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Metric Invariants in Dynamical Systems*

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The scaling function and the linear model for a circle endomorphism are two important smooth invariants under conjugacy. We discuss these two invariants and some relations between them. Furthermore, we use these relations to discuss some realization results in this direction. The discussion in this paper avoids quasiconformal mapping theory and Gibbs theory and g-measure theory, which are used in our previous discussions, therefore, is straightforward and simple.

KEY WORDS: Expanding circle endomorphism; linear model; scaling function.

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1. INTRODUCTION

Let M be a smooth Riemanian manifold. Suppose $f: M \to M$ be a map. For a positive integer n, $f^n = \underbrace{f \circ \cdots \circ f}_{n}$ denotes the n composition of f. Set $f^0 = id$, the identity map. The semigroup $\{f^n\}_{n=0}^{\infty}$ gives us a dynamical system. We simply call f a dynamical system. Then f and g are said to be topologically conjugate if there is a homeomorphism h of M such that

$$h \circ f = g \circ h.$$

Furthermore, if the conjugacy h is a C^r-diffeomorphism for $r \geqslant 1$, we call f and g smoothly conjugate. One of the important research problems in dynamical systems is to understand invariants under conjugacy.

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* This paper is dedicated to Professor Shui-Nee Chow on the occasion of his 60th Birthday.

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An object associated to f is said to be a topological invariant if it is invariant under topological conjugacy and to be a smooth invariant if it is invariant under smooth conjugacy. For examples, the topological entropy of f is a topological invariant and the eigenvalue of f at a periodic point is a smooth invariant (refer to [7]).

In this paper, we study two smooth invariants associated to a circle endomorphism. One is called the scaling function. The other is called the linear model. Most importantly, we study some relations between these two smooth invariants. Using these relations, we study some realization problems. One is for linear models which used to be studied in [3, 5] by employing some quasiconformal theory (refer to [1]) and the other is for scaling functions which used to be studied in [4] by employing g-measure theory (refer to [12, 15]) and Gibbs measure theory (refer to [2]). The discussion in this paper avoids any result from quasiconformal theory, g-measure theory, and Gibbs theory (these are heavily used in [4, 5]). Therefore, it is straightforward and simple and could be addressed to a wider class of audiences. Another point of this paper is to show a useful technique to construct a desired dynamical system by information from smooth invariants. For simplicity, we only discuss a circle endomorphism of degree two, although many aspects in this paper can be generalized to a higher degree. However, a complete study in this direction for a circle endomorphism of degree >2 is still an interesting research problem.

2. DISTORTION PROPERTY

Let $\mathbb{R}$ be the real line and $S^1=\mathbb{R}/\mathbb{Z}$ be the circle. Consider the translation $T(x)=x+1$. Then S^1 can be viewed as a quotient space: define an equivalent relation $x\sim y$ if $y=T^k(x)$, let $[x]$ mean the equivalence class of x, then

$$S^1=\{[x]\mid x\in\mathbb{R}\}.$$

Let $\pi:\mathbb{R}\to S^1$ be the covering map sending 0 to [0].

Suppose $f: S^1\to S^1$ is an orientation-preserving covering map of degree two. We call it a degree two circle endomorphism. Its lifts $F:\mathbb{R}\to\mathbb{R}$ are orientation-preserving homeomorphisms such that

$$\pi F=f\pi \quad \text{and} \quad F(x+1)=F(x)+2. \tag{1}$$

We write the late equality as

$$FT=T^2F. \tag{2}$$

Without loss of generality, assume [0] is a fixed point of f. We take the lift F fixing 0.

We say f is $C^{1+\alpha}$, $0<\alpha\leqslant 1$, if F is C^1 and its derivative F' is α-Hölder continuous, i.e.,

$$\sup_{x\neq y}\frac{|F'(x)-F'(y)|}{|x-y|^{\alpha}}<\infty.$$

We call f expanding if there are two constants $C>0$ and $\lambda>1$ such that

$$(F^m)'(x)\geqslant C\lambda^m,\quad \forall x\in\mathbb{R},\ \forall m>0. \tag{3}$$

Suppose f is $C^{1+\alpha}$ expanding for some $0<\alpha\leqslant 1$. Then F is a $C^{1+\alpha}$-diffeomorphism of $\mathbb{R}$ satisfying (2). (In this case f has only one fixed point [0].) Let $m=\max_{x\in\mathbb{R}}F'(x)>0$ be the maximum value of F'. Let F^{-1} be the inverse of F. Let

$$A=\sup_{x\neq y\in\mathbb{R}}\frac{|(F^{-1})'(x)-(F^{-1})'(y)|}{|x-y|^{\alpha}}<\infty$$

be the Hölder constant of F^{-1}. For any $x,y\in\mathbb{R}$, let $x_i=F^{-i}(x)$ and $y_i=F^{-i}(y)$ for $i=0,1,\dots$.

Lemma 1 (Distortion)**.** *Let* $B=(mAC^{-\alpha}\lambda^{\alpha})/(\lambda^{\alpha}-1)$ *be a constant associated to f. Then for any* $x,y\in\mathbb{R}$,

$$\left|\log\frac{(F^n)'(F^{-n}(y))}{(F^n)'(F^{-n}(x))}\right|=\left|\log\frac{(F^{-n})'(x)}{(F^{-n})'(y)}\right|\leqslant B|x-y|^{\alpha}. \tag{4}$$

The reader may refer to [7] for the proof of this lemma.

3. LINEAR MODELS

Suppose f is a $C^{1+\alpha}$ expanding degree two circle endomorphism for some $0<\alpha\leqslant 1$. Since 0 is an expanding fixed point of F, its eigenvalue $\delta=F'(0)>1$. Define a sequence of $C^{1+\alpha}$-diffeomorphisms $\{\gamma_n=\delta^n F^{-n}\}_{n=1}^{\infty}$. Since $F(0)=0$,

$$\log\gamma_n'(x)=\log\frac{(F^{-n})'(x)}{(F^{-n})'(0)}.$$

For any x and $m > n \geqslant 1$, let $x_n = F^{-n}(x)$, then

$$\log \gamma_m'(x) = \log \frac{(F^{-m})'(x)}{(F^{-m})'(0)} = \log \frac{(F^{-(m-n)})'(x_n)(F^{-n})'(x)}{(F^{-(m-n)})'(0)(F^{-n})'(0)}$$
$$= \log \frac{(F^{-(m-n)})'(x_n)}{(F^{-(m-n)})'(0)} + \log \frac{(F^{-n})'(x)}{(F^{-n})'(0)} = \log \frac{(F^{-(m-n)})'(x_n)}{(F^{-(m-n)})'(0)} + \log \gamma_n'(x).$$

From Lemma 1,

$$|\log \gamma_m'(x) - \log \gamma_n'(x)| = \left| \log \frac{(F^{-(m-n)})'(x_n)}{(F^{-(m-n)})'(0)} \right| \leqslant B|x_n|^{\alpha}.$$

This implies that for any $x \in \mathbb{R}$, $\{\log \gamma_n'(x)\}_{n=1}^{\infty}$ is a Cauchy sequence since x_n tends to 0 as n goes to infinity. Therefore, it converges to a function $\psi(x)$. Furthermore,

$$|\log \gamma_n'(x) - \log \gamma_n'(y)| = \left| \log \frac{(F^{-n})'(x)}{(F^{-n})'(y)} \right|$$

for any x and y. From Lemma 1 again,

$$|\log \gamma_n'(x) - \log \gamma_n'(y)| \leqslant B|x-y|^{\alpha}$$

for any x and y. This implies that

$$|\psi(x) - \psi(y)| \leqslant B|x-y|^{\alpha}.$$

So $\psi(x)$ is an α-Hölder continuous function on $\mathbb{R}$. Moreover, we see that $\log \gamma_n'$ converges to ψ uniformly on any compact set in $\mathbb{R}$. Let $\phi = e^{\psi}$. Then γ_n' converges to ϕ and this convergence is uniform on any compact set in $\mathbb{R}$. Let $\gamma(x) = \int_0^x \phi(\xi)\mathrm{d}\xi$. Then γ is a $C^{1+\alpha}$-diffeomorphism of $\mathbb{R}$ such that γ_n converges to γ and this convergence is uniform on any compact set in $\mathbb{R}$. Since $\delta\gamma_n = \gamma_{n+1} \circ F$, $\delta\gamma = \gamma \circ F$ on $\mathbb{R}$. We normalize γ by $\gamma(0) = 0$ and $\gamma(1) = 1$. Then the normalized conjugacy γ is unique because if $\tilde{\gamma}$ is another normalized conjugacy such that $\delta\tilde{\gamma} = \tilde{\gamma} \circ F$, then $\delta\tilde{\gamma}(\gamma^{-1}(x)) = \tilde{\gamma}(\gamma^{-1}(\delta x))$. This implies that

$$(\tilde{\gamma} \circ \gamma^{-1})'(x) = (\tilde{\gamma} \circ \gamma^{-1})(\delta x) = (\tilde{\gamma} \circ \gamma^{-1})'(0)$$

for any x. So the difference between $\tilde{\gamma}$ and γ is a multiplication by a constant. Because $\tilde{\gamma}(1) = \gamma(1) = 1$, we have $\tilde{\gamma} = \gamma$. Let $P(x) = \delta x$. Then γ is the unique conjugacy between F and P, i.e.,

$$P\gamma = \gamma F. \tag{5}$$

All unit intervals $[n, n+1)$ bounded by integers are copies of S^1 in $\mathbb{R}$ identified by $T(x) = x+1$. Under the conjugacy γ, these copies are mapped to $[\gamma(n), \gamma(n+1))$ identified by

$$L = \gamma T \gamma^{-1}. \tag{6}$$

We would like to make a comment at this point. When we talk about a nonlinear dynamical system f on the circle, we mean that the nonlinearity is calculated by the Lebesgue metric introduced by the Lebesgue metric on $\mathbb{R}$ and the linear equivalence $T(x) = x+1$. Therefore, the nonlinear dynamical system f can be viewed as a pair (F, T). On the other hand, we can also view f as a linear map P and a nonlinear equivalence L on the real line as follows. Define an equivalence relation on the real line, $x \sim y$ if $x = L^k(y)$. Let $[x]_L$ mean the L-equivalent class of x. Then $S^1 = \{[x]_L | x \in \mathbb{R}\}$. Let $\pi_L \colon \mathbb{R} \to S^1$ be the universal covering map sending 0 to $[0]_L$. Then

$$f \pi_L = \pi_L P.$$

Thus f can be viewed as a pair (P, L).

Definition 1. The $C^{1+\alpha}$ diffeomorphism L is called the linear model of f.

Theorem 1. *The linear model L is a complete smooth invariant.*

Proof. Let f and g be two $C^{1+\alpha}$ degree two expanding circle endomorphisms. Suppose they are C^1 conjugate. That means there is a C^1 diffeomorphism h of S^1 such that

$$hf = gh.$$

Let F and G be the lifts of f and g fixing 0 and let H be the lift of h fixing 0. Then $H(x+1) = H(x)+1$ and

$$HF = GH.$$

So $\delta = F'(0) = G'(0)$. Let γ_f and γ_g be the normalized diffeomorphisms such that

$$\delta \gamma_f = \gamma_f F \quad \text{and} \quad \delta \gamma_g = \gamma_g G.$$

Then

$$\gamma_g H = \gamma_f \quad \text{and} \quad H T H^{-1} = T.$$

So

$$L_f = \gamma_f T \gamma_f^{-1} = \gamma_g H T H^{-1} \gamma_g^{-1} = \gamma_g T \gamma_g^{-1} = L_g.$$

This implies that L is a smooth invariant.

Now suppose f and g have the same linear model, i.e., $L_f = L_g$. Let $H = \gamma_g^{-1}\gamma_f$. Then H is $C^{1+\alpha}$ diffeomorphism satisfying $H(x+1) = H(x) + 1$. So it induces a circle diffeomorphism h such that $hf = gh$. That means f and g are C^1 conjugate. □

The pair (P, L) gives a representation of the smooth conjugacy class of f. Moreover, L must satisfy the relation

$$\delta L = L^2 \delta, \qquad L(0) = 1, \ \ \delta > 1. \tag{7}$$

If the reader is familiar with the renormalization theory for folding maps, he can discover that this equation is similar to the Civitanovic–Feigenbaum equation there (for example, see [7]).

The converse problem was stated in [6, p. 298]. That is, given a linear map $P(x) = \delta x, \delta > 1$, and an analytic diffeomorphism L of $\mathbb{R}$ satisfying (7), Douady and Hubbard asked that under what condition does the pair (P, L) represent a degree two expanding circle endomorphism?

Suppose L is the linear model of a degree two $C^{1+\alpha}$ expanding circle endomorphism f. Since

$$L = \gamma T \gamma^{-1} \quad \text{for} \quad \gamma = \lim_{n\to\infty} \delta^n F^{-n}, \ \ \delta = F'(0) > 1,$$

then from Lemma 1,

$$L'(x) = \lim_{n\to\infty} \frac{(F^{-n})'(y+1)}{(F^{-n})'(y)}, \quad \text{where } \gamma(y) = x.$$

Thus

$$\sup_{x\in\mathbb{R}} |\log L'(x)| < \infty. \tag{8}$$

Actually, this is also a sufficient condition. This was first studied by Cui [3] by employing some results in quasiconformal theory (refer to [1]) and in Mãne's paper [11]. Here we show a much simple proof just by employing the distortion property in Section 2. The proof is obtained by a understanding of the dual symbolic labeling to all intervals in Markov partitions (refer to [7, p. 76] and also see Section 4).

Theorem 2. *Suppose* $P(x)=\delta x$ *is a linear map of* $\mathbb{R}$ *for* $\delta>1$ *and* L *is an orientation-preserving* $C^{1+\alpha}$*-diffeomorphism of* $\mathbb{R}$ *for some* $0<\alpha\leqslant 1$. *Then* (P, L) *is a representation of a smooth class of degree two* $C^{1+\alpha}$ *expanding circle endomorphisms if and only if they satisfy* (7) *and* (8).

Proof. We have seen that (7) and (8) are necessary. Now let us prove that they are also sufficient. Let γ be any $C^{1+\alpha}$ diffeomorphism of $\mathbb{R}$ satisfying (6). Then $F=\gamma^{-1}P\gamma$ satisfies (2) since L satisfies (7). Therefore, F gives a degree two $C^{1+\alpha}$ circle endomorphism f. We must prove that f is also expanding.

The map f has two inverse branches

$$g_0=F^{-1}\colon [0,1]\to I_0=[0,a], \qquad g_1=F^{-1}T\colon [0,1]\to I_1=[a,1],$$

where $F(a)=1$. For any finite string w_n of 0's and 1's of length n, we read it from right to left (the dual labeling, see Section 4), i.e., $\nu_n=j_{n-1}\cdots j_1 j_0$. Let $g_{\nu_n}=g_{j_{n-1}}\cdots g_{j_1}g_{j_0}$ and $I_{\nu_n}=g_{\nu_n}([0,1])$. In other words, I_{ν_n} is the maximal closed interval in S^1 such that f^n restricted to the interior of I_{ν_n} is injective and $f^k(I_{\nu_n})$ is in $I_{j_{n-k-1}}$ for every $0\leqslant k<n$. Let η_n be the set of intervals I_{ν_n} for all strings ν_n of length n.

Another way to view I_{ν_n} in η_n is by using F directly. Because of (2), $F^n([0,1])=[0,2^n]$. Let $k=\sum_{q=0}^{n-1} j_q 2^q$. We have that $I_{\nu_n}=F^{-n}([k,k+1])$. Let $\nu'_{n-1}=j_{n-1}\cdots j_1$ and $m=\sum_{q=0}^{n-1} j_{q+1}2^{q-1}$. Then $I_{\nu'_{n-1}}=F^{-(n-1)}([m,m+1])\supset I_{\nu_n}$ and $k=2m+i_0$. Consider

$$\frac{|I_{\nu'_{n-1}}|}{|I_{\nu_n}|}=\frac{|F^{-(n-1)}([m,m+1])|}{|F^{-n}([k,k+1])|}.$$

Because of (5), (6), and (7)

$$\frac{|\gamma(I_{\nu'_{n-1}})|}{|\gamma(I_{\nu_n})|}=\frac{|\gamma F^{-(n-1)}([m,m+1])|}{|\gamma F^{-n}([k,k+1])|}=\frac{|P^{-(n-1)}(\gamma([m,m+1]))|}{|P^{-n}(\gamma([k,k+1]))|}$$
$$=\frac{|\delta\gamma([m,m+1])|}{|\gamma([k,k+1])|}=\frac{|\gamma([2m,2m+2])|}{|\gamma([k,k+1])|}.$$

So if $i_0=0$, then

$$\frac{|\gamma(I_{\nu'_{n-1}})|}{|\gamma(I_{\nu_n})|}=\frac{|\gamma([k,k+1])|+|\gamma([k+1,k+2])|}{|\gamma([k,k+1])|}$$
$$=\frac{|\gamma([k,k+1])|+|L\gamma([k,k+1])|}{|\gamma([k,k+1])|}=1+\frac{|L(\gamma([k,k+1]))}{|\gamma([k,k+1])|}$$

and if $i_0=1$, then

$$\begin{aligned}\frac{|\gamma(I_{\nu'_{n-1}})|}{|\gamma(I_{\nu_n})|} &= \frac{|\gamma([k-1,k])|+|\gamma([k,k+1])|}{|\gamma([k,k+1])|} \\ &= \frac{|L^{-1}(\gamma([k,k+1]))|+|\gamma([k,k+1])|}{|\gamma([k,k+1])|} = \frac{|L^{-1}(\gamma([k,k+1]))|}{|\gamma([k,k+1])|}+1.\end{aligned}$$

Since there is a constant $D>0$ such that $D^{-1}\leqslant L'(x)\leqslant D$ for all x,

$$\frac{|\gamma(I_{\nu'_{n-1}})|}{|\gamma(I_{\nu_n})|} \geqslant \lambda = 1+D^{-1} > 1.$$

This implies that

$$\kappa_n = \max\{|\gamma(I_{\nu_n})| \,\big|\, I_{\nu_n}\in\eta_n\} \leqslant \lambda^{-n}.$$

Since γ is diffeomorphic on [0,1],

$$\tau_n = \max\{|I_{w_n}| \,\big|\, I_{w_n}\in\eta_n\} \leqslant E_0\lambda^{-n}$$

for some constant $E_0>0$.

Following this fact and applying the proof of Lemma 1, we have a constant $E_1>0$ such that

$$E_1^{-1} < \frac{(F^n)'(x)}{(F^n)'(y)} \leqslant E_1$$

for any $n>0$, and $I_{\nu_n}\in\eta_n$, and any $x, y\in I_{\nu_n}$. (The reader may refer to [7] for a full amount of this argument.) Since $F^n(I_{\nu_n})=[0,1]$ for any $I_{\nu_n}\in\eta_n$, there is a $y_{\nu_n}\in I_{\nu_n}$ such that $(F^n)'(y_{\nu_n})\geqslant E\lambda^n$, where $E>0$ is a constant. For any x in [0,1] and any $n>0$, let I_{ν_n} be an interval in η_n containing x. Then

$$(F^n)'(x) \geqslant E_1^{-1}(F^n)'(y_{\nu_n}) \geqslant E_1^{-1}E\lambda^n = C\lambda^n, \quad C=E_1^{-1}E, \;\; x\in\mathbb{R}.$$

Therefore, f is expanding. □

4. SCALING FUNCTIONS

From Section 3, the reader has seen the power of the dual labeling of a sequence of Markov partitions $\{\eta_n\}_{n=0}^{\infty}$. Let us now formally introduce the symbolic representation and the dual symbolic representation of degree two expanding circle endomorphisms.

Suppose f is a degree two $C^{1+\alpha}$ expanding circle endomorphism. Let us first introduce the symbolic representation. We still consider two inverse branches of f as

$$g_0 = F^{-1}\colon [0,1] \to I_0 = [0,a], \quad \text{and} \quad g_1 = F^{-1}T\colon [0,1] \to I_1 = [a,1].$$

Let

$$\Sigma^+ = \prod_{n=0}^{\infty}\{0,1\} = \{\omega = i_0 i_1 \cdots i_{n-1} \cdots \mid i_n = 0, \text{ or } 1, \ n = 1,2,\ldots\}$$

with the product topology. Let $\sigma^+ : \Sigma^+ \to \Sigma^+$ be the shift, $\sigma^+(\omega) = i_1 \cdots i_{n-1} i_n \cdots$ for $\omega = i_0 i_1 \cdots i_{n-1} i_n \cdots$. Then

$$\sigma^+ \colon \Sigma^+ \to \Sigma^+$$

is called the symbolic dynamical system. For each $\omega = i_0 i_1 \cdots i_{n-1} \cdots \in \sum^+$, let $\omega_n = i_0 i_1 \cdots i_{n-1}$. Define

$$g_{\omega_n} = g_{i_0} g_{i_1} \cdots g_{i_{n-1}} \quad \text{and} \quad I_{\omega_n} = g_{\omega_n}([0,1]).$$

Then

$$\cdots \subset I_{\omega_n} \subset I_{\omega_{n-1}} \subset \cdots I_{\omega_2} \subset I_{\omega_1}.$$

Since f is expanding, we have $|I_{\omega_n}| \leqslant C^{-1}\lambda^{-n}$. Thus $\cap_{n=1}^{\infty} I_{\omega_n} = \{x_\omega\}$ is a singleton. Define

$$\theta(\omega) = x_\omega : \Sigma^+ \to S^1.$$

Then $\theta(\omega)$ is a continuous and onto map and one-to-one except for the countable set consisting of ω eventually all zero's or all one's. On this countable set, $\theta(\omega)$ is two-to-one. Moreover,

$$\theta\sigma^+ = f\theta.$$

Therefore, θ is a semi-conjugacy between f and σ^+. From this construction, one can easily get

Proposition 1. *Any two $C^{1+\alpha}$ degree two expanding circle endomorphisms are topologically conjugate.*

Proof. Suppose f and g are $C^{1+\alpha}$ degree two expanding circle endomorphisms. Then

$$\theta_f \sigma^+ = f\theta_f \quad \text{and} \quad \theta_g \sigma^+ = g\theta_g.$$

Let $h = \theta_f \theta_g^{-1}$. The map h is first defined on S^1 but on a countable set. But one can easily extend it to the whole S^1 as a homeomorphism and $fh = hg$. □

We conclude the above argument as follows: The symbolic dynamical system $\sigma^+ : \Sigma^+ \to \Sigma^+$ is a representation of the topological conjugacy class. So we call (σ^+, Σ^+) the symbolic representation of f.

Next let us see the dual symbolic representation and the scaling function. The purpose here is to give a representation for each smooth conjugacy class. Let

$$\Sigma^- = \prod_{n=-\infty}^{0} \{0, 1\} = \{\nu = \cdots j_{n-1} \cdots j_1 j_0 \cdots \mid j_{n-1} = 0, \text{ or } 1, \; n = 1, 2, \ldots\}$$

with the product topology. Let $\sigma^- : \Sigma^- \to \Sigma^-$ be the shift, $\sigma^-(\nu) = \cdots j_{n-1} \cdots j_1 j_0$ for $\nu = \cdots j_{n-1} \cdots j_1 j_0$. Then

$$\sigma^- : \Sigma^- \to \Sigma^-$$

is called the dual symbolic dynamical system.

For each $\nu = \cdots j_{n-1} \cdots j_1 j_0 \in \Sigma^-$, let $\nu = j_{n-1} \cdots j_1 j_0$. Define

$$g_{\nu_n} = g_{j_{n-1}} \cdots g_{j_1} g_{j_0} \quad \text{and} \quad I_{\nu_n} = g_{\nu_n}([0, 1]).$$

Let $\eta_n = \{I_{\nu_n}\}$. Then η_n is a partition. It is actually a Markov partition (see [7]). Thus for $n = 1, 2, \ldots$, we get a sequence of Markov partitions. Let $\sigma^-(\nu_n) = (\sigma^-(\nu))_{n-1}$. Then

$$I_{\nu_n} \subset I_{\sigma^-(\nu_n)}, \quad \text{where } I_{\nu_n} \in \eta_n \quad \text{and} \quad I_{\sigma^-(\nu_n)} \in \eta_{n-1}.$$

Define

$$S(\nu_n) = \frac{|I_{\nu_n}|}{|I_\sigma - (\nu_n)|}.$$

We say a function S on Σ^- is Hölder continuous if there is a constant $C > 0$ and $0 < \tau < 1$ such that

$$|S(\nu) - S(\nu')| \leqslant C\tau^n$$

for any $\nu = \cdots j_{n-1} \cdots j_0$ and $\nu' = \cdots j_{n-1} \cdots j_0 \in \Sigma^-$.

Lemma 2. *For any* $\nu = \cdots \nu_n \in \Sigma^-$, *the limit*

$$S(\nu) = \lim_{n \to \infty} S(\nu_n)$$

exists. And moreover, S *is a Hölder continuous function.*

Proof. For any $m > n > 0$, $F^{m-n}(I_{\nu_m}) = (I_{\nu_n})$ and $F^{m-n}(I_{\sigma^-(\nu_m)}) = (I_{\sigma^-(\nu_n)})$. Moreover, $F^{n-1}(I_{\sigma^-(\nu_n)}) = [0,1]$ and $F^{n-1}(I_{\nu_n}) = [0,a]$ or $[a,1]$ depending on $j_0 = 0$ or 1. From Lemma 1 and the mean value theorem, there is a constant $E > 0$ such that

$$E^{-1} \leqslant \frac{|I_{\nu_n}|}{|I_{\sigma^-(\nu_n)}|} \leqslant E \text{ and } \left|1 - \frac{F^{m-n}(\eta)}{F^{m-n}(\xi)}\right| \leqslant E|I_{\sigma^-(\nu_n)}|^\alpha \leqslant EC^{-\alpha}\lambda^{-\alpha(n-1)}, \eta, \xi \in I_{\sigma^-(\nu_n)}.$$

So

$$|S(\nu_m) - S(\nu_n)| = \left|1 - \frac{F^{m-n}(\eta)}{F^{m-n}(\xi)}\right| S(\nu_n) \leqslant E^2 C^{-\alpha}\lambda^{-\alpha(n-1)}.$$

Thus $\{S(\nu_n)\}_{n=0}^{\infty}$ is a Cauchy sequence and the limit $S(\nu) = \lim_{n\to\infty} S(\nu_n)$ exists and defines a function on Σ^-.

For any $\nu = \cdots \nu_n$ and $\nu' = \cdots \nu_n$ in Σ^- and any $m > n$,

$$|S(\nu_m) - S(\nu'_m)| = \left|\frac{F^{m-n}(\eta)}{F^{m-n}(\xi)} - \frac{F^{m-n}(\eta')}{F^{m-n}(\xi')}\right| S(\nu_n) \leqslant 2E^2 C^{-\alpha}\lambda^{-\alpha(n-1)}.$$

Thus

$$|S(\nu) - S(\nu')| \leqslant A\tau^n$$

for some constants $A > 0$ and $0 < \tau < 1$. This says that S is Hölder continuous. □

Definition 2. The function $S\colon \Sigma^- \to \mathbb{R}^+$ is called the scaling function of f.

Theorem 3. *The scaling function S is a complete smooth invariant.*

Proof. Suppose f and g are $C^{1+\alpha}$ degree two expanding circle endomorphisms. From Proposition 1, they are topologically conjugate. So there is a homeomorphism h of S^1 such that $hf = gh$. For every $\nu = \cdots \nu_n \in \Sigma^-$, let $I_{\nu_n, f}$ and $I_{\nu_n, g}$ be intervals for f and g. Then $h(I_{\nu_n, f}) = I_{\nu_n, g}$.

If h is a C^1 diffeomorphism h, then

$$S_g(\nu_n) = \frac{|I_{\nu_n, g}|}{|I_{\sigma^-(\nu_n), g}|} = \frac{|h(I_{\nu_n, f})|}{|h(I_{\sigma^-(\nu_n), f})|} = \frac{h'(\eta)}{h'(\xi)} \frac{|I_{\nu_n, f}|}{|I_{\sigma^-(\nu_n), f}|} = \frac{h'(\eta)}{h'(\xi)} S_f(\nu_n).$$

Since $h'(\eta)/h'(\xi)$ tends to 1 uniformly as $\eta - \xi$ goes to 0. We have $S_f(\nu) = S_g(\nu)$. So S is a smooth invariant.

Now suppose $S = S_f = S_g$. Write

$$|h(I_{\nu_n,f})| = |I_{\nu_n,g}| = \prod_{i=0}^{n-1} S_g((\sigma^-)^i(\nu_n))$$

and

$$|I_{\nu_n,f}| = \prod_{i=0}^{n-1} S_f((\sigma^-)^i(\nu_n)).$$

Then

$$\frac{|h(I_{\nu_n,f})|}{|I_{\nu_n,f}|} = \prod_{i=0}^{n-1} \frac{S_g((\sigma^-)^i(\nu_n))}{S_f((\sigma^-)^i(\nu_n))}.$$

Since $S_g((\sigma^-)^i(\nu_n))$ and $S_f((\sigma^-)^i(\nu_n))$ both exponentially converges to the same positive function S, $S_g((\sigma^-)^i(\nu_n))/S_f((\sigma^-)^i(\nu_n))$ exponentially converges to 1. Thus there is a constant $E > 0$ such that

$$E^{-1} \leqslant \frac{|h(I_{\nu_n,f})|}{|I_{\nu_n,f}|} \leqslant E$$

for all $n > 0$ and all ν_n. The set of endpoints of $I_{\nu_n,f}$ for all $n > 0$ and all ν_n is dense in [0, 1]. So the additive formula implies that

$$E^{-1} \leqslant \frac{|h(x) - h(y)|}{|x - y|} \leqslant E, \forall x, y \in [0, 1].$$

(The additive formula means that if $E^{-1} \leqslant a_i/b_i \leqslant E$ for sequences of positive numbers a_i and b_i, then $E^{-1} \leqslant \sum a_i / \sum b_i \leqslant E$.)

A bi-Lipschitz homeomorphism is absolutely continuous. It now follows from the result in [14] that if h is absolutely continuous, then it is $C^{1+\alpha}$ diffeomorphism. So S is a complete smooth invariant. □

A more general result than Theorem 3 is proved in [8, 9] (see also [7, 10]) where we study the smooth classification of geometrically finite dynamical systems by using scaling functions. The reader who is interested in this topic could go to these papers. Some other rigidity results are also discussed in these papers.

A function S on Σ^- is said to satisfy the summation condition if

$$S(\nu 0) + S(\nu 1) = 1, \quad \forall \nu \in \Sigma^-. \tag{9}$$

Define a sequence of functions on Σ^-:

$$C_{S,N}(\nu) = \prod_{n=0}^{N} \frac{S(\nu 1 \overbrace{0 \cdots 0}^{n})}{S(\nu 0 \underbrace{1 \cdots 1}_{n})}.$$

We say S satisfies the compatibility condition if $C_{S,N}(\nu)$ converges exponentially to a constant function on Σ^-, i.e.,

$$\prod_{n=0}^{\infty} \frac{S(\nu 1 \overbrace{0 \cdots 0}^{n})}{S(\nu 0 \underbrace{1 \cdots 1}_{n})} = \text{const on} \quad \Sigma^-. \tag{10}$$

The next result indicates that these two conditions are important in the scaling function theory.

Theorem 4. *Let S be a Hölder continuous function on Σ^-. Then S is the scaling function of a $C^{1+\alpha}$ degree two expanding circle endomorphism for some $0 < \alpha \leqslant 1$ if and only if S satisfies the summation condition* (9) *and the compatibility condition* (10).

The theorem is first proved in [4] by employing some results in g-measure theory and Gibbs measure theory. However, in Section 5, we will study some relations between the scaling function and the linear model. These relations provide an alternative technique to study the theorem and other similar problems in this direction.

5. RELATIONS BETWEEN *S* AND *L*.

Notice that $L^k([0,1]) = [\gamma(k), \gamma(k+1)]$. Embed the set of nonnegative integers into Σ^-: For each natural number $k = \sum_{q=0}^{n-1} j_q 2^q$ with $j_q = 0$ or 1, let $\nu = \nu(k) = \cdots 000 j_{n-1} \cdots j_0$. Conversely, for each $\nu = \cdots 000 j_{n-1} \cdots j_0$, let $k = k(\nu) = \sum_{q=0}^{n-1} j_q 2^q$. Define

$$\text{sol}(\nu) = \text{sol}(k) = \frac{|L^k([0,1])|}{|L^{k-1}([0,1])|}.$$

The function sol(k) is similar to the solenoid function (refer to [13]). For $\nu = \cdots \nu_n = \cdots j_{n-1} \cdots j_0 \in \Sigma^-$,

$$\frac{1}{S(\nu)} = \lim_{n\to\infty} \frac{|I_{\sigma^-(\nu_n)}|}{|I_{\nu_n}|} = \lim_{n\to\infty} \frac{|\gamma(I_{\sigma^-(\nu_n)})|}{|\gamma(I_{\nu_n})|}.$$

Let $k = k(\cdots 000 j_{n-1} \cdots j_1 j_0)$ and $l = l(\cdots 000 j_{n-1} \cdots j_1)$. Then $k = 2l + j_0$. Note that

$$\begin{aligned} I_{\nu_n} &= g_{j_{n-1}} \circ \cdots \circ g_{j_0}(I) = F^{-1} \circ T^{j_{n-1}} \circ \cdots \circ F^{-1} \circ T^{j_0}([0, 1]) \\ &= F^{-n}(T^{j_0 + 2j_1 + \cdots + 2^{n-1} j_{n-1}}([0, 1])) = F^{-n}([k, k+1]), \end{aligned}$$

and similarly,

$$I_{\sigma^-(\nu_n)} = F^{-n+1}([l, l+1]).$$

Therefore, since $\gamma(F^{-n}(x)) = \delta^{-n}\gamma(x)$ and

$$\delta\gamma(l) = \gamma(F(l)) = \gamma(F(T^l(0))) = \gamma T^{2l}(F(0)) = \gamma(2l),$$

we have

$$\begin{aligned} \frac{|\gamma(I_{\sigma-(\nu_n)})|}{|\gamma(I_{\nu_n})|} &= \frac{|\gamma(F^{-n+1}([l, l+1]))|}{|\gamma(F^{-n}([k, k+1]))|} = \frac{\delta|\gamma([l, l+1])|}{|\gamma([k, k+1])|} \\ &= \frac{|\gamma([k - j_0, k - j_0 + 2])|}{|\gamma([k, k+1])|}. \end{aligned}$$

Since $\delta I = [0, \delta] = I \cup L(I)$ and $\gamma(k) = L^k(0)$, we can rewrite

$$\frac{|\gamma([k - j_0, k - j_0 + 2])|}{|\gamma([k, k+1])|} = \frac{|L^{k-j_0}(I + L(I))|}{|L^k(I)|} = \left(1 + \frac{|L^{(-1)^{j_0}}(L^k(I))|}{|L^k(I)|}\right).$$

Thus we get

$$\frac{1}{S(\nu)} = \frac{|L^{k-j_0}(I + L(I))|}{|L^k(I)|} = \left(1 + \frac{|L^{(-1)^{j_0}}(L^k(I))|}{|L^k(I)|}\right). \tag{11}$$

Consider $\nu = \nu' j_0 = 000 j_{n-1} \cdots j_0$ and consider the two cases, $j_0 = 0$ and $j_0 = 1$. From (11), in the first case,

$$\frac{1}{S(\nu' 0)} = \left(1 + \frac{|L^{k+1}(I)|}{|L^k(I)|}\right) = 1 + \mathrm{sol}(\nu' 1); \tag{12}$$

and, in the second case,

$$\frac{1}{S(\nu' 1)} = \left(1 + \frac{|L^{k-1}(I)|}{|L^k(I)|}\right) = 1 + \frac{1}{\mathrm{sol}(\nu' 1)}. \tag{13}$$

Thus we have

$$\mathrm{sol}(\nu' 1) = \frac{S(\nu' 1)}{S(\nu' 0)}. \tag{14}$$

Suppose

$$\nu = \cdots 000\nu_n = \cdots 000 j_{n-1} \cdots j_0 \in \Sigma^-.$$

Let $k = k(\nu) = \sum_{q=0}^{n-1} j_q 2^q$. Define the "add-one" function add ν by

$$\mathrm{add}(\nu(k)) = \nu(k+1).$$

Note that

$$\mathrm{sol}(\nu) = \frac{L^k(I)}{L^{k-1}(I)}$$

and

$$\delta L^k(I) = L^{2k}(\delta I) = L^{2k}(I + L(I)).$$

Therefore,

$$\mathrm{sol}(\nu) = \frac{\delta L^k(I)}{\delta L^{k-1}(I)} = \frac{L^{2k}(\delta I)}{L^{2k-2}(\delta I)} = \frac{L^{2k}(I) + L^{2k+1}(I)}{L^{2k-2}(I) + L^{2k-1}(I)}.$$

We find the following formula:

$$\frac{\mathrm{sol}(\nu)}{\mathrm{sol}(\nu 0)} = \frac{(L^{2k}(I) + L^{2k+1}(I))/L^{2k}(I)}{(L^{2k-2}(I) + L^{2k-1}(I))/L^{2k-1}(I)} = \frac{1 + \mathrm{sol}(\nu 1)}{1 + (L^{2k-2}(I)/L^{2k-1}(I))}.$$

This implies that

$$\frac{\mathrm{sol}(\nu)}{\mathrm{sol}(\nu 0)} = \frac{1 + \mathrm{sol}(\nu 1)}{1 + [\mathrm{sol}(\mathrm{add}^{-1}(\nu 0))]^{-1}}. \tag{15}$$

From (4),

$$\frac{S(\nu 1)}{S(\nu 0)} = \mathrm{sol}(\nu 1).$$

From (12), (13), and (15),

$$\frac{S(\nu 10)}{S(\nu 01)} = \frac{1 + [\mathrm{sol}(\nu 01)]^{-1}}{1 + [\mathrm{sol}(\nu 11)]} = \frac{\mathrm{sol}(\nu 10)}{\mathrm{sol}(\nu 1)}.$$

Similarly,

$$\frac{S(\nu 100)}{S(\nu 011)} = \frac{1 + [\mathrm{sol}(\nu 011)]^{-1}}{1 + [\mathrm{sol}(\nu 101)]} = \frac{1 + [\mathrm{sol}(\mathrm{add}^{-1}(\nu 100))]^{-1}}{1 + \mathrm{sol}(\nu 101)} = \frac{\mathrm{sol}(\nu 100)}{\mathrm{sol}(\nu 10)}.$$

Proceeding by the induction, we conclude

$$\mathrm{sol}(\nu 1 \underbrace{0\cdots 0}_{n-1}) = \prod_{i=0}^{n-1} \frac{S(\nu 1 \overbrace{0\cdots 0}^{i})}{S(\nu 0 \underbrace{1\cdots 1}_{i})}.$$

Thus $C_{S,N}(\nu)$ converges exponentially to $\mathrm{sol}(\cdots 0\cdots 0)$ for all $\nu\in\Sigma^-$. This is just the compatibility condition (10). Since for any $\nu=\cdots\nu_n\in\Sigma^-$,

$$I_{\nu_n} = I_{\nu_n 0} \cup I_{\nu_n 1}$$

the scaling function S of f also satisfies the summation condition (9). Thus the above relations provide a proof of the "only if" part in Theorem 4.

Now suppose we are given a Hölder continuous function S: $\Sigma^- \to \mathbb{R}^+$ satisfying (9) and (10). Consider

$$\Sigma^- = \Sigma_0^- \cup \Sigma_1^-,$$

where

$$\Sigma_0^- = \{\nu = \cdots j_{n-1}\cdots j_1 0\}, \qquad \Sigma_1^- = \{\nu = \cdots j_{n-1}\cdots j_1 1\}.$$

For any $\nu\in\Sigma^-$, define

$$\mathrm{sol}(v1) = \frac{S(\nu 1)}{S(\nu 0)}.$$

It is a Hölder continuous function on Σ_1^-. Define

$$\mathrm{sol}(\nu 1 \underbrace{0\cdots 0}_{n}) = \prod_{i=0}^{n} \frac{S(\nu 1 \overbrace{0\cdots 0}^{i})}{S(\nu 0 \underbrace{1\cdots 1}_{i})}.$$

It is clearly Hölder continuous at any point $\nu\neq\cdots 000$. For $n>m$ the ratio

$$\frac{\mathrm{sol}(\nu 1 \overbrace{0\cdots 0}^{n})}{\mathrm{sol}(\nu 1 \underbrace{0\cdots 0}_{m})} = \prod_{i=m+1}^{n} \frac{S(\nu 1 \overbrace{0\cdots 0}^{i})}{S(\nu 0 \underbrace{1\cdots 1}_{i})}.$$

From (10), these partial products converge to 1 exponentially fast. Thus sol is also Hölder continuous at $\cdots 000$. Since Σ^- is a compact space, we conclude that sol is a positive Hölder continuous function on Σ^-.

In the following we will see that the function sol satisfies (15). Consider $\nu' 1 \in \Sigma_1^-$. Then

$$\operatorname{sol}(\nu) = \operatorname{sol}(\nu' 1) = \frac{S(\nu' 1)}{S(\nu' 0)},$$
$$\operatorname{sol}(\nu 0) = \operatorname{sol}(\nu' 10) = \frac{S(\nu' 1)}{S(\nu' 0)} \frac{S(\nu' 10)}{S(\nu' 01)},$$
$$\operatorname{sol}(\nu 1) = \operatorname{sol}(\nu' 11) = \frac{S(\nu' 11)}{S(\nu' 10)}$$

and

$$\operatorname{sol}(\operatorname{add}^{-1}(\nu 0)) = \operatorname{sol}(\operatorname{add}^{-1}(\nu' 10)) = \operatorname{sol}(\nu' 01) = \frac{S(\nu' 01)}{S(\nu' 00)}.$$

Thus we have

$$\frac{\operatorname{sol}(\nu)}{\operatorname{sol}(\nu 0)} = \frac{S(\nu' 01)}{S(\nu' 10)}$$

and

$$\frac{1 + \operatorname{sol}(\nu 1)}{1 + [\operatorname{sol}(\operatorname{add}^{-1}(\nu 0))]^{-1}} = \frac{1 + (S(\nu' 11)/S(\nu' 10))}{1 + (S(\nu' 00)/S(\nu' 01))}$$
$$= \frac{(S(\nu' 10) + S(\nu' 11))/S(\nu' 10)}{(S(\nu' 01) + S(\nu' 00))/S(\nu' 01)} = \frac{S(\nu' 01)}{S(\nu' 10)}.$$

The last equality follows from (9). This says that (15) holds for $\nu = \nu' 1 \in \Sigma_1^-$.

Now suppose $\nu = \nu' 1 \underbrace{0 \cdots 0}_{n}$. Then

$$\operatorname{sol}(\nu) = \operatorname{sol}(\nu' 1 \underbrace{0 \cdots 0}_{n}) = \prod_{i=0}^{n} \frac{S(\nu' 1 \overbrace{0 \cdots 0}^{i})}{S(\nu' 0 \underbrace{1 \cdots 1}_{i})}$$

and

$$\operatorname{sol}(\nu 0) = \operatorname{sol}(\nu' 1 \underbrace{0 \cdots 0}_{n+1}) = \prod_{i=0}^{n+1} \frac{S(\nu' 1 \overbrace{0 \cdots 0}^{i})}{S(\nu' 0 \underbrace{1 \cdots 1}_{i})}.$$

So

$$\frac{\mathrm{sol}(\nu)}{\mathrm{sol}(\nu 0)} = \frac{S(\nu' 0 \overbrace{1 \cdots 1}^{n+1})}{S(\nu' 1 \underbrace{0 \cdots 0}_{n+1})}.$$

But

$$\mathrm{sol}(\nu 1) = \mathrm{sol}(\nu' 1 \underbrace{0 \cdots 0}_{n} 1) = \frac{S(\nu' 1 \overbrace{0 \cdots 0}^{n} 1)}{S(\nu' 1 \underbrace{0 \cdots 0}_{n+1})}$$

and

$$\mathrm{sol}(\mathrm{add}^{-1}(\nu 0)) = \mathrm{sol}(\mathrm{add}^{-1}(\nu' 1 \underbrace{0 \cdots 0}_{n+1})) = \mathrm{sol}(\nu' 0 \underbrace{1 \cdots 1}_{n+1}) = \frac{S(\nu' 0 \overbrace{1 \cdots 1}^{n+1})}{S(\nu' 0 \underbrace{1 \cdots 1}_{n} 0)}.$$

We get

$$1 + \mathrm{sol}(\nu 1) = \frac{S(\nu' 1 \overbrace{0 \cdots 0}^{n+1}) + S(\nu' 1 \overbrace{0 \cdots 0}^{n} 1)}{S(\nu' 1 \underbrace{0 \cdots 0}_{n+1})} = \frac{1}{S(\nu' 1 \underbrace{0 \cdots 0}_{n+1})}$$

and

$$1 + [\mathrm{sol}(\mathrm{add}^{-1}(\nu 0))]^{-1} = 1 + \frac{S(\nu' 0 \overbrace{1 \cdots 1}^{n} 0)}{S(\nu' 0 \underbrace{1 \cdots 1}_{n+1})} = \frac{1}{S(\nu' 0 \underbrace{1 \cdots 1}_{n+1})}.$$

So

$$\frac{1 + \mathrm{sol}(\nu 1)}{1 + [\mathrm{sol}(\mathrm{add}^{-1}(\nu 0))]^{-1}} = \frac{S(\nu' 0 \overbrace{1 \cdots 1}^{n+1})}{S(\nu' 1 \underbrace{0 \cdots 0}_{n+1})}$$

and (15) holds for $\nu = \nu' 1 \underbrace{0 \cdots 0}_{n} \in \Sigma_0^-$.

Embed all nonnegative numbers k into Σ^- by the formula $\nu(k) = \cdots 000 j_{n-1} \cdots j_1 j_0$ where $k = j_0 + j_1 2 + \cdots + j_{n-1} 2^{n-1}$. Then define $a_k = L^k(0)$ inductively by $a_0 = 0, a_1 = 1$ and a_k by

$$a_{k+1} - a_k = \mathrm{sol}(\nu(k))(a_k - a_{k-1}), \quad k \geqslant 1.$$

We need to check that L on $\{a_k\}_{k\geqslant 0}$ satisfies (7). However, this follows (15) due to the following reason.

$$\frac{(a_{k+1}-a_k)/(a_k-a_{k-1})}{(a_{2k+1}-a_{2k})/(a_{2k}-a_{2k-1})} = \frac{1+(a_{2k+2}-a_{2k+1})/(a_{2k+1}-a_{2k})}{1+(a_{2k-1}-a_{2k-2})/(a_{2k}-a_{2k-1})} = \frac{(a_{2k+2}-a_{2k})/(a_{2k+1}-a_{2k})}{(a_{2k}-a_{2k-2})/(a_{2k}-a_{2k-1})}.$$

Thus

$$\frac{a_{k+1}-a_k}{a_k-a_{k-1}} = \frac{(a_{2k+2}-a_{2k})/(a_{2k+1}-a_{2k})}{(a_{2k}-a_{2k-2})/(a_{2k}-a_{2k-1})} \cdot \frac{a_{2k+1}-a_{2k}}{a_{2k}-a_{2k-1}} = \frac{a_{2k+2}-a_{2k}}{a_{2k}-a_{2k-2}}.$$

One can check that $a_2 = \delta$ and

$$a_2 - a_1 = \frac{a_4 - a_2}{a_2}.$$

This implies that

$$a_4 = \delta^2 = \delta a_2.$$

Inductively, suppose $a_{2k} = \delta a_k$ holds for all $0 \leqslant k \leqslant n$. Then

$$\frac{a_{2n+2}-a_{2n}}{a_{2n}-a_{2n-2}} = \frac{a_{n+1}-a_n}{a_n-a_{n-1}} = \frac{\delta a_{n+1}-\delta a_n}{\delta a_n - \delta a_{n-1}} = \frac{\delta a_{n+1}-a_{2n}}{a_{2n}-a_{2n-2}}.$$

Therefore, $a_{2n+2} = \delta a_{n+1}$ for all $n \geqslant 0$.

Using $\{a_k\}_{k=0}^{\infty}$, we give a sequence of nested partitions $\{\eta_n\}$ on $I = [0, 1]$ as follows. For each $n \geqslant 0$, consider the interval $[0, a_{2^n}] = [0, \delta^n]$. For each $0 \leqslant k \leqslant 2^n$, define $I^n_{\nu(k)} = \delta^{-n}[a_k, a_{k+1}]$. Then $\eta_n = \{I_{\nu(k)}\}_{k=0}^{2^{n-1}-1}$ gives a partition of [0, 1]. Since

$$I^n_{\nu(k)} = I^{n+1}_{\nu(2k)} \cup I^{n+1}_{\nu(2k+1)},$$

η_{n+l} is a finer partition of η_n. Thus we get a sequence of Markov partitions $\{\eta_n\}$. For this sequence of Markov partitions and the dual labeling on it, the scaling function is just S. Moreover, define

$$L(\delta^{-n} a_k) = \delta^{-n} a_{k+2^n}, \quad \forall n, \ \ k \geqslant 0.$$

So we defined L on $\mathbb{R}^+$.

Since

$$\frac{a_{k+1}-a_k}{a_k-a_{k-1}}=\lim_{n\to\infty}\frac{a_{k+1+2^n}-a_{k+2^n}}{a_{k+2^n}-a_{k+2^n-1}}=\lim_{n\to\infty}\mathrm{sol}(w(k+2^n))$$

must be true for a linear model L, we can use it to define a_k for all negative integers $k<0$. For example, $a_{-1}=(\mathrm{sol}(\cdots 000))^{-1}$. Similarly, a_k for $k\leqslant -2$ can be defined inductively. So similarly, L can be extended to $\mathbb{R}$.

Consider $x=\delta^{-n}a_k$. Then one can check that $x=\delta^{-n-m}a_{2^m k}$ for $m>0$. Let $\epsilon=\delta^{-n-m}a_{2^m k+1}-x$. Then

$$\frac{|L(x+\epsilon)-L(x)|}{\epsilon}=\frac{|L(\delta^{-n-m}(a_{2^m k+1}-a_{2^m k}))|}{\delta^{-n-m}(a_{2^m k+1}-a_{2^m k})}$$

$$=\frac{\delta^{-n-m}(a_{2^{n+m}+2^m k+1}-a_{2^{n+m}+2^m k})}{\delta^{-n-m}(a_{2^m k+1}-a_{2^m k})}=\frac{|I_{1j_{n-1}\cdots j_0\overbrace{0\cdots 0}^{m}}|}{|I_{0j_{n-1}\cdots j_0\underbrace{0\cdots 0}_{m}}|}$$

$$=\prod_{i=0}^{m}\frac{S(\bar{0}1j_{n-1}\cdots j_0\overbrace{0\cdots 0}^{m-i})}{S(\bar{0}0j_{n-1}\cdots j_0\underbrace{0\cdots 0}_{m-i})}\prod_{i=0}^{n-1}\frac{S(\bar{0}1j_{n-1}\cdots j_i)}{S(\bar{0}0j_{n-1}\cdots j_i)}\frac{S(\bar{0}1)}{S(\bar{0}0)},$$

Where $\bar{0}=\cdots 000$. So

$$L'(x)=\prod_{k=0}^{\infty}\frac{S(\bar{0}1j_{n-1}\cdots j_0\overbrace{0\cdots 0}^{m-i})}{S(\bar{0}0j_{n-1}\cdots j_0\underbrace{0\cdots 0}_{m-i})}\prod_{k=0}^{n-1}\frac{S(\bar{0}1j_{n-1}\cdots j_k)}{S(\bar{0}0j_{n-1}\cdots j_k)}\frac{S(\bar{0}1)}{S(\bar{0}0)}.$$

Since S is Hölder, $\sup_{x\in\mathbb{R}}|\log L'(x)|<\infty$. Now $L'(x)$ is Hölder continuous on $\{\delta^{-n}a_k\}$ since S is Hölder. Furthermore, since $\{\delta^{-n}a_k\}$ and $\{\delta^{-n}a_{k+2^n}\}$ are both dense in $\mathbb{R}, L$ can be extended to a $C^{1+\alpha}$ diffeomorphism on $\mathbb{R}$. Therefore, from S, first construct L. Then using Theorem 2, construct a degree two $C^{1+\alpha}$ dynamical system f whose linear model is L and the scaling function is S. This method provides another way to prove the "if" part in Theorem 4.

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