+1}} : U_{i+1} \to V_{i+1}$$

is a quadratic-like map. Then its Julia set is $J_{i+1}$. Let $\xi^i = \{\eta^i_n\}_{n=0}^{\infty}$. It is the two-dimensional puzzle for $J_i$. We call it the $i^{\text{th}}$ partition.

Let $m_i = \prod_{j=1}^{i} k_i$, $1 \leqslant i < \infty$. We have thus constructed a most natural infinite sequence of simple renormalizations of $F : U \to V$,

$$\{F_i = F^{\circ m_i} : U_i \to V_i\}_{i=1}^{\infty},$$

and the nested-nested sequence $\{\xi^i\}_{i=0}^{\infty}$ of partitions for $\{J_i\}_{i=0}^{\infty}$ (where $J_0 = J$) which we call the three-dimensional puzzle. Henceforth, we will fix all notations.

An annulus $A$ is a double connected domain. The definition of the modulus $(A)$ of an annulus $A$ is defined in many books in complex analysis (see, for example, ?? [14]). It is $\log r$ if $A$ is holomorphically diffeomorphic to the annulus $A_r = D_r \setminus \overline{D}_1$ where $D_r$ is the open disk centered at $0$ with radius $r > 1$. The sets $U \setminus \overline{U'}$ and $V \setminus \overline{U}$ are annuli.

**Definition 2.** We say an infinitely renormalizable quadratic polynomial $P(z) = z^2 + c$ has complex bounds if there are a constant $\lambda > 0$ and an infinite sequence of simple renormalizations $\{F_{i_s} = F^{\circ m_{i_s}} : U_{i_s} \to V_{i_s}\}_{s=1}^{\infty}$ such that the modulus $(V_{i_s} \setminus \overline{U}_{i_s})$ is greater than $\lambda$ for every $s \geqslant 1$.

**Definition 3.** We say an infinitely renormalizable quadratic polynomial $P(z) = z^2 + c$ is unbranched if there are an infinite subsequence of renormalizations $\{J_{i_l}\}_{l=1}^{\infty}$ of the Julia set $J$ of $P$, neighborhoods $W_l$ of $J_{i_l}$ for every $l > 0$, and a constant $\mu > 0$ such that the modulus $(W_l \setminus J_{i_l})$ is greater than $\mu$ and $W_l \setminus J_{i_l}$ contains no point in the critical orbit $CO$ of $P$.

Using the the three-dimensional puzzle and the extended three-dimensional puzzle, the following theorem is proved in ref. [7] (see also ref. [8]).

**Theorem B.** The Julia set of an unbranched infinitely renormalizable quadratic polynomial having complex bounds is locally connected.

The main purpose of this paper is to use the three-dimensional puzzle $\{\xi^i\}_{i=0}^{\infty}$ $P$ to study a conformal measure on the Julia set of $P$.

## 3 Ergodicity of a conformal measure

Suppose $P$ is an unbranched infinitely renormalizable quadratic polynomial having complex bounds. Without loss of generality, we can just think $U_i = C^i_{n_i+k_i}$ and $V_i = C^i_{n_i}$ are two puzzle pieces. (In Section 2, $U_i$ and $V_i$ are open and $C^i_{n_i+k_i}$ and $C^i_{n_i}$ are closed. However, by abusing the notation of renormalization a little bit, we can assume $U_i = C^i_{n_i+k_i}$ and $V_i = C^i_{n_i}$ in the renormalization.)

From the proof of Theorem B in ref. [7], we have the following lemma.

**Lemma 1.** There are a fixed constant $\lambda > 0$, a subsequence of integers $n_i \to \infty$ and domains $0 \in U_i \subset V_i$ satisfying

(i) $(V_i \setminus \overline{U}_i) \geqslant \lambda$;

(ii) $F_i = P^{m_i} : U_i \to V_i$ is a quadratic-like map with a connected Julia set $J_i$;

(iii) diam$(V_i) \to 0$ as $i \to \infty$.

**Remark 1.** Since $f$ is unbranched, when we choose $V_i = C^i_{n_i}$ in Lemma 1, we assume that they satisfy the unbranched condition, that is, there are $W_i \supset \overline{V}_i$ such that $(W_i \setminus \overline{V}_i) \geqslant \lambda = \mu/2$ and $W_i \setminus \overline{V}_i$ is disjoint with the critical orbit $CO$, where $\mu$ is the constant in Definition 3.

In ref. [15], Blokh and Misiurewicz defined a graph critical rational map as follows. A finite graph $G$ is a one-dimensional compact branched manifold. A rational map $R$ is called graph critical if any critical point $c$ either belongs to an invariant finite graph $G$ or has a minimal limit set or is non-recurrent and disjoint from $G$. They then proved that for a graph critical rational map $R$ and a conformal measure $\mu$ for $R$, either for $\mu$-almost every point $z \in J$, $\omega(z) = J$, or for $\mu$-almost every point $z \in J$, there is a critical point $c$ of $R$ such that $\omega(z) = \omega(c)$. Following this result, we have the following lemma. Let $\mu$ be a $t$-conformal measure for $P$. Let $\mathcal{O} = \{z \in J \mid 0 \in \omega(z)\}$.

**Lemma 2.** The set $\mathcal{O}$ has full measure, i.e. $\mu(\mathcal{O}) = 1$.

Lemma 2 is a direct corollary of the above Blokh-Misiurewicz's result. The reason is as follows. Since $P$ is infinitely renormalizable, the limit set $\overline{CO}$ is minimal. But $P$ has only two critical points 0 and $\infty$. Since $P^{-1}(\infty) = \{\infty\}$ and the limit set of 0 is a minimal set, $P$ is graph critical. Then for $\mu$-almost every $z \in J$, either $\omega(z) = J$ or $\omega(z) = \omega(0)$. In both cases, $0 \in \omega(z)$.

Suppose $\Lambda_0$ is the set of all repelling periodic points of $P$. Let $\Lambda = \cup_{n=0}^{\infty} P^{-n}(\Lambda_0)$. Then the set $\Lambda$ has zero $\mu$ measure, i.e. $\mu(\Lambda) = 0$. The reason is that suppose $p$ is a repelling periodic point of $P$. Let

$$A_p = \{z \in J \mid \text{there is an integer } i \geqslant 0 \text{ such that } P^i(z) = p\}.$$

For any $x \in A_p$, suppose $P^i(x) = p$. For any $n \geqslant i$, $(P^n)'(x) = (P^{n-i})'(p)(P^i)'(x)$. Thus $|(P^n)'(x)| \to \infty$, $n \to \infty$. Since $\mu$ is a $t$-conformal measure,

$$\mu(P^n(A_p)) = \int_{A_p} |(P^n)'|^t d\mu.$$

(If $P^n$ is injective on $A_p$, then we have the above formula; otherwise, we partition $A_p$ into a union of injective pieces.) If $\mu(A_p) > 0$, then $\int_{A_p} |(P^n)'|^t d\mu \to \infty$, $n \to \infty$. But $\mu(P^n(A_p)) \leqslant 1$. This is a contradiction. Therefore, $\mu(A_p) = 0$. And moreover, $\mu(\cup_{n=0}^{\infty} P^{-n}(A_p)) = 0$. Since $\Lambda = \cup_{p \in \Lambda_0} \cup_{n=0}^{\infty} P^{-n}(A_p)$, we have $\mu(\Lambda) = 0$. This property was not really used in our proof of Theorem 1. The reason mentioned here is just because we would like to point out that one only needs to think subsets in $J \setminus \Lambda$.

Consider any $P$-invariant set $X \subset \mathcal{O}$ (or $X \subset \mathcal{O} \setminus \Lambda$), i.e. $P^{-1}(X) = X$ and a fixed $V_i$. For every $x \in X$, there is the smallest non-negative integer $m \geqslant 0$ such that $P^m(x)$ first falls in $V_i$. Just like what we constructed in the three-dimensional puzzle, pulling back $V_i$ along the orbit $x, P(x), \cdots, P^m(x)$, we get $V_i^j(x) = P^{-(m-j)}(V_i)$ containing $P^j(x)$. In particular, $x \in V_i^m(x)$. Thus $\mathcal{B}_i = \{V_i^0(x)\}_{x \in X}$ consists of puzzle pieces. We have the following properties for the sequence $\{\mathcal{B}_i\}_{i=1}^{\infty}$:

(i) Each $\mathcal{B}_i$ is a cover of $X$.

(ii) The cover $\mathcal{B}_{i+1}$ is a sub-partition of $\mathcal{B}_i$, i.e. each puzzle piece of $\mathcal{B}_{i+1}$ is a subset of a puzzle piece of $\mathcal{B}_i$. Therefore, let $B_i = \cup_{E \in \mathcal{B}_i} E$, then $B_{i+1} \subset B_i$.

(iii) The intersection $\cap B_i = X$.

(iv) The measure $\mu(B_i) \to \mu(X)$ as $i \to \infty$.

The properties (i)-(iv) are clear by considering Lemma 1.

The condition measure $\mu(X|Y)$ of $X$ in $Y$ is by the definition,

$$\mu(X|Y) = \frac{\mu(X \cap Y)}{\mu(Y)}.$$

**Lemma 3.** If $\mu(X) > 0$, then there is a sequence of puzzle pieces $E_i \in \mathcal{B}_i$ such that the condition measure $\mu(X|E_i)$ tends to 1 as $i$ goes to $\infty$.

**Proof.** Let $B_i = \cup_{E\in\mathcal{B}_i} E$. Then we have $\mu(X|B_i) \to 1$ as $i \to \infty$. Since

$$\mu(X|B_i) = \frac{\sum_{E\in\mathcal{B}_i} \mu(X\cap E)}{\sum_{E\in\mathcal{B}_i} \mu(E)},$$

by the summation inequality, for each $i$, there is an $E_i \in \mathcal{B}_i$ such that $\mu(X|E_i)$ tends to 1 as $i$ goes to $\infty$. It completes the proof.

The following is the Koebe distortion lemma and the readers may find its proof in many books in one-complex variable or in complex dynamical systems, see, for example, refs. [1, 2, 8].

**Lemma 4.** Let $D_i \subset D_i'$, $i \in \mathcal{I}$, be a family of two Jordan disks. Suppose $A_i = D_i' \setminus D_i$ is an annulus and $(A_i)$ is the modulus. Suppose

$$a = \inf_{i\in\mathcal{I}} (D_i' \setminus \overline{D}_i) > 0.$$

Then there is a constant $C > 0$ only depending on $a$ such that for any conformal map $f$ defined on $D_i'$ and any $z, w \in D_i$,

$$C^{-1} \leqslant \frac{|f'(z)|}{|f'(w)|} \leqslant C.$$

Now we are ready to prove the main result in this paper.

**Proof of Theorem 1.** Suppose $X \subset J$ is a $P$-invariant set, i.e. $P^{-1}(X) = X$. Since $\mathcal{O} = \{z \in J \mid 0 \in \omega(z)\}$ has full $\mu$-measure, we may assume that $X \subset \mathcal{O}$. Suppose $\mu(X) > 0$. Let $\mathcal{B}_i$ be the sequence of covers of $X$. Then we have a sequence of puzzle pieces $E_i \in \mathcal{B}_i$ such that $\mu(X|E_i) \to 1$ as $i \to \infty$.

For each $i > 0$, consider $P^{l_i} : E_i \to V_i$ and its inverse $g_i : V_i \to E_i$. Since $P$ is unbranched and our choice of $V_i$ (see Remark 1), there are a constant $\lambda > 0$ independent of $i$ and a domain $W_i \supset \overline{V}_i$ such that $(W_i \setminus \overline{V}_i) \geqslant \lambda$ and $W_i \setminus \overline{V}_i$ is disjoint with the critical orbit $CO$.

Therefore, $g_i$ can be extended to a conformal map on $W_i$. From Lemma 4, there is a constant $C > 0$ independent of $i$ such that

$$C^{-1} \leqslant \frac{|(P^{l_i})'(z)|}{|(P^{l_i})'(w)|} = \frac{|g'(w')|}{|g'(z')|} \leqslant C$$

for $z, w \in E_i$ and $z' = P^{l_i}(z), w' = P^{l_i}(w) \in V_i$. Denote $X^c = J \setminus X$, since $P^{l_i}(X^c) = X^c$ and $P^{l_i}(B_i) = V_i$, we have

$$C^{-t}\mu(X^c|E_i) \leqslant \mu(X^c|V_i) \leqslant C^t\mu(X^c|E_i).$$

Clearly, $\mu(X^c|E_i) \to 0$ as $i \to \infty$ since $\mu(X|E_i) \to 1$ as $i \to \infty$. So we get $\mu(X^c|V_i) \to 0$ and $\mu(X|V_i) \to 1$ as $i \to \infty$.

The above arguments are true for all $P$-invariant sets. So we consider $X$ and $X^c$. If $\mu(X^c) > 0$ too, then we have $\mu(X^c|V_i) \to 1$ as $i \to \infty$ again. But

$$\mu(X|V_i) + \mu(X^c|V_i) = 1.$$

This is a contradiction. So $\mu(X^c) = 0$ and $\mu(X) = 1$. We completed the proof.

**Acknowledgements** The first and third authors were supported in part by grants from the National Natural Science Foundation of China and the NBRP of China and the second author was supported in part by grants from the National Natural Science Foundation of China and the PSC-CUNY and the Hundred Talents Program from the Chinese Academy of Sciences.

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