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Analyticity of the susceptibility function for unimodal Markovian maps of the interval

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Abstract
We study the expression (susceptibility)

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_I \rho(\mathrm{d}x) X(x) \frac{\mathrm{d}}{\mathrm{d}x}(A(f^n x)),$$

where f is a unimodal Markovian map of the interval I, $\rho = \rho_f$ is the corresponding absolutely continuous invariant measure and A is a C^1 function defined on I. We show that $\Psi(\lambda)$ is analytic near $\lambda = 1$, where $\Psi(1)$ is formally the derivative of $\int_I \rho(\mathrm{d}x) A(x)$ with respect to f in the direction of the vector field X.

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In a previous note [Ru] the susceptibility function was analysed for some examples of maps of the interval. The purpose of the present note is to give a concise treatment of the general unimodal Markovian case (assuming f real analytic). We hope that it will similarly be possible to analyse maps satisfying the Collet–Eckmann condition. Eventually, as explained in [Ru], application of a theorem of Whitney [Wh] should prove differentiability of the map $f \mapsto \rho_f$ restricted to a suitable set.

Set-up

Let I be a compact interval of $\boldsymbol{R}$ and $f : I \to I$ be real analytic. We assume that there is c in the interior of I such that $f'(c) = 0$, $f'(x) > 0$ for $x < c$, $f'(x) < 0$ for $x > c$ and

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$f''(c) < 0$. Replacing I by a possibly smaller interval, we assume that $I = [a, b]$ where $a = f^2(c)$, $b = f(c)$. We assume that the postcritical orbit $P = \{f^n c : n \geqslant 1\}$ is finite: $P = \{p_1, \ldots, p_m\}$; in particular, f is Markovian. We shall assume that f is *analytically expanding* in the sense of assumption A below; in particular, the periodic orbits of f are assumed to be repelling, and therefore c cannot be periodic. We also assume that f is topologically mixing (this can always be achieved by replacing I by a smaller interval and f by some iterate f^N).

Theorem. *Under the above conditions, and assumption A stated later, there is a unique f-invariant probability measure ρ absolutely continuous with respect to Lebesgue measure on I. If X is real analytic on I and $A \in \boldsymbol{C}^1(I)$, then*

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_I \rho(\mathrm{d}x) X(x) \frac{\mathrm{d}}{\mathrm{d}x} A(f^n x)$$

extends to a meromorphic function in $\boldsymbol{C}$, without poles on $\{\lambda : |\lambda| = 1\}$.

Change of variable

The finite set $\{c\} \cup P$ decomposes I into m subintervals I_j, with $2m$ endpoints (we 'double' the endpoints of consecutive subintervals, distinguishing between a – endpoint at the right of an interval and a + endpoint at the left). Note that $\eta = \{I_j : j = 1, \ldots, m\}$ is a Markov partition for the map f. Consider the critical values of f^n. Then for large $n > 0$, the set of critical values will be stabilized and is always P. We define *polar* endpoints as follows:

(1) $p \in P$ is a polar $-$ endpoint of an interval in η if p is a local maximum value of f^n for n large.
(2) $v \in P$ is a polar + endpoint of an interval in η if p is a local minimum value of f^n for n large.

Every $p \in P$ is a polar $-$ or + endpoint and may be both, c is a nonpolar endpoint on both sides.

We define now an increasing continuous map $\varpi : I \to \boldsymbol{R}$ so that $J = \varpi I$ is a compact interval. We write $\varpi I_j = J_j$ for $1 \leqslant j \leqslant m$; denote by ω the inverse of ϖ. We assume that $\omega | J_j$ extends to a holomorphic function in a complex neighbourhood of J_j for $1 \leqslant j \leqslant m$ and that for $q \in \{c\} \cup P$, ω has the property

$$\omega(\varpi q \pm \xi) = \omega(\varpi q) \pm \frac{\xi^2}{2} + O(\xi^4)$$

if q is a $\pm$ polar endpoint, and

$$\omega(\varpi q \pm \xi) = \omega(\varpi q) \pm \xi + O(\xi^2)$$

if q is a nonpolar endpoint. (We should really consider disjoint copies of the I_j and J_j, and disjoint neighbourhoods of these in $\boldsymbol{C}$ or in a Riemann surface two-sheeted near polar endpoints. This would lead to notational complications that we prefer to omit.)

Applications of this singular change of coordinate have been used in [Ji1, BJR, Ru]; the reference [Ji2] contains some more relevant study regarding the method of singular change of coordinates in one-dimensional dynamical systems. The reader is encouraged to compare this method with orbifold metrics in [Th, chapter 13]. Another relevant application of this method in complex dynamical systems can be found in [DH].

From now on we shall say that ϖq is a $\pm$polar (nonpolar) endpoint if q is $\pm$polar (nonpolar).

The dynamical system viewed after the change of variable

For any two intervals $I_j, I_k \in \eta$ with $fI_j \supset I_k$, we define

$$\psi_{jk} = \varpi \circ (f|I_j)^{-1} \circ (\omega|J_k).$$

Note that the ψ_{jk} are restrictions of inverse branches of $g = \varpi \circ f \circ \omega : J \to J$ to intervals in $\omega\eta$. The function $\psi_{jk} : J_k \to J_j$ extends holomorphically to a complex neighbourhood of J_k. Indeed, note that $(f|I_j)^{-1}$ is holomorphic except if I_j is one of the two intervals around c, in which case the singularity is corrected by $\omega|J_n$, where J_n is the rightmost interval in $\omega\eta$. In other cases $\omega|J_k$ cancels the singularity of $\varpi|I_j$ by our definition of ω. (Note that $\psi'_{jk}(x) \geqslant 0$ or $\leqslant 0$ on J_k and may vanish only at an interval endpoint.)

Assumption A. *Each J_k, for $k = 1, \ldots, m$, has a bounded open connected neighbourhood U_k in $\boldsymbol{C}$ such that $\psi_{jk} : J_k \to J_j$ extends to a continuous function $\psi_{jk} : \bar{U}_k \to \boldsymbol{C}$ holomorphic in U_k and with $\psi_{jk}\bar{U}_k \subset U_j$.*

One checks that the sets U_k can be assumed to be in ϵ-neighbourhoods of the J_k. Also, assumption A implies that periodic points for g are strictly repelling. The smoothness of ω, ϖ in the interior of subintervals shows that the same property holds for f, apart from interval endpoints where we however also assume the property to hold:

the periodic orbits of f are strictly repelling.

Markovian graph

Consider the Markov partition $\eta = \{I_j\}$. Let us write $j \succ k$ (j covers k) if $fI_j \supset I_k$ (we allow $j \succ j$). This defines a directed graph with vertex set $\{1, \ldots, m\}$ and oriented edges (j, k) for $j \succ k$. Since we have assumed our dynamical system f to be topological mixing, our graph is also mixing in the sense that there is $N \geqslant 1$ such that for all $j, k \in \{1, \ldots, m\}$ we have $j \succ \cdots \succ k$ (N edges) corresponding to $f^N I_j \supset I_k$.

Transfer operators

For a function $\Phi = (\Phi_j)$ defined on $\sqcup J_j$, we write

$$(\mathcal{L}\Phi)_k(z) = \sum_{j:j\succ k} \operatorname{sgn}(j)\psi'_{jk}(z)\Phi(\psi_{jk}z),$$

$$(\mathcal{L}_0\Phi)_k(z) = \sum_{j:j\succ k} \operatorname{sgn}(j)\Phi(\psi_{jk}z),$$

where $\operatorname{sgn}(j)$ is $+1$ if ψ_{jk} is increasing on J_k and -1 if ψ_{jk} is decreasing on J_k. If H is the Hilbert space of functions on $\sqcup_{j\in L}\bar{U}_j$ which are square integrable with respect to Lebesgue measure, and have holomorphic restrictions to the U_j, then $\mathcal{L}$ and $\mathcal{L}_0$ acting on H are holomorphy improving, hence compact and of trace class.

Properties of $\mathcal{L}$ (refer to [B])

For $x \in J_k$ we have

$$(\mathcal{L}\Phi)_k(x) = \sum_{j\succ k} |\psi'_{jk}(x)|\Phi_j(\psi_{jk}x),$$

hence $\Phi \geqslant 0$ implies $\mathcal{L}\Phi \geqslant 0$ ($\mathcal{L}$ preserves positivity) and

$$\int_J \mathrm{d}x (\mathcal{L}\Phi)(x) = \sum_k \int_{J_k} \mathrm{d}x (\mathcal{L}\Phi)_k(x) = \sum_j \int_{J_j} \mathrm{d}x\, \Phi_j(x) = \int_J \mathrm{d}x\, \Phi(x)$$

($\mathcal{L}$ preserves mass). Using mixing one finds that $\mathcal{L}$ has a simple eigenvalue $\mu_0 = 1$ corresponding to an eigenfunction $\sigma_0 > 0$. The other eigenvalues μ_ℓ satisfy $|\mu_\ell| < 1$ and their (generalized) eigenfunctions σ_ℓ satisfy $\int_J \mathrm{d}x\, \sigma_\ell(x) = 0$. If we normalize σ_0 by $\int_J \mathrm{d}x\, \sigma_0(x) = 1$, then $\sigma_0(\mathrm{d}x) = \sigma_0(x)\mathrm{d}x$ is the unique g-invariant probability measure absolutely continuous with respect to Lebesgue measure on J. In particular, $\sigma_0(x)\mathrm{d}x$ is ergodic.

Let now, $H_1 \subset H$ consist of those $\Phi = (\Phi_k)$ such that the derivative Φ' vanishes at the (polar) endpoints $\varpi a, \varpi b$ of J and such that at the common endpoint ϖq ($q \in \{c\} \cup P \backslash \{a, b\}$) of two subintervals we have equality on both sides of a quantity which is either

- the value of Φ for a nonpolar endpoint or
- the value of $\pm\Phi'$ for a polar $\pm$ endpoint.

We note that $\mathcal{L}H_1 \subset H_1$ (this requires a case by case discussion). Furthermore $\sigma_0 \in H_1$ (take $\phi \in H$ such that $\phi \geqslant 0$, $\int_J \mathrm{d}y\, \phi(y) = 1$ and ϕ, ϕ' vanishes at subinterval endpoints; then $\phi \in H_1$ and $\sigma_0 = \lim_{n\to\infty} \mathcal{L}^n \phi \in H_1$).

Evaluating $\Psi(\lambda)$

The image $\rho(\mathrm{d}x) = \rho(x)\mathrm{d}x$ of $\sigma_0(y)\mathrm{d}y$ by ω is the unique f-invariant probability measure absolutely continuous with respect to Lebesgue measure on I. We have

$$\rho(x) = \sigma_0(\varpi x)\varpi'(x).$$

Consider now the expression

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_I \rho(\mathrm{d}x) X(x) \frac{\mathrm{d}}{\mathrm{d}x} A(f^n x),$$

where we assume that X extends to a holomorphic function in a neighbourhood of each I_k and takes the same value on both sides of common endpoints of intervals in η (continuity). Also assume that $A \in \mathcal{C}^1(I)$. For sufficiently small $|\lambda|$, the series defining $\Psi(\lambda)$ converges. Writing $B = A \circ \omega$ (B has piecewise continuous derivative) and $x = \omega y$ we have

$$X(x) \frac{\mathrm{d}}{\mathrm{d}x} A(f^n x) = X(\omega y) \frac{1}{\omega'(y)} \frac{\mathrm{d}}{\mathrm{d}y} B(g^n y),$$

hence

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_J \mathrm{d}y\, \sigma_0(y) \frac{X(\omega y)}{\omega'(y)} \frac{\mathrm{d}}{\mathrm{d}y} B(g^n y).$$

Defining $Y(y) = \sigma_0(y)X(\omega y)/\omega'(y)$, we see that Y extends to a holomorphic function in a neighbourhood of each J_k, which we may take to be U_k, except for a simple pole at each polar endpoint of J_k. Since $\sigma_0 \in H_1$, the properties assumed for ω imply that also $(X \circ \omega) \times \sigma_0 \in H_1$. Note that near a nonpolar subinterval endpoint ϖq

$$\omega'(\varpi q \pm \xi) = 1 + O(\xi)$$

and near a $\pm$ polar endpoint

$$\omega'(\varpi q \pm \xi) = \xi + O(\xi^3).$$

Therefore

$$Y(\varpi q \pm \xi) = A^{\pm}\frac{1}{\xi} + B^{\pm} + O(\xi),$$

where $B^+ = B^-$ for the two sides of ϖq and $B^+ = 0$ at the left endpoint ϖa of J, $B^- = 0$ at the right endpoint ϖb of J. We may write

$$\int_J \mathrm{d}y\, \sigma_0(y) \frac{X(\omega y)}{\omega'(y)} \frac{\mathrm{d}}{\mathrm{d}y} B(g^n y) = \int_J \mathrm{d}y\, Y(y) g'(y) \cdots g'(g^{n-1} y) B'(g^n y)$$
$$= \int_J \mathrm{d}s (\mathcal{L}_0^n Y)(s) B'(s),$$

where $\mathcal{L}_0$ has been defined above, and we have thus

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_J \mathrm{d}s (\mathcal{L}_0^n Y)(s) B'(s).$$

Properties of $\mathcal{L}_0$

We let now $H_0 \subset H$ be the space of functions $\Phi = (\Phi_k)$ vanishing at the endpoints ϖa, ϖb of J and such that the values of Φ on both sides of common endpoints of intervals J_j coincide (continuity). Therefore $\mathcal{L}_0 H_0 \subset H_0$.

There is a periodic orbit $\gamma_1, \ldots, \gamma_p$ (with $g\gamma_j = \gamma_{j+1(\mathrm{mod}\, p)}$) of polar endpoints where γ_α is the $\pm$ endpoint of some subinterval $J_{k(\alpha)}$. Choose P_α to be 0 on subintervals different from $J_{k(\alpha)}$ and to be holomorphic on a complex neighbourhood of $J_{k(\alpha)}$ except at γ_α. Also assume that

$$P_\alpha(\gamma_\alpha \pm \xi) = \frac{1}{\xi} + O(\xi)$$

and that P_α vanishes at the endpoint of $J_{k(\alpha)}$ different from γ_α. Then

$$\mathcal{L}_0 P_\alpha - |f'(\gamma(\alpha))|^{1/2} P_{\alpha+1(\mathrm{mod}\, p)} \in H_0.$$

Therefore $\mathcal{L}_0^p P_1 - \Lambda P_1 = u \in H_0$ where $\Lambda = \prod_{\alpha=1}^p |f'(\gamma(\alpha))|^{1/2} > 1$. Since the spectrum of $\mathcal{L}$ acting on H is contained in the closed unit disk and since the derivative u' is in H, we may define $v = (\mathcal{L}^p - \Lambda)^{-1} u' \in H$. Since $\int_J \mathrm{d}y\, u'(y) = 0$ we also have $\int_J \mathrm{d}y\, v(y) = 0$ and we can take $w \in H_0$ such that $w' = v$. We have thus

$$((\mathcal{L}_0^p - \Lambda) w)' = (\mathcal{L}^p - \Lambda) w' = (\mathcal{L}^p - \Lambda) v = u'$$

so that $(\mathcal{L}_0^p - \Lambda) w = u$ (there is no additive constant of integration since $(\mathcal{L}_0^p - \Lambda) w$ and u are in H_0). Finally

$$(\mathcal{L}_0^p - \Lambda)(P_1 - w) = 0.$$

There is thus a $\mathcal{L}_0$-invariant p-dimensional vector space spanned by vectors $P_\alpha - w_\alpha$ with $w_\alpha \in H_0$, such that the spectrum of $\mathcal{L}_0$ restricted to this space consists of eigenvalues ω_ℓ with

$$\omega_\ell = \Lambda^{1/p} \mathrm{e}^{2\pi \ell \mathrm{i}/p} = \left| \prod_{\alpha=1}^p f'(\gamma_\alpha) \right|^{1/2p} \mathrm{e}^{2\pi \ell \mathrm{i}/p}$$

for $\ell = 0, \ldots, p-1$.

For the postcritical but nonperiodic polar points $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_q$ define $\tilde{P}_\beta$ like P_α above, with γ_α replaced by $\tilde{\gamma}_\beta$. For each β there is $\alpha = \alpha(\beta)$ with

$$\mathcal{L}_0^q (\tilde{P}_\beta - \tilde{\Lambda}_\beta P_\alpha) \in H_0$$

with some $\tilde{\Lambda}_\beta \neq 0$, hence

$$\mathcal{L}_0^q (\tilde{P}_\beta - \tilde{\Lambda}_\beta (P_\alpha - w_\alpha)) = \tilde{Y}_\beta \in H_0.$$

Poles of $\Psi(\lambda)$

We may now write

$$Y = Y_0 + Y_1 + Y_2,$$

where

$$Y_0 \in H_0,$$
$$Y_1 = \sum_{\alpha=1}^{p} c_\alpha (P_\alpha - w_\alpha),$$
$$Y_2 = \sum_{\beta=1}^{q} \tilde{c}_\beta (\tilde{P}_\beta - \tilde{\Lambda}_\beta (P_{\alpha(\beta)} - w_{\alpha(\beta)}))$$

and there is a corresponding decomposition $\Psi(\lambda) = \Psi_0(\lambda) + \Psi_1(\lambda) + \Psi_2(\lambda)$. Here $\Psi_1(\lambda)$ is a sum of terms $C_\ell/(\lambda - \omega_\ell)$ where $\omega_\ell = \Lambda^{1/p} \times p$th root of unity; $\Psi_2(\lambda)$ = polynomial of degree $q-1$ in λ plus $\lambda^q \sum_{\beta=1}^{q} \tilde{c}_\beta \tilde{\Psi}_\beta(\lambda)$ where $\tilde{\Psi}_\beta$ is obtained if we replace Y by $\tilde{Y}_\beta$ in the definition of Ψ. The poles of $\Psi(\lambda)$ are thus those of $\Psi_1(\lambda)$ at the ω_ℓ and those of $\Psi_0(\lambda)$ and $\tilde{\Psi}_\beta(\lambda)$. The discussion is the same for Ψ_0 and the $\tilde{\Psi}_\beta$, we shall thus only consider Ψ_0. Since $Y_0 \in H_0$ and $\mathcal{L}_0 H_0 \subset H_0$ we have

$$\Psi_0(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_J \mathrm{d}s (\mathcal{L}_0^n Y_0)(s) B'(s) = -\sum_{n=0}^{\infty} \lambda^n \int_J \mathrm{d}s (\mathcal{L}_0^n Y_0)'(s) B(s)$$
$$= -\sum_{n=0}^{\infty} \lambda^n \int_J \mathrm{d}s (\mathcal{L}^n Y_0')(s) B(s).$$

It follows that $\Psi_0(\lambda)$ extends meromorphically to $\boldsymbol{C}$ with poles at the μ_k^{-1}. We want to show that the residue of the pole at $\mu_0^{-1} = 1$ vanishes. Since $\int_J \mathrm{d}y\, \sigma_k(y) = 0$ for $k \geqslant 1$, the coefficient of σ_0 in the expansion of Y_0' is proportional to

$$\int_J \mathrm{d}y\, Y_0'(y) = Y(\varpi b) - Y(\varpi a) = 0$$

because $Y_0 \in H_0$. Therefore $\Psi_0(\lambda)$ is holomorphic for $|\lambda| = 1$ and the same holds for the $\tilde{\Psi}_\beta(\lambda)$, concluding the proof of the theorem. In fact we know that the poles of $\Psi(\lambda)$ are located at μ_k^{-1} for $k \geqslant 1$ and at ω_ℓ^{-1} for $\ell = 0, \ldots, p-1$, so that $|\mu_k^{-1}| > 1$, $|\omega_\ell^{-1}| < 1$.

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