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The Thurston Type Theorem for Branched Coverings of Two-Sphere

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Abstract

I give a survey about a program which intends to find topological characterizations of a rational map and then use them to study the rigidity problem for rational maps. Thurston started this program by considering critically finite branched coverings and gave a necessary and sufficient combinatorial condition for a critically finite rational maps among all critically finite branched coverings. Douady and Hubbard gave a complete proof of this result, that is, a critically finite branched covering is combinatorially equivalent to a unique rational map (up to conjugations by automorphisms of the Riemann sphere) if and only if it has no Thurston obstruction. McMullen showed that no Thurston obstruction is essentially true for rational maps. Cui, Jiang, and Sullivan constructed a counter-example of a geometrically finite branched covering such that it has no Thurston obstruction but is not combinatorially equivalent to a rational map. Thus Thurston's condition fails for a geometrically finite rational maps among all geometrically finite branched coverings. Following this work, classes of semi-rational and sub-hyperbolic semi-rational branched coverings and the CLH-equivalence are introduced into this study in Cui, Jiang, Sullivan's paper. They further showed that a semi-rational branched covering is always combinatorially equivalent to a sub-hyperbolic semi-rational branched covering. Following this, the Thurston type theorem is proved for sub-hyperbolic semi-rational branched coverings by using some combinatorial methods, that is, a sub-hyperbolic semi-rational branched covering is CLH-equivalent to a unique rational map (up to conjugations by automorphisms of the Riemann sphere) if and only if it has no Thurston obstruction. Jiang and Zhang further studied in this direction from the bounded geometry point of view and gave a comprehensive understanding simultaneously for both theorems. This survey article intends to give a complete picture about this development.

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1 Branched coverings and combinatorial equivalence

Suppose $\mathbb{P}^1$ is the Riemann sphere. We also consider it as the two-sphere if we only consider it as a topological object. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an orientation-preserving branched covering of degree $d > 1$. Let

$$\Omega_f = \{z \in \mathbb{P}^1 \mid \deg_z f > 1\}$$

be the set of branched points of f, where $deg_p f$ means the local degree of f at p. It consists of a finite number of points. Points in $f(\Omega_f)$ are called critical values. The set

$$P_f = \cup_{n \geqslant 1}^{\infty} f^n(\Omega_f)$$

is called the set of post-critical orbits of f. A point in P_f is called a post-critical point. Let P'_f be the set of limiting points of P_f. Then $\overline{P}_f = P_f \cup P'_f$.

Since $\Omega_f \cup \overline{P}_f \subset f^{-1}(\overline{P}_f)$,

$$f : \mathbb{P}^1 \setminus f^{-1}(\overline{P}_f) \to \mathbb{P}^1 \setminus \overline{P}_f$$

is a covering.

Suppose f and g are two branched coverings. We say that f and g are combinatorially equivalent if there exist homeomorphisms $\phi, \psi : (\mathbb{P}^1, \overline{P}_f) \to (\mathbb{P}^1, \overline{P}_g)$ such that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1, \overline{P}_f) & \xrightarrow{\psi} & (\mathbb{P}^1, \overline{P}_g) \\ \downarrow f & & \downarrow g \\ (\mathbb{P}^1, \overline{P}_f) & \xrightarrow{\phi} & (\mathbb{P}^1, \overline{P}_g) \end{array}$$

commutes and ϕ is isotopic to ψ rel $\overline{P}_f$. That is, $\phi \circ f = g \circ \psi$ and there is a continuous map

$$H(z,t) : \mathbb{P}^1 \times [0,1] \to \mathbb{P}^1$$

such that

- for each $0 \leqslant t \leqslant 1$, $h_t = H(\cdot, t)$ is a homeomorphism of $\mathbb{P}^1$;
- $h_0 = \phi$ and $h_1 = \psi$;
- $h_t(z) = \phi(z) = \psi(z), \quad \forall z \in \overline{P}_f, \ \ 0 \leqslant t \leqslant 1.$

2 Critical finiteness and geometrical finiteness

A branched covering f is said to be *critically finite* if P_f is finite. It is said to be *geometrically finite* if P_f itself is infinite but P'_f is finite.

For a positive integer n, we use f^n to mean the n^{th} iteration $\underbrace{f \circ \cdots \circ f}_{n}$. As usual, f^0 means the identity map. A point p in $\mathbb{P}^1$ is called a periodic point of period $k \geqslant 1$ if

$$f^k(p) = p \quad \text{and} \quad f^i(p) \neq p, \quad 0 < i < k.$$

A point is called eventually periodic if there is an integer $l > 0$ such that $f^l(p)$ is periodic.

If f is critically finite, then every point in P_f is eventually periodic since $f(P_f) \subseteq P_f$. For a geometrically finite branched covering, every point in P'_f is periodic (this is an easy exercise, we leave it to the reader).

3 Thurston's theorem

Suppose f is a critically finite branched covering. Then $\overline{P}_f = P_f$. Suppose γ is a simple closed curve on $\mathbb{P}^1 \setminus P_f$. Then the set $f^{-1}(\gamma)$ is a union of disjoint simple closed curves on $\mathbb{P}^1 \setminus f^{-1}(P_f)$. If γ moves continuously, so does each component of $f^{-1}(\gamma)$. A simple closed curve γ is non-peripheral if each component of $\mathbb{P}^1 \setminus \gamma$ contains at least two points of P_f. Consider a multi-curve

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\}$$

of simple, closed, disjoint, non-homotopic, and non-peripheral curves on $\mathbb{P}^1 \setminus P_f$. It is said to be f-stable if for any $\gamma \in \Gamma$, all the non-peripheral components of $f^{-1}(\gamma)$ are homotopic in $\mathbb{P}^1 \setminus P_f$ to elements of Γ.

For each f-stable multi-curve Γ, define a linear transformation,

$$f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma,$$

as follows: Let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{P}^1 \setminus P_f$ and let $d_{i,j,\alpha}$ be the degree of $f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j$. Then

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} \gamma_i.$$

Since the transition matrix $A(\Gamma)$ of f_Γ is non-negative, there exist a maximal non-negative eigenvalue $\lambda(\Gamma, f)$ and a corresponding non-negative eigenvector. It is easy to check that

$$(f^n)_\Gamma = (f_\Gamma)^n.$$

An f-stable multi-curve Γ is called a *Thurston obstruction* if $\lambda(\Gamma, f) \geqslant 1$.

Let $v_f : \mathbb{P}^1 \to Z^+ \cup \{\infty\}$ be the minimal positive function such that

- $v_f(x) = 1$ for all $x \notin P_f$.
- $v_f(x)$ is the integer multiple of $v_f(y) \cdot deg_y f$ for all $y \in f^{-1}(x)$.

Then $\mathcal{O}_f = (\mathbb{P}^1, v_f)$ is called an orbifold. It is called hyperbolic if its Euler characteristic

$$\chi(\mathcal{O}_f) = 2 - \sum_{x \in \mathbb{P}^1} (1 - \frac{1}{v_f(x)}) < 0.$$

Theorem 1 (Thurston's Theorem). *A critically finite orientation-preserving branched covering f with hyperbolic $\mathcal{O}_f$ is combinatorially equivalent to a unique rational map (up to conjugations by automorphisms of the Riemann sphere) if and only if f has no Thurston obstruction.*

In 1982 Thurston gave a necessary and sufficient combinatorial condition for a critically finite rational maps among all critically finite branched coverings. A complete proof of the above theorem was written by Douady and Hubbard in their paper [8].

4 A local theory

The topological characterization in Thurston's theorem gives a global property of dynamics of a critically finite rational map. However, there are other local topological characterizations for many other rational maps such as Kœnig's theorem, Böttcher's theorem, and the Fatou linearization theorem.

Kœnig's theorem says that a holomorphic germ $g(z)$ at 0 fixing 0 and having $\lambda = g'(0)$ with $0 < |\lambda| < 1$ or $|\lambda| > 1$ can be written as $z \to \lambda z$ under an appropriate conformal coordinate. Böttcher's theorem says that a holomorphic germ $g(z)$ at 0 fixing 0 and having $g'(0) = 0$ can be written as $z \to z^n$ for an integer $n \geqslant 2$ under an appropriate conformal coordinate. The Leau-Fatou flower theorem says that a holomorphic germ $g(z)$ at 0 fixing 0 such that $\lambda = g'(0)$ is a q-th root of unity but $g^q \neq id$ has a flower structure locally. That is, g has kq pairwise attracting petals at 0 and the same number of pairwise repelling petals at 0. The union of these petals forms a neighborhood of 0. The Fatou linearization theorem says that the restriction of g^q to any attracting petal can be written as $z \to z + 1$ under an appropriate conformal coordinate. The reader may refer to Milnor's book [21] and Carlson and Gamlin's book [2] for these theorems. He may also refer to [14, 15, 16] for some new proofs of these theorems and their generalizations from the holomorphic motions point of view.

Therefore, we have to first understanding some local topological characterization in order to have the Thurston type theorem for a general branched covering. Cui, Jiang, and Sullivan studied a local theory in the paper [3] first circulated in 1994. The paper then divided into two parts and published in a workshop proceedings as [4, 5]. The paper [4] more concerns about the setting of an appropriate way to study a Thurston type theorem for geometrically finite rational map and development of a local theory. An improved version of [4] is presented as [6]. We outline this local theory in this section. The reader who is interested in this local theory may refer to [6] for more details.

Suppose that f is a geometrically finite branched covering of $\mathbb{P}^1$. Then P_f' consists of finitely many periodic cycles. Suppose $z_0 \in P_f'$ has the period $k \geqslant 1$.

Definition 1. *We say f is locally combinatorially attracting at z_0 if there exists a combinatorial equivalence (ϕ, ψ) from f to a branched covering g of $\mathbb{P}^1$ such that*

i) *g^k is holomorphic at a neighborhood of $w_0 = \phi(z_0)$ and*
ii) *w_0 is an attractive fixed point of g^k, that is, $0 < |(g^k)'(w_0)| < 1$, or w_0 is a super-attractive fixed point of g^k, that is, $(g^k)'(w_0) = 0$.*

The locally combinatorially attracting property at the point z_0, which is not a branched point of f^k, can be described by a *combinatorially invariant shrinking family of curves nested at z_0* as we define now.

Suppose $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ is a family of disjoint pairwise simple closed curves in $\mathbb{P}^1 \setminus \overline{P}_f$. We call Γ a combinatorially invariant family of curves nested at z_0 if for all $n \geqslant 0$, γ_{n+1} separates γ_n from z_0 and there is a component of $f^{-k}(\gamma_{n+1})$ homotopic to γ_n rel $\overline{P}_f$. A combinatorially invariant family of curves Γ nested at $z_0 \in P'_f$ is called shrinking if for every neighborhood U about z_0, there is a simple closed curve $\beta \subset U \setminus \overline{P}_f$ such that β is homotopic to some $\gamma_n \in \Gamma$ rel $\overline{P}_f$.

Lemma 1. *Suppose* $\deg_{z_0} f^k = 1$. *Then* f *is locally combinatorially attracting at* z_0 *if and only if there is a combinatorially invariant shrinking family of curves nested at* z_0.

From Kœnig's theorem, we have the following

Lemma 2. *Suppose* $k = 1$ *and* $\deg_{z_0} f = 1$. *If* f *is locally combinatorially attracting at* z_0, *then there is a neighborhood* V *of* z_0 *such that any combinatorially invariant family of curves nested at* z_0 *in* V *is shrinking.*

The number of combinatorially invariant shrinking families of curves is essentially determined by the number of critical orbits approaching z_0.

Lemma 3. *Suppose* $k = 1$ *and* $deg_a f = 1$. *If there is only one branched point of* f *whose forward orbit converges to* z_0, *then there is at most one combinatorially invariant shrinking family of curves nested at* z_0 *up to isotopy rel* $\overline{P}_f$. *This means that if* $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ *and* $\Gamma' = \{\beta_n\}_{n=0}^{\infty}$ *are combinatorially invariant shrinking families of curves nested at* z_0, *then there exist integers* l *and* $m \geqslant 1$ *such that* γ_{n+l} *is isotopic to* β_n *rel* $\overline{P}_f$ *for all* $n \geqslant m$.

Using the above two lemmas, we constructed a counter-example in [4].

Theorem 2. *There exists a geometrically finite branched covering* f *which has no Thurston obstruction and which is not combinatorially equivalent to a rational map.*

Let me give an outline for our construction. Take $g(z) = \lambda z + z^2$ with $0 < |\lambda| < 1$. Then $c = -\lambda/2$ is the unique branched point in the complex plane. Let $P = \{c_n = g^n(c)\}_{n=1}^{\infty}$ be the post-critical orbit of c. Then $c_n \to 0$ as $n \to \infty$.

There is a combinatorially invariant shrinking family of curves $\Gamma = \{\alpha_n\}_{n=0}^{\infty}$ nested at 0 such that for U_n, the domain bounded by α_n, we have that $q|U_0$ is univalent, $U_0 \supset P$, and $U_n = q^n(U_0)$. There is an arc β in U_0 which connects c_1 with 0 and passes over every point c_n such that $q(\beta) \subset \beta$ and β intersects with each α_n $(n > 0)$ at only one point.

Denote by A the annulus bounded by α_0 and α_2. Take δ to be a Jordan curve in A such that δ intersects with β at only one point and δ bounds a disk D in A containing c_1 and c_2.

Let T_0 be the Dehn twist along δ. Here T_0 can be picked as a homeomorphism which is the identity out of an annulus neighborhood $C(\delta)$ of δ in $A - \{c_1, c_2\}$ and which is $h^{-1} \circ R \circ h$ on $C(\delta)$, where h is a conformal map from $C(\delta)$ to a standard annulus $C = \{z | 0 < r < |z| < 1\}$ and R is a homeomorphism from C to itself defined as

$$R(\rho e^{i\theta}) = \rho e^{i(\theta + 2\pi \frac{\rho - r}{1 - r})}, \quad r < \rho < 1.$$

Define $T_{2n} = q^{2n} T_0 q^{-2n}$ on $q^{2n}(C(\delta))$ and the identity elsewhere. Let

$$f_n = q \circ T_0^2 \circ T_2^2 \circ \cdots \circ T_{2n}^2.$$

Then $\{f_n\}$ uniformly converges to a branched covering f of $\mathbb{P}^1$ as n goes to ∞. In particular, $P_f = P \cup \{\infty\}$ and $P_f' = \{0\}$.

Let $\gamma_n^n = \alpha_{2n}$ and $\gamma_{n+k}^n = f^k(\gamma_n^n) \subset U_{2n}$ for $k > 0$. Since $\gamma_{n+1}^n = f(\gamma_n^n) = \alpha_{2n+1}$, we see that $\Gamma^n = \{\gamma_{n+k}^n\}_{k=1}^{\infty}$ is a combinatorially invariant family of curves nested at 0 for all $n \geqslant 0$.

Since $f(\gamma_{n+1}^n)$ is not isotopic to γ_{n+1}^{n+1} rel $\overline{P}_f$, each curve in Γ^n is not isotopic to any curve in Γ^{n+1} rel $\overline{P}_f$. If Γ^m is shrinking for some m, so is Γ^{m+1} since γ_{m+1}^{m+1} separates γ_m^m from 0. Thus Γ^n is not shrinking for all $n \geqslant 0$ by Lemma 3. Because γ_n^n converges to 0, f has no combinatorially invariant shrinking family of curves nested at 0 and hence is not locally combinatorially attracting at 0 by Lemma 1 and Lemma 2.

The branched covering f has no Thurston obstruction. This is because $f_0^{-2n}(U_{2n}) \supset P$ for all $n \geqslant 0$ by the induction, where f_0^{-2n} denotes the branch of f^{-2n} at 0. Take any f-stable multi-curve Γ. Let $A(\Gamma)$ be the transition matrix of f_Γ. For any element $\gamma \in \Gamma$, since γ_n^n converges to zero as n goes to ∞, there is an integer $m > 0$ such that γ is outside the closure of U_{2m}. Since $f_0^{-2m}(U_{2m}) \supset P$ is a Jordan domain, every component of $f^{-2m}(\gamma)$ is peripheral. This shows that $A(\Gamma)^{2m}$ is the zero matrix. Hence its spectral radius is 0. This implies that the spectral radius of $A(\Gamma)$ is less than one. So Γ is not a Thurston obstruction.

This example shows that not only the Thurston obstruction condition has to be considered but also the locally combinatorially attracting condition at any point in P_f' has to be considered in order to prove the Thurston type theorem for geometrically finite branched coverings.

5 Sub-hyperbolic semi-rational branched coverings

Definition 2. *A geometrically finite branched covering f is said to be semi-rational if*

- *f is holomorphic in a neighborhood of P_f';*
- *each cycle in P_f' is either attractive or super-attractive or parabolic; and*
- *each attracting petal associated with a parabolic cycle in P_f' contains a post-critical point from P_f.*

Clearly, every geometrically finite rational map is a semi-rational branched covering. Furthermore, we have the following

Definition 3. *A semi-rational branched covering f is said to be sub-hyperbolic if P_f' contains only attractive or super-attractive periodic cycles.*

From the definitions, one can check that a geometrically finite branched covering f is locally combinatorially attracting at every point of P_f' if and only if it is combinatorially equivalent to a sub-hyperbolic semi-rational branched covering. More interestingly, from the local theory in the previous section, we proved in [6] (see also [4]) the following

Theorem 3. *A semi-rational branched covering f is always locally combinatorially attracting at every point of P'_f.*

Following this theorem, we have

Corollary 1. *A semi-rational branched covering f is always combinatorially equivalent to a sub-hyperbolic semi-rational branched covering.*

Thus, to prove the Thurston type theorem for geometrically finite branched coverings, the correct class is the sub-hyperbolic semi-rational branched coverings.

6 CJS' theorem for sub-hyperbolic semi-rational branched coverings

Suppose f is a sub-hyperbolic semi-rational branched covering of $\mathbb{P}^1$. For each point $z \in P'_f$, let $k \geqslant 1$ be the period of f at z. There is an open round disk $D(z)$ centered z such that

$$f^k(w) = z + \lambda(w-z) + h.o.t, \quad w \in D(z)$$

for some λ with $0 < |\lambda| < 1$ or

$$f^k(w) = z + \alpha(w-z)^n + h.o.t,$$

for some $n \geqslant 2$ and $\alpha \neq 0$, where h.o.t. means higher order terms.

We defined the *CLH-equivalence* for sub-hyperbolic semi-rational geometrically finite branched coverings in [6] (see also [4]). Here CLH stands combinatorial(ly) but locally holomorphical(ly).

Definition 4. *Two sub-hyperbolic semi-rational branched coverings f and g are said to be CLH-equivalent if there is a pair of homeomorphisms $(\phi, \psi) : \mathbb{P}^1 \to \mathbb{P}^1$ such that*

- *ψ is isotopic to ϕ rel $\overline{P}_f$,*
- *$\phi f = g\psi$,*
- *$\phi|U_f = \psi|U_f$ is holomorphic on some open set $U_f \supset P'_f$.*

Suppose f is a sub-hyperbolic semi-rational branched covering. Suppose $P'_f = \{c_i\}$. It contains finitely many periodic cycles. For each c_i, let $D_i \subset U_f$ be a disk such that $c_i \in D_i$. We say that an open annulus A is attached to an open disk D if A and D are disjoint but the boundary circle ∂D of the disk is one of boundary components of the annulus A. Then $\overline{D \cup A}$ is a larger closed disk. The following lemma is proved in [26] (see also [17]).

Lemma 4 (Shielding Ring Lemma). *There is a collection $\{D_i\}$ of open (topological) disks and a collection of open (topological) annuli $\{A_i\}$ such that*

1. *$c_i \in D_i$,*
2. *$\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$,*
3. *$\overline{D_i} \setminus \{c_i\}$ contains no branched point and no critical value,*

4. *for each i, A_i is attached to D_i such that $\overline{A_i}$ contains no point from the set P_f of post-critical orbits,*
5. *f on $\overline{D_i} \cup A_i$ is holomorphic,*
6. *the image $f(A_i)$ of every annulus A_i is contained in some D_j.*

We call a disk D_i and an annulus A_i in the above lemma a holomorphic disk and a shielding ring. Let

$$P_1 = P_f \setminus (\cup_i \overline{D}_i).$$

Then P_1 consists of finitely many points. Choose D_i and A_i small such that the closed disks $\{\overline{D}_i \cup \overline{A}_i\}$ are pairwise disjoint and such that P_1 contains at least three points. Let

$$Q_f = P_1 \cup \cup_i \overline{D}_i.$$

Consider the surface

$$S = \mathbb{P}^1 \setminus Q_f.$$

A simple closed curve γ is non-peripheral if each component of $\mathbb{P}^1 \setminus \gamma$ contains at least two points of P_1 or at least one holomorphic disk from $\{D_i\}$. Consider a multi-curve

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\}$$

of simple, closed, disjoint, non-homotopic, and non-peripheral curves on S. It is said to be f-stable if for any $\gamma \in \Gamma$, all the non-peripheral components of $f^{-1}(\gamma)$ are homotopic in S to elements of Γ.

For each f-stable multi-curve Γ, define a linear transformation,

$$f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma,$$

as follows: Let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in S and $d_{i,j,\alpha} = deg f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j$. Then

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} \gamma_i.$$

Since the transition matrix $A(\Gamma)$ of f_Γ is non-negative, there exist a maximal non-negative eigenvalue $\lambda(\Gamma, f)$ and a correspondent non-negative eigenvector. It is easy to check that

$$(f^n)_\Gamma = (f_\Gamma)^n.$$

An f-stable multi-curve Γ is called a *Thurston obstruction* if $\lambda(\Gamma, f) \geqslant 1$.

Theorem 4 (CJS' Theorem). *Suppose f is a sub-hyperbolic semi-rational branched covering. Then f is CLH-equivalent to a unique rational map (up to conjugations by automorphisms of the Riemann sphere) if and only if f has no Thurston obstruction.*

This theorem and the local theory in §4 are first studied in the paper [3] circulated in 1994. The paper then divided into two parts and published in a workshop proceedings as [4, 5]. The paper [4] more concerns about the setting of an appropriate way to study a Thurston type theorem for geometrically finite rational map and development of a local theory. An improved version of [4] is presented as [6]. The paper [5] more concerns about a proof of Theorem 4. An improved version of [5] is presented as [7]. The proof given in [5] as well as [7] is quite involved-a combinatorially complex and expositionally formidable surgery argument is used to reduce the problem to that of Thurston's original postcritically finite setup (see Theorem 1), together with checking that certain gluing data are analytically realizable. In a sequel paper [26] (see also [17]) to [4, 5], a new and simpler proof is given by completely giving up the idea in the paper [5]. This new proof adapted some arguments used in the proof in Douady and Hubbard's paper [8] directly and combined with an idea of constructing a shielding ring (see Lemma 4) around every accumulation point in the limiting set of the post-critical set. Then we proved that the bounded geometry is held due to these shielding rings. This new proof is easier and checkable and also gives some new interpretation of the proof of Thurston's Theorem in Douady-Hubbard's paper [8] (see §8 for an outline). Thus a complete and understandable proof of Theorem 4 is available in [26]. We give an outline of this proof in §9. The reader who is interested in this proof can go to [26] for details.

7 CJS' theorem for semi-rational branched covering

Combining Corollary 1 and Theorem 4, we eventually obtained a Thurston type theorem for geometrically finite rational maps among all semi-rational branched coverings of the two sphere in [6] as follows.

Theorem 5 (CJS' Theorem). *Suppose f is a semi-rational branched covering of the two sphere. Then f is combinatorially equivalent to a rational map R if and only if f has no Thurston obstructions. However, the rational map R is not unique up to conjugation by Möbius transformations of the Riemann sphere.*

8 Bounded geometry and a proof of Thurston's theorem

The proof of the "only if" part of the Thurston's theorem is an application of Jenkins-Strebel's theorem. The most difficult part of the theorem is the proof of the "if" part. Douady and Hubbard wrote a beautiful paper [8] to give a complete proof of Thurston's theorem. The reader may also refer to [23] for canonical Thurston obstructions. In this section, I will outline a proof of the "if" part given in the paper [17] from the bounded geometry point of view. Most important, the idea in the proof can be also used to have a similar proof of the "if" part of CJS' theorem in [26] which I will outline it in the next section.

Suppose f is a critical finite branched covering. Then P_f has only finitely many points. Suppose P_f contains at least four points and $0, 1, \infty \in P_f$.

The Teichmüller space $\mathcal{T}_f = \mathcal{T}(P_f)$ is the Teichmüller space with the base point $\mathbb{P}^1 \setminus P_f$. It can be defined as

1. the space of complex structures on $\mathbb{P}^1$ modulo the equivalence relation that $\mu \sim \nu$ if $\mu = h^*\nu$ for some quasiconformal homeomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ with $h|_{P_f} = id$ and h is isotopic to the identity rel P_f (we use $[\mu]$ or simply μ to mean the equivalence class of μ) or
2. the space of quasiconformal homeomorphisms ϕ of the sphere to itself modulo the equivalence relation $\phi \sim \psi$, where $\phi \sim \psi$ means that there exists an analytic isomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1, P_f) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(P_f)) \\ \downarrow id & & \downarrow h \\ (\mathbb{P}^1, P_f) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(P_f)) \end{array}$$

commutes on P_f and commutes up to isotopy rel P_f (we use $[\phi]$ or simply ϕ to mean the equivalence class of ϕ).

The Teichmüller distance $d_T(\tau, \tau')$ for $\tau, \tau' \in \mathcal{T}_f$ is defined as

$$d_T(\tau, \tau') = \frac{1}{2} \inf_{\mu \in \tau, \mu' \in \tau'} \log K(\phi \circ (\phi')^{-1})$$

where ϕ and ϕ' are quasiconformal homeomorphisms of $\mathbb{P}^1$ whose Beltrami coefficients are μ and μ', respectively, and $K(\phi \circ (\phi')^{-1})$ is the quasiconformal dilatation of $\phi \circ (\phi')^{-1}$. The Teichmüller space $\mathcal{T}_f$ is a finite dimensional contractible complex Banach manifold.

Then f induces a pull-back operator $f^* : \mathcal{T}_f \to \mathcal{T}_f$ by $f^*\tau = [f^*\mu]$ for $\tau = [\mu] \in \mathcal{T}_f$, where μ is a complex structure on $\mathbb{P}^1$. Thurston's theorem is equivalent to the existence and uniqueness of a fixed point of f^* in $\mathcal{T}_f$.

It is known that $(f^*)^2$ contracts the Teichmüller distance in this case (see Proposition 3.3 of [8]). Our purpose is to study the bounded geometry property and to use this property to prove that $(f^*)^2$ is strongly contractive. Our proof in [26] starts from the following two lemmas.

Lemma 5. *Let $X \subset \mathbb{P}^1$ be a finite subset such that $0, 1, \infty \in X$. Let m be the cardinality of X. Let $b > 0$ be a constant. If every simple closed geodesic in $\mathbb{P}^1 \setminus X$ has the hyperbolic length greater than b then the spherical distance between any two distinct points in X has a positive lower bound a which depends only on b and m.*

Let $\mathcal{R}^d$ be the set of all the rational functions with degree $d > 1$. For a sequence $\{g_n\} \subset \mathcal{R}^d$ and $g \in \mathcal{R}^d$, we say $g_n \to g$ if g_n converges to g uniformly in the spherical metric on $\mathbb{P}^1$. Therefore we have a topological space $\mathcal{R}^d$. We use $\#(\cdot)$ to denote the cardinality of a set.

For $g \in \mathcal{R}^d$, let Ω_g be the set of branched points of g. Suppose $a > 0$ is a constant. Let $\mathcal{F}_{d,a}$ be a subset of $\mathcal{R}^d$ such that for any $g \in \mathcal{F}_{d,a}$, there exists a set, say X_g, such that

1. $m = \#(X_g) \geqslant 4$ is fixed,
2. $0, 1, \infty \in X_g$,
3. $\Omega_g \cup \{0, 1, \infty\} \subseteq g^{-1}(X_g)$,
4. $d(x, y) \geqslant a$ for all $x \neq y \in X_g$, where $d(\cdot, \cdot)$ means the spherical distance.

Lemma 6. *For fixed $d, a > 0$, $\mathcal{F}_{d,a}$ is compact.*

We define a subset $\mathcal{T}_{f,b}$ of points of $\mathcal{T}_f$ having bounded geometry determined by $b > 0$. Here $\mathcal{T}_{f,b}$ consists of all points $\mu \in \mathcal{T}_f$ such that the hyperbolic length of every simple closed geodesic in $\mathbb{P}^1 \setminus \phi_\mu(P_f)$ is greater than or equal to b, where ϕ_μ is the corresponding quasiconformal homeomorphism of $\mathbb{P}^1$ fixing $0, 1, \infty$ whose Beltrami coefficient is μ.

Let $T_\mu \mathcal{T}_f$ be the tangent space of $\mathcal{T}_f$ at a point μ and let $T^*_\mu \mathcal{T}_f$ be the co-tangent space of $\mathcal{T}_f$ at μ. Using the above two lemmas, we have in [26] the following

Lemma 7. *At any point*

$$\mu \in \mathcal{T}_{f,b} \cap (f^*)^{-1}(\mathcal{T}_{f,b}) \cap (f^*)^{-2}(\mathcal{T}_{f,b}),$$

the dual operator (or the co-tangent operator)

$$f^2_* : T^*_{(f^*)^2\mu} \mathcal{T}_f \to T^*_\mu \mathcal{T}_f$$

as well as the tangent operator

$$d(f^*)^2 : T_\mu \mathcal{T}_f \to T_{(f^*)^2\mu} \mathcal{T}_f$$

are strictly contracting. More precisely, there is a $0 < \tau < 1$ dependent only on b and f such that $\|d(f^*)^2\| \leqslant \|f^2_*\| \leqslant \tau$.

And, furthermore,

Lemma 8. *For any point*

$$\mu \in \mathcal{T}_{f,b} \cap (f^*)^{-1}(\mathcal{T}_{f,b}) \cap (f^*)^{-2}(\mathcal{T}_{f,b}),$$

the pull-back operator $(f^*)^2$ strictly contracts the Teichmüller distance $d_T(\cdot, \cdot)$ between μ and $f^*\mu$. More precisely, there is a constant $0 < \tau' < 1$, depending only on $b, D > 0$ such that if $d_T(\mu, f^*\mu) \leqslant D$, then

$$d_T((f^*)^2\mu, (f^*)^2(f^*\mu)) \leqslant \tau' d_T(\mu, f^*\mu).$$

For a point $\mu_0 \in \mathcal{T}_f$, let $\mu_n = (f^*)^n \mu_0$ for $n \geqslant 0$. We say that μ_0 satisfies the bounded geometry property if $\{\mu_n\}_{n=0}^\infty \subset \mathcal{T}_{f,b}$ for some constant $b > 0$. The following lemma says that Thurston's theorem follows from the bounded geometry property.

Lemma 9. *Suppose there is a constant $b > 0$ and a point $\mu_0 \in \mathcal{T}_f$ such that $\{\mu_n\}_{n=0}^\infty \subset \mathcal{T}_{f,b}$. Then f^* has a unique fixed point in $\mathcal{T}_f$.*

Finally, we proved that no Thurston obstruction implies bounded geometry. Suppose γ is a simple closed curve on $\mathbb{P}^1 \setminus P_f$. For every $\mu \in \mathcal{T}_f$, let ϕ_μ be the corresponding quasiconformal homeomorphism fixing $0, 1, \infty$ whose Beltrami coefficeint is μ. The μ-norm $\|\gamma\|_\mu$ of γ is defined as the hyperbolic length of the simple closed geodesic homotopic to $\phi_\mu(\gamma)$ in the hyperbolic Riemann surface $\mathbb{P}^1 \setminus \phi_\mu(P_f)$.

Since there are finitely many distinct f-stable multi-curves on $\mathbb{P}^1 \setminus P_f$, there are only finitely many distinct Thurston linear transformations f_Γ. If f has no Thurston obstruction, then from $(f^n)_\Gamma = f_\Gamma^n$ for all integers $n > 0$, we have a universal number k such that

$$\|A(\Gamma)^k\| < 1/2$$

for any f-stable multi-curve Γ. Without loss of generality, we can assume that $\|A(\Gamma)\| < 1/2$ for any f-stable multi-curve Γ; that means,

$$\max_j \sum_i b_{i,j} < \frac{1}{2}$$

where $A(\Gamma) = (b_{ij})$. (The reason is that, if $(f^*)^k$ has a fixed point μ, then, since $(f^*)^2$ contracts the Teichmüller distance, we have $f^*\mu = \mu$. That is, μ is a fixed point of f^*.)

Suppose Γ is an f-stable multi-curve in $\mathbb{P}^1 \setminus P_f$. For any $\mu \in \mathcal{T}_f$, we define the μ-length of Γ as follows.

$$\|\Gamma\|_\mu = \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_\mu}.$$

For any $\mu \in \mathcal{T}_f$, let $\tilde{\mu} = f^*\mu \in \mathcal{T}_f$. Let $A = -\log\log(\sqrt{2}+1)$ be the magic number in the theory of hyperbolic Riemann surfaces.

Lemma 10. *Let $D > 0$ and J be two constants satisfying the condition that $J > \log d + 2D$, where $d > 1$ is the degree of f. Let $p_0 = \#(P_f)$ and $p_1 = \#(f^{-1}(P_f)) - p_0 > 0$. Then there is a constant $M > 0$ depending only on D, J, A, d, p_0, p_1 such that for any maximal f-stable multi-curve Γ and any $\mu \in \mathcal{T}_f$ satisfying $d_T(\mu, \tilde{\mu}) < D$,*

$$||\Gamma||_{\tilde{\mu}} \leqslant \frac{1}{2}||\Gamma||_\mu + M.$$

From the above lemma, the following lemma can be proven from the bounded geometry property in [26] for any $\mu_0 \in \mathcal{T}_f$.

Lemma 11. *For any $\mu_0 \in \mathcal{T}_f$, there is a constant $b > 0$ such that*

$$\{\mu_n = (f^*)^n \mu_0\}_{n=0}^{\infty} \subset \mathcal{T}_{f,b}.$$

Lemmas 9 and 11 complete the proof of Thurston's theorem.

9 Bounded geometry and a proof of CJS' theorem

The proof of the "only if" part is an application of Jenkins-Strebel's theorem, and the reader may refer to McMullen's book [20, Appendix B]. The most difficult part is to prove the "if" part of this theorem. We outline a proof given in [26] (see also [17]) by using the bounded geometry property and Teichmüller theory as we did for the proof of Thurston's Theorem in §8.

Suppose f is a sub-hyperbolic semi-rational branched covering. Since $Q_f = P_1 \cup \overline{D}_f$ is a closed subset of $\mathbb{P}^1$, we have the Teichmüller space $\mathcal{T}(Q_f)$ of Q_f. It can be defined as

1. the space of complex structures on $\mathbb{P}^1$ modulo the equivalence relation that $\mu \sim \nu$ if $\mu = h^*\nu$ for some quasiconformal homeomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ with $h|_{Q_f} = id$ and h is isotopic to the identity rel Q_f (we use $[\mu]_{Q_f}$ or simply μ to mean the equivalence class of μ) or
2. the space of quasiconformal homeomorphisms ϕ of the sphere to itself modulo the equivalence relation $\phi \sim \psi$, where $\phi \sim \psi$ means that there exists an analytic isomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1, Q_f) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(Q_f)) \\ \downarrow id & & \downarrow h \\ (\mathbb{P}^1, Q_f) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(Q_f)) \end{array}$$

commutes on Q_f and commutes up to isotopy rel Q_f (we use $[\phi]$ or simply ϕ to mean the equivalence class of ϕ).

The Teichmüller distance $d_T(\tau, \tau')$ for $\tau, \tau' \in \mathcal{T}_f$ is defined as

$$d_T(\tau, \tau') = \frac{1}{2} \inf_{\mu \in \tau, \mu' \in \tau'} \log K(\phi \circ (\phi')^{-1}),$$

where ϕ and ϕ' are quasiconformal homeomorphisms of $\mathbb{P}^1$ whose Beltrami coefficients are μ and μ', respectively, and $K(\phi \circ (\phi')^{-1})$ is the quasiconformal dilatation of $\phi \circ (\phi')^{-1}$.

It is known that $\mathcal{T}(Q_f)$ is a contractible complex Banach manifold and

$$\mathcal{T}(Q_f) = \mathcal{T}(\mathbb{P}^1 \setminus Q_f) \times \mathcal{M}(Q_f),$$

where $\mathcal{T}(\mathbb{P}^1 \setminus Q_f)$ is the classical Teichmüller space of the hyperbolic Riemann surface $\mathbb{P}^1 \setminus Q_f$ with the ideal boundary $\partial Q_f = P_1 \cup \partial D_f$, and $\mathcal{M}(Q_f)$ is the set of all complex structures on Q_f. The reader may refer to [9, 11, GL, 19, 22] for a theory of the Teichmüller space $\mathcal{T}(Q_f)$ of a closed subset Q_f of the Riemann sphere $\mathbb{P}^1$. Most interesting, the Teichmüller space $\mathcal{T}(Q_f)$ has been studied from the holomorphic motions point of view in [22, 12, 18].

We take

$$\mathcal{T}_f = \mathcal{T}_0(Q_f) = \mathcal{T}(\mathbb{P}^1 \setminus Q_f) \times \{0\}$$

as the Teichmüller space associated to f in this case. It is the space of all equivalence classes $\tau = [\mu]_{Q_f} \in \mathcal{T}(Q_f)$ such that $\mu = 0$ on Q_f. For each $\tau = [\mu]_{Q_f} \in \mathcal{T}_f$, let ϕ_μ be the unique quasiconformal homeomorphism of $\mathbb{P}^1$ fixing 0, 1, ∞ whose Beltrami coefficient is μ. Then $\phi_\mu|Q_f$ is conformal. We call ϕ_μ the corresponding quasiconformal homeomorphism for μ. The Teichmüller space $\mathcal{T}_f$ is a finite dimensional contractible complex Banach space. As in the critically finite case, the map f induces an operator f^* from $\mathcal{T}_f$ to itself.

Lemma 12. *The map f is CLH-equivalent to a unique rational map (up to conjugations by automorphisms of $\mathbb{P}^1$) if and only if f^* has a unique fixed point.*

The bounded geometry in this case is given by

Definition 5. *Suppose $d(\cdot,\cdot)$ is the spherical distance. Let $b>0$ be a constant. Let $\mathcal{T}_{f,b}$ be the subspace of $\mu \in \mathcal{T}_f$ satisfying the following conditions:*

1. *for all $z_i \neq z_{i'} \in P_1$,*
$$d(\phi_\mu(z_i), \phi_\mu(z_{i'})) \geqslant b;$$
2. *for all $z_j \in P_1$ and all $D_i \in \Lambda_1$,*
$$d(\phi_\mu(z_j), \phi_\mu(D_i)) \geqslant b;$$
3. *for all $D_i \neq D_{i'} \in \Lambda_1$,*
$$d(\phi_\mu(D_i), \phi_\mu(D_{i'})) \geqslant b;$$
4. *every $D_i \in \Lambda_1$, $\phi_\mu(D_i)$ contains a round disk of radius b centered at $\phi_\mu(c_i)$.*

We call $\mathcal{T}_{f,b}$ the subspace having bounded geometry determined by b.

For each holomorphic disk D_i, we fix a point b_i on the boundary ∂D_i. Let
$$E = P_1 \cup \cup_i \{b_i, c_i\}.$$
Assume that $0, 1, \infty \in E$.

Since $\Omega_f \cup Q_f \subset f^{-1}(Q_f)$, we let $X = f^{-1}(Q_f) \setminus Q_f$. Then $\#(X) > 0$. For each $\mu \in \mathcal{T}_f$, let ϕ_μ be the corresponding quasiconformal homeomorphism fixing $0, 1, \infty$ whose Beltrami coefficeint is μ. We consider three hyperbolic Riemann surfaces,
$$S_\mu = \mathbb{P}^1 \setminus \phi_\mu(Q_f),$$
$$S_{\mu,E} = \mathbb{P}^1 \setminus \phi_\mu(E),$$
and
$$S_{\mu,X} = S_\mu \setminus \phi_\mu(X) = \mathbb{P}^1 \setminus \phi_\mu(f^{-1}(Q_f)).$$
It is clear that
$$S_{\mu,X} \subset S_\mu \subset S_{\mu,E}.$$

Suppose γ is a simple closed curve on $\mathbb{P}^1 \setminus Q_f$. The μ-norm $\|\gamma\|_\mu$ of γ is defined as the hyperbolic length of the simple closed geodesic homotopic to $\phi_\mu(\gamma)$ in S_μ. Suppose γ is a simple closed curve on $\mathbb{P}^1 \setminus E$. The (μ, E)-norm $\|\gamma\|_{\mu,E}$ of γ is defined as the hyperbolic length of the simple closed geodesic homotopic to $\phi_\mu(\gamma)$ in $S_{\mu,E}$. Suppose γ is a simple closed curve on $\mathbb{P}^1 \setminus f^{-1}(Q_f)$. The (μ, X)-norm $\|\gamma\|_\mu$ of γ is defined as the hyperbolic length of the simple closed geodesic homotopic to $\phi_\mu(\gamma)$ in $S_{\mu,X}$. If γ is a simple closed curve on $\mathbb{P}^1 \setminus f^{-1}(Q_f)$, then we have the following

$$\|\gamma\|_{\mu,E} \leqslant \|\gamma\|_\mu \leqslant \|\gamma\|_{\mu,X}$$

since the embedding $i : S_{\mu,X} \hookrightarrow S_\mu$ and the embedding $i : S_\mu \hookrightarrow S_{\mu,E}$ are analytic and thus decrease the hyperbolic metrics on $S_{\mu,X} \subset S_\mu \subset S_{\mu,E}$.

Our proof in [26] of CJS' theorem starts from the following three lemmas:

Lemma 13. *For each constant $b > 0$, there is a constant $a > 0$ only depending on b such that for any $\mu \in \mathcal{T}_{f,b}$ and any simple closed curve γ in $\mathbb{P}^1 \setminus Q_f$, the μ-norm $\|\gamma\|_\mu \geqslant a$.*

On the other hand, we have

Lemma 14. *For each constant $a > 0$, there is a constant $b > 0$ depending only on a such that if every simple closed geodesic ξ of $S_{\mu,E}$ has its hyperbolic length $\geqslant a$, then $\mu \in \mathcal{T}_{f,b}$.*

Lemma 15. *Suppose $\gamma \subset S_{\mu,E}$ is a simple closed geodesic. Then $\gamma \subset S_\mu$ provided $\|\gamma\|_{\mu,E}$ is small. More precisely, there is a universal number $C_2 > 0$ small such that $\gamma \subset S_\mu$ whenever $\|\gamma\|_{\mu,E} \leqslant C_2$, and moreover, there is another universal number $C_7 > 0$ such that*

$$\frac{1}{\|\gamma\|_\mu} \leqslant \frac{1}{\|\gamma\|_{\mu,E}} \leqslant \frac{1}{\|\gamma\|_\mu} + C_7.$$

From the above three lemmas and by using the property of the shielding rings $\{A_i\}$, we first proved in [26] that the dual operator (or the co-tangent operator) f_* of f^* and the tangent operator df^* of f^* are strictly contracting. (We note that in the critically finite case, one can only prove that the second iterations $(f_*)^2$ and $(df^*)^2$ are strictly contractive but in the sub-hyperbolic semi-rational case, the following lemma says that f_* and df^* themselves are strictly contractive.)

Lemma 16. *Suppose $b > 0$ is a fixed constant. There is a constant $a > 0$ such that for every $\mu \in (f^*)^{-1}(\mathcal{T}_{f,b})$ and $\tilde{\mu} = f^*\mu$ and for every holomorphic quadratic differential $\tilde{q}$ over $(S_{\tilde{\mu}}, \phi_{\tilde{\mu}}(P_1))$ with $\|\tilde{q}\| = 1$,*

$$\int_{\cup \phi_{\tilde{\mu}}(A_i)} |\tilde{\psi}(w)| dw d\overline{w} \geqslant a.$$

Therefore, the dual operator (or the tangent operator)

$$f_* : T^*_{\tilde{\mu}}\mathcal{T}_f \to T^*_\mu \mathcal{T}_f$$

strictly contracts the Teichmüller norm of every co-tangent vector at $\tilde{\mu}$ and the tangent map

$$df^* : T_\mu \mathcal{T}_f \to T_{\tilde{\mu}}\mathcal{T}_f$$

strictly contracts the Teichmüller norm of every tangent vector at μ.

Furthermore, we proved in [26] that the pull-back operator f^* itself strictly contracts the Teichmüller distance by assuming the bounded geometry property.

Lemma 17. *Suppose $D > 0$ and $b > 0$ are two constants. Then there is a constant $0 < \tau'' < 1$, depending only on D and b such that for any $\mu \in (f^*)^{-1}(\mathcal{T}_{f,b})$ and any $\nu \in \mathcal{T}_f$ with $d_T(\mu, \nu) \leqslant D$,*

$$d_T(f^*\mu, f^*\nu) \leqslant \tau'' d_T(\mu, \nu).$$

For $\mu_0 \in \mathcal{T}_f$, let $\mu_n = (f^*)^n \mu_0$ for $n \geqslant 0$. We say that μ_0 has the bounded geometry property if $\{\mu_n\}_{n=0}^{\infty} \subset \mathcal{T}_{f,b}$ for some $b > 0$. Thus the following lemma says that if we have a μ_0 having the bounded geometry property, then we have a unique fixed point for f^* in $\mathcal{T}_f$.

Lemma 18. *Suppose there is a constant $b > 0$ and a point $\mu_0 \in \mathcal{T}_f$ such that $\{\mu_n\}_{n=0}^{\infty} \subset \mathcal{T}_{f,b}$. Then f^* has a unique fixed point in $\mathcal{T}_f$.*

Furthermore, we proved in [26] that no Thurston obstruction implies the bounded geometry property for any μ_0.

Remember that $S = \mathbb{P}^1 \setminus Q_f$. There can be only finitely many distinct f-stable multi-curves on S and thus finitely many distinct linear transformations f_Γ. Since $(f_\Gamma)^n = f_\Gamma^n$ for all integers $n > 0$, we have a universal number k such that

$$A(\Gamma)^k = (b_{ij})$$

with $\|A^k\| < 1/2$ for any f-stable multi-curve Γ. As in the critically finite case, without loss of generality, we can assume that $\|A(\Gamma)\| < 1/2$ for any f-stable multi-curve Γ, which means,

$$\max_j \sum_i b_{i,j} < \frac{1}{2}.$$

Suppose Γ is an f-stable multi-curve in S. For any $\mu \in \mathcal{T}_f$, we define the μ-length of Γ as follows.

$$\|\Gamma\|_\mu = \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu,E}}.$$

For any $\mu \in \mathcal{T}_f$, let $\tilde{\mu} = f^*\mu \in \mathcal{T}_f$. Let $B > A = -\log\log(\sqrt{2}+1)$ be a number from Lemma 15 such that for any $\|\gamma\|_{\mu,E} \leqslant e^{-B}$,

$$\frac{1}{\|\gamma\|_{\mu,E}} \leqslant \frac{1}{\|\gamma\|_\mu} + \text{a universal constant}.$$

Let $p_0 = \#(E)$ and $p_1 = \#(X)$.

Lemma 19. *Let $D > 0$ and J be two constants satisfying the condition that $J > \log d + 2D$, where d is the degree of f. Then there is a constant $M > 0$ depending only on D, J, B, d, p_0, p_1 such that for any maximal f-stable multi-curve Γ on S and any $\mu \in \mathcal{T}_f$ such that $d_T(\mu, \tilde{\mu}) < D$,*

$$||\Gamma||_{\tilde{\mu}} \leqslant \frac{1}{2}||\Gamma||_\mu + M.$$

From the above lemma, the following lemma can be proven from the bounded geometry property in [26] as follows.

Lemma 20. *For any $\mu_0 \in \mathcal{T}_f$, there is a constant $b > 0$ such that*

$$\{\mu_n = (f^*)^n \mu_0\}_{n=0}^{\infty} \subset \mathcal{T}_{f,b}.$$

Now Lemmas 12, 18, and 20 complete the proof of CJS' theorem.

10 Remarks and further study

One of important problems in complex dynamical systems is to study the rigidity problem for rational maps. A rational map f is said to be rigid if any another rational map g combinatorially equivalent to f must be conjugate to f by an automorphism of the Riemann sphere.

Thurston's theorem implies that any critically finite rational map f with a hyperbolic orbifold $\mathcal{O}_f = (\mathbb{P}^1, v_f)$ is rigid. Here $v_f : \mathbb{P}^1 \to Z^+ \cup \{\infty\}$ is the minimal positive function such that

- $v_f(x) = 1$ for all $x \notin P_f$.
- $v_f(x)$ is the integer multiple of $v_f(y) \cdot deg_y f$ for all $y \in f^{-1}(x)$.

Then $\mathcal{O}_f$ is hyperbolic if its Euler characteristic

$$\chi(\mathcal{O}_f) = 2 - \sum_{x \in \mathbb{P}^1} (1 - \frac{1}{v_f(x)}) < 0.$$

We know that $\chi(\mathcal{O}_f) \leqslant 0$ for any critically finite f and if $\#(P_f) \geqslant 5$, then $\chi(\mathcal{O}_f) < 0$. So a rational map is rigid if $\#(P_f) \geqslant 5$. When $\#(P_f) \leqslant 3$, the Teichmüller space $\mathcal{T}_f$ consists of only one point. So f^* always has a unique fixed point. Therefore, a rational map f with $\#(P_f) \leqslant 3$ is rigid. There is a family of rational maps (the Lattès maps (see [21])) with $\#(P_f) = 4$ such that they are topologically conjugate to each other but never conjugate to each other by automorphisms of the Riemann sphere. So they are not rigid.

Note that from $\#(P_f) \geqslant 2$ and the equation $\chi(\mathcal{O}_f) = 0$, we have that if $\#(P_f) = 2$, v_f takes values (∞, ∞) on P_f; if $\#(P_f) = 3$, v_f takes values $(3, 3, 3)$ or $(2, 2, \infty)$ or $(2, 4, 4)$ or $(2, 3, 6)$ on P_f; if $\#(P_f) = 4$, v_f takes values $(2, 2, 2, 2)$ on P_f.

CJS' theorem implies that a sub-hyperbolic geometrically finite rational map f is rigid modulo the local structures at P'_f. However, the local structures at P'_f are flexible and can even be deformed to a non-sub-hyperbolic geometrically finite rational map.

Both the critically finite rational maps and the sub-hyperbolic geometrically finite rational maps have a finiteness property from the topological point of view. For rational maps with this kind of finiteness property, except for our work I mentioned in this survey, I would like also to mention that Brown (a PhD student of Hubbard) has done some work in this direction for quadratic polynomials in his PhD thesis [1].

Another class of rational maps which may have this kind of finiteness property from the topological point of view is a class of rational maps which have Siegel disks and are critically finite outside the union of the closures of Siegel disks. Note that a rational map can only have finitely many Siegel disks. I used to raise a problem about the study of the Thurston type theorem for a class of branched coverings, each of which has finitely many rotation disks and is critically finite outside the union of the rotation disks. Progress was made in my PhD student Zhang's thesis [24] (see also [25]) for a class of simple Siegel disk type branched coverings. That is, each map in this class has only one rotation disk and outside this rotation disk it is critically finite. Some other conditions which are required for each map in this class are that the boundary of the rotation disk contains at least one branched point and that the rotation number of the map on the rotation disk is of bounded type.

For entire functions have this kind of finiteness property from topological point of view, a Thurston type theorem is studied in Hubbard, Schleicher, and Shishikuras paper [13] for the exponential family.

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