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SOME APPLICATIONS OF UNIVERSAL HOLOMORPHIC MOTIONS

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Abstract

For a closed subset E of the Riemann sphere, its Teichmüller space $T(E)$ is a universal parameter space for holomorphic motions of E over a simply connected complex Banach manifold. In this paper, we study some new applications of this universal property.

1. Introduction

Definition 1.1. Let V be a connected complex manifold with a basepoint x_0 and let E be a subset of the Riemann sphere $\hat{\mathbf{C}}$. A *holomorphic motion of E over V* is a map $\phi : V \times E \to \hat{\mathbf{C}}$ that has the following three properties:
(a) $\phi(x_0, z) = z$ for all z in E,
(b) the map $\phi(x, \cdot) : E \to \hat{\mathbf{C}}$ is injective for each x in V, and
(c) the map $\phi(\cdot, z) : V \to \hat{\mathbf{C}}$ is holomorphic for each z in E.

We say that V is the *parameter space* of the holomorphic motion ϕ.

Definition 1.2. Let V and W be connected complex manifolds with basepoints, and f be a basepoint preserving holomorphic map of W into V. If ϕ is a holomorphic motion of E over V its *pullback* by f is the holomorphic motion

$$f^*(\phi)(x, z) = \phi(f(x), z) \quad \forall (x, z) \in W \times E \tag{1.1}$$

of E over W.

Unless otherwise stated, we will assume that E is a closed subset of $\hat{\mathbf{C}}$ and that $0, 1, \infty \in E$. Associated to each such set E in $\hat{\mathbf{C}}$, there is a contractible complex Banach manifold which we call the Teichmüller space of the closed set E, denoted by $T(E)$. This was first studied by G. Lieb in his doctoral dissertation (see [15]). We can also define a holomorphic motion

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$$\Psi_E : T(E) \times E \to \hat{\mathbf{C}}$$

of the closed set E over the parameter space $T(E)$. In [18] it was shown that $T(E)$ is a universal parameter space for holomorphic motions of the closed set E over a simply connected complex Banach manifold. This universal property has found several interesting applications to the study of holomorphic motions; we refer the reader to the papers [10], [18], [19], and [20]. For another application of holomorphic motions to complex dynamical systems, see [14]. In this paper, we study some more applications of the universal property.

An important topic in the study of holomorphic motions is the question of extensions.

Definition 1.3. If E is a proper subset of $\hat{E}$ and $\phi : V \times E \to \hat{\mathbf{C}}$, $\hat{\phi} : V \times \hat{E} \to \hat{\mathbf{C}}$ are two maps, we say that $\hat{\phi}$ *extends* ϕ if $\hat{\phi}(x,z) = \phi(x,z)$ for all (x,z) in $V \times E$.

If $\phi : V \times E \to \hat{\mathbf{C}}$ is a holomorphic motion, a natural question is whether there exists a holomorphic motion $\hat{\phi} : V \times \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ that extends ϕ. For $V = \Delta$ (the open unit disk), important results were obtained in [2] and in [23]. Later, in his fundamental paper [22], Slodkowski showed that any holomorphic motion of E over Δ can be extended to the whole sphere. Slodkowski's theorem cannot be generalized to higher dimensional parameter spaces. This was shown by Hubbard with a two-dimensional Teichmüller space as a parameter space; we refer to [4] for the details. That example crucially depends on the main theorem in Hubbard's thesis (on the nonexistence of holomorphic sections of universal Teichmüller curve; see [12], and [9]). A detailed discussion is also given in [13]. In his paper [4], Douady discusses maximal holomorphic motions. For some other examples of maximal holomorphic motions, see [7].

Our first application (in §4) is to give an alternative proof of Hubbard's example. The distinctive feature of our example is that it is a simple application of universal holomorphic motions and a theorem of Earle ([6]), and avoids the deep theorem in Hubbard's thesis.

Theorem 1. *Let E be the finite set $\{0, 1, \infty, \zeta_1, \dots, \zeta_n\}$, where $\zeta_i \neq \zeta_j$ for $i \neq j$, and $n \geq 2$. Then, the universal holomorphic motion $\Psi_E : T(E) \times E \to \hat{\mathbf{C}}$ cannot be extended to a holomorphic motion of $\hat{\mathbf{C}}$.*

Remark 1.4. Theorem 8.1 in [10] gives an example of a holomorphic motion of a proper subset E of $\hat{\mathbf{C}}$ over a contractible domain B in $\mathbf{C}^2$ that cannot be extended to a holomorphic motion of $\hat{E}$ (over B) where E is a proper subset of $\hat{E}$. However, the set E in that example is an infinite set. Our example gives a new proof of the fact that even for a finite set, Slodkowski's theorem cannot be extended to higher-dimensional parameter spaces.

Let E_0 be any subset of $\hat{\mathbf{C}}$, not necessarily closed; as usual, we assume that $0, 1, \infty \in E_0$. Let E be the closure of E_0. In [16], it was shown that a holomorphic motion of E_0 over the open unit disk can always be extended to E. Our second application (in §5) generalizes this fact for holomorphic motions over any complex Banach manifold. We prove the following theorem.

THEOREM 2. *Let $\phi : V \times E_0 \to \hat{\mathbf{C}}$ be a holomorphic motion where V is any complex Banach manifold with a basepoint. There exists a holomorphic motion $\hat{\phi} : V \times E \to \hat{\mathbf{C}}$ such that $\hat{\phi}$ extends ϕ.*

In an earlier version of this paper we proved Theorem 2 when V is simply connected. We thank Clifford J. Earle for suggesting that we should extend this theorem to the non-simply connected case.

Our third application (in §7) is to study a holomorphic family of hyperbolic dynamical systems (see §6 for the definition). We show that the corresponding family of Julia sets moves holomorphically over the same simply connected complex Banach manifold. Moreover, if we consider a basepoint in the simply connected complex Banach manifold, the Kobayashi pseudometric between any point and the basepoint controls the quasiconformal distance between the Julia sets corresponding to this point and the basepoint.

Let V be a simply connected complex Banach manifold with a basepoint x_0. Let ρ_V denote the Kobayashi pseudometric on V.

THEOREM 3. *Suppose $R(x, z)$ is a hyperbolic family of holomorphic dynamical systems over V. Then for any $x \in V$, the Julia set J_x is quasiconformally equivalent to J_{x_0} by a quasiconformal map of $\hat{\mathbf{C}}$ whose dilatation does not exceed $\exp(2\rho_V(x, x_0))$. Moreover, J_x depends holomorphically on x over V.*

Remark 1.5. This theorem has its root in [16] and has also been studied by others (see, for example, [4]). The purpose of Theorem 3 in this paper is to consider a family defined over an infinite dimensional parameter space and to show a direct application of universal holomorphic motions. In addition, by using universal holomorphic motions, we give an estimate of the quasiconformal distance in terms of the Kobayashi pseudometric ρ_V on the parameter space.

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2. The Teichmüller space of E

2.1. Definition of $T(E)$. Recall that a homeomorphism of $\hat{\mathbf{C}}$ is called *normalized* if it fixes the points 0, 1, and ∞.

The normalized quasiconformal self-mappings f and g of $\hat{\mathbf{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The *Teichmüller space* $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\hat{\mathbf{C}}$.

The basepoint of $T(E)$ is the E-equivalence class of the identity map.

Let $M(\mathbf{C})$ denote the open unit ball of the complex Banach space $L^\infty(\mathbf{C})$. Each μ in $M(\mathbf{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\hat{\mathbf{C}}$ onto itself. The basepoint of $M(\mathbf{C})$ is the zero function.

We define the quotient map

$$P_E : M(\mathbf{C}) \to T(E)$$

by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ, written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbf{C})$ to the basepoint of $T(E)$.

In his doctoral dissertation ([15]), G. Lieb proved that $T(E)$ is a complex Banach manifold such that the projection map P_E from $M(\mathbf{C})$ to $T(E)$ is a holomorphic split submersion. (This result is also proved in [10].)

The space $T(E)$ is simply connected; for other properties of $T(E)$ see [10] and [18].

2.2. When E is finite. Let E be a finite set. Its complement $E^c = \Omega$ is the Riemann sphere with punctures at the points of E. There is a natural identification of $T(E)$ with the classical Teichmüller space $Teich(\Omega)$ which will be very useful in our paper.

Recall that two quasiconformal mappings f and g with domain Ω belong to the same Teichmüller class if and only if there is a conformal map h of $f(\Omega)$ onto $g(\Omega)$ such that the self-mapping $g^{-1} \circ h \circ f$ of Ω is isotopic to the identity rel the boundary of Ω. (The isotopy condition means that $g^{-1} \circ h \circ f$ extends to a homeomorphism of the closure of Ω onto itself that is isotopic to the identity by an isotopy that fixes the boundary pointwise.) The Teichmüller space $Teich(\Omega)$ is the set of Teichmüller classes of quasiconformal mappings with domain Ω.

The Teichmüller class of f depends only on its Beltrami coefficient, which is a function μ in the open unit ball $M(\Omega)$ of the complex Banach space $L^\infty(\Omega)$. The standard projection Φ of $M(\Omega)$ onto $Teich(\Omega)$ maps μ to the Teichmüller class of any quasiconformal map whose domain is Ω and whose Beltrami coefficient is μ. The basepoints of $M(\Omega)$ and $Teich(\Omega)$ are 0 and $\Phi(0)$ respectively. It is well-known that $Teich(\Omega)$ is a complex manifold and that $\Phi : M(\Omega) \to Teich(\Omega)$ is a holomorphic split submersion. See [11] or [21] for basic results in Teichmüller theory.

We define a map θ from $T(E)$ into $Teich(\Omega)$ by setting $\theta(P_E(\mu))$ equal to the Teichmüller class of the restriction of w^μ to Ω. The map θ can be easily shown to be biholomorphic; see Example 3.1 in [18] for the details. This gives a canonical identification of $T(E)$ with the classical Teichmüller space $Teich(\Omega)$.

3. Universal holomorphic motion of E

The *universal holomorphic motion* Ψ_E of E over $T(E)$ is defined as follows:

$$\Psi_E(P_E(\mu), z) = w^\mu(z) \quad \text{for } \mu \in M(\mathbf{C}) \text{ and } z \in E.$$

The definition of P_E in §2.1 implies that Ψ_E is well-defined. It is a holomorphic motion since P_E is a holomorphic split submersion and $\mu \mapsto w^\mu(z)$ is a holomorphic map from $M(\mathbf{C})$ to $\hat{\mathbf{C}}$ for every fixed z in $\hat{\mathbf{C}}$ (by Theorem 11 in [1]). This holomorphic motion is "universal" in the following sense:

THEOREM 3.1. *Let $\phi : V \times E \to \hat{\mathbf{C}}$ be a holomorphic motion. If V is simply connected, then there exists a unique basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

For a proof see Section 14 in [18].

The following result is an easy consequence of Theorem 3.1; see Section 17 in [18] for a proof.

THEOREM 3.2. *Let $\phi : V \times E \to \hat{\mathbf{C}}$ be a holomorphic motion, where V is a simply connected complex Banach manifold with basepoint x_0. Then, for every $x \in V$, $\phi(x, \cdot)$ is the restriction to E of a quasiconformal self map of $\hat{\mathbf{C}}$ with dilatation not exceeding* $\exp(2\rho_V(x_0, x))$, *where ρ_V is the Kobayashi pseudometric on V.*

(We are assuming that the hyperbolic metric has constant curvature -4.)

Remark 3.3. We consider the special case when $E = \hat{\mathbf{C}}$. The quotient map $P_{\hat{\mathbf{C}}} : M(\mathbf{C}) \to T(\hat{\mathbf{C}})$ is bijective, and so we use it to identify $T(\hat{\mathbf{C}})$ biholomorphically with $M(\mathbf{C})$. We have the universal holomorphic motion $\Psi_{\hat{\mathbf{C}}} : M(\mathbf{C}) \times \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ as follows:

$$\Psi_{\hat{\mathbf{C}}}(\mu, z) = w^\mu(z)$$

for all z in $\hat{\mathbf{C}}$.

4. Proof of theorem 1

Suppose E_1, E_2 are closed subsets of $\hat{\mathbf{C}}$ such that $E_1 \subset E_2$ and 0, 1, and ∞ are in E_1. If μ is in $M(\mathbf{C})$, then the E_2-equivalence class of w^μ is contained in the E_1-equivalence class of w^μ. Therefore, there is a well-defined 'forgetful map' p_{E_2, E_1} from $T(E_2)$ to $T(E_1)$ such that $P_{E_1} = p_{E_2, E_1} \circ P_{E_2}$. It is easy to see that this forgetful map is a basepoint preserving holomorphic split submersion. We have the universal holomorphic motions $\Psi_{E_1} : T(E_1) \times E_1 \to \hat{\mathbf{C}}$ and $\Psi_{E_2} : T(E_2) \times E_2 \to \hat{\mathbf{C}}$. The following lemma and its easy corollary will be very useful in our paper. They are proved in Section 13 in [18]. For the reader's convenience, we give the precise statements.

LEMMA 4.1. *Let V be a connected complex Banach manifold with basepoint x_0 and let F and G be basepoint preserving holomorphic maps from V into $T(E_1)$*

and $T(E_2)$, *respectively. Then* $p_{E_2,E_1} \circ G = F$ *if and only if* $G^*(\Psi_{E_2})$ *extends* $F^*(\Psi_{E_1})$.

Corollary 4.2. *Let* V *be as above and let* F *and* G *be basepoint preserving holomorphic maps from* V *to* $T(E)$ *and* $M(\mathbf{C})$, *respectively. Then* $P_E \circ G = F$ *if and only if* $G^*(\Psi_{\hat{\mathbf{C}}})$ *extends* $F^*(\Psi_E)$.

Theorem 1 in our paper is a remarkably simple application of the above corollary. Let $E = \{0, 1, \infty, \zeta_1, \dots, \zeta_n\}$, where $\zeta_i \neq \zeta_j$ for $i \neq j$ and n is a positive integer ≥ 2. Consider the universal holomorphic motion

$$\Psi_E : T(E) \times E \to \hat{\mathbf{C}}.$$

By our discussion in §2.2, $T(E)$ and the classical Teichmüller space $Teich(\hat{\mathbf{C}} \backslash E)$ are canonically identified. Consider the identity map $I : T(E) \to T(E)$ (which is obviously a basepoint preserving holomorphic map). Let $\phi : T(E) \times E \to \hat{\mathbf{C}}$ be the holomorphic motion $I^*(\Psi_E)$ (which is the same as Ψ_E). Suppose ϕ extends to a holomorphic motion $\hat{\phi} : T(E) \times \hat{\mathbf{C}} \to \hat{\mathbf{C}}$. Then, by Theorem 3.1, and Remark 3.3, there exists a basepoint preserving holomorphic map $F : T(E) \to M(\mathbf{C})$ such that $F^*(\Psi_{\hat{\mathbf{C}}}) = \hat{\phi}$. Since $\hat{\phi}$ extends ϕ, it follows by Corollary 4.2 that $P_E \circ F = I$. That means, the map P_E has a (global) holomorphic section F. Since $T(E)$ and $Teich(\hat{\mathbf{C}} \backslash E)$ are naturally identified, this is impossible by Earle's theorem in [6]. □

5. Proof of theorem 2

Our method uses a construction that was central in proving Theorem 3.1 (see [18]). To make our paper self-contained, we give the outline of that construction.

In this section E_0 is a subset of $\hat{\mathbf{C}}$ (not necessarily closed) such that 0, 1, ∞ belong to E_0. Let E denote the closure of E_0.

We begin by noting the following fact, which is Lemma 14.1 in [18].

Lemma 5.1. *If* ϕ *is a holomorphic motion of* E_0 *over* V, *then, for each* x *in* V, *the map* $\phi(x, \cdot)$ *from* E_0 *into* $\hat{\mathbf{C}}$ *is continuous.*

Let $\{E_n\}$ be an increasing sequence of finite subsets of $\hat{\mathbf{C}}$ such that 0, 1, and ∞ belong to each E_n and $\bigcup_n E_n$ is dense in E_0. For each $n \geq 1$, let $S_n = \hat{\mathbf{C}} \backslash E_n$. Recall from §2.2, that $T(E_n)$ are $Teich(S_n)$ are naturally identified. Let 0_n denote the basepoint of $Teich(S_n)$, and let d_n be the Teichmüller metric on $Teich(S_n)$ (see [11] or [21] for the definition of Teichmüller metric).

Let $S = \coprod_n S_n$ be the disjoint union of the S_n. The *product Teichmüller space* $Teich(S)$ is the set of all sequences $t = \{t_n\}$ such that $t_n \in Teich(S_n)$ for each n and

$$\sup\{d_n(0_n, t_n) : n \geq 1\} < \infty.$$

The basepoint of $Teich(S)$ is the sequence $0 = \{0_n\}$ whose nth term is the basepoint of $Teich(S_n)$. We know that $Teich(S)$ is a complex Banach manifold. The following fact will be crucial in our discussion (see Corollary 7.6 in [10] or Corollary 5.5 in [18]).

LEMMA 5.2. *Let V be a connected complex Banach manifold and, for each $n \geq 1$, let f_n be a holomorphic map of V into $Teich(S_n)$. For each v in V, let $f(v)$ be the sequence $\{f_n(v)\}$. If $f(v_0)$ belongs to $Teich(S)$ for some v_0 in V, then $f(v)$ also belongs to $Teich(S)$ for all v in V, and the map $v \mapsto f(v)$ from V to $Teich(S)$ is holomorphic.*

For other properties of $Teich(S)$ the interested reader is referred to [8], [10], and [18].

For each $n \geq 1$, let π_n be the forgetful map p_{E,E_n} from $T(E)$ to $Teich(S_n)$ and let p_n be the forgetful map p_{E_{n+1},E_n} from $Teich(S_{n+1})$ to $Teich(S_n)$. (The map p_n is the usual puncture-forgetting map in classical Teichmüller theory; see [21]).

It is clear that

$$\pi_n = p_n \circ \pi_{n+1} \quad \text{for all } n \geq 1. \tag{5.1}$$

Each forgetful map π_n preserves basepoints. Therefore, by Lemma 5.2, the sequence $\{\pi_n(\tau)\}$ belongs to $Teich(S)$ for each τ in $T(E)$ and the map $\pi : T(E) \to Teich(S)$ defined by setting

$$\pi(\tau) = (\pi_1(\tau), \ldots, \pi_n(\tau), \ldots) \quad \text{for all } \tau \in T(E)$$

is holomorphic. Equation (5.1) implies that π maps $T(E)$ into the closed subset

$$T' = \{x = (x_1, x_2, \ldots) \in Teich(S) : p_n(x_{n+1}) = x_n \text{ for all } n \geq 1\}$$

of $Teich(S)$. It is proved in §8 in [18] that π maps $T(E)$ homeomorphically onto T'.

We are now ready to prove Theorem 2.

STEP 1. Let $\phi : V \times E_0 \to \hat{\mathbf{C}}$ be a holomorphic motion, where V is a simply connected complex Banach manifold with a basepoint. For each $n \geq 1$, the restriction ϕ_n of ϕ to $V \times E_n$ is a holomorphic motion of the finite set E_n. By Theorem 3.1, there exists a basepoint preserving holomorphic map $f_n : V \to T(E_n)$ such that $f_n^*(\Psi_{E_n}) = \phi_n$. Recall that each $T(E_n)$ can be identified with the classical Teichmüller space $Teich(S_n)$. By Lemma 5.2, the formula

$$\hat{F}(x) = (f_1(x), \ldots, f_n(x), \ldots), \quad x \in V,$$

defines a basepoint preserving holomorphic map $\hat{F} : V \to Teich(S)$.

Clearly, each ϕ_{n+1} extends ϕ_n, for each $n \geq 1$. Hence, by Lemma 4.1, we have $p_n \circ f_{n+1} = f_n$ for all $n \geq 1$ (here, $p_n : Teich(S_{n+1}) \to Teich(S_n)$ is the puncture-forgetting map). Therefore, $\hat{F}$ maps V into T'. Since π maps $T(E)$ homeomorphically onto T', we get a unique map $F : V \to T(E)$ such that $\hat{F} =$

$\pi \circ F$. This map F is clearly basepoint preserving. By Theorem 7.3 in [18], it is holomorphic.

Finally, define $\hat{\phi} = F^*(\Psi_E)$. Since $\pi_n \circ F = f_n$ for each $n \geq 1$, it follows by Lemma 4.1 that $\hat{\phi}$ extends ϕ_n for each $n \geq 1$. Therefore, $\hat{\phi} = \phi$ on $V \times \bigcup_{n=1}^{\infty} E_n$. Since $\bigcup_n E_n$ is dense in E_0 it follows by Lemma 5.1 that $\hat{\phi} = \phi$ on $V \times E_0$, which means $\hat{\phi}$ extends ϕ.

Step 2. Now suppose V is any complex Banach manifold. Choose a holomorphic universal covering $\eta : U \to V$ such that $\eta(t_0) = x_0$ where x_0 is the basepoint of V and t_0 is a basepoint of U.

Consider the holomorphic motion $\psi := \eta^*(\phi)$ of E_0 over U. Since U is simply connected, by Step 1, there exists a holomorphic motion $\tilde{\psi} : U \times E \to \hat{\mathbf{C}}$ that extends ψ, i.e.

$$\tilde{\psi}(t, z) = \psi(t, z) = \phi(\eta(t), z) \tag{5.2}$$

for all t in U and z in E_0.

By Lemma 5.1, the map $\tilde{\psi}(t, \cdot)$, from E to $\hat{\mathbf{C}}$ is continuous. By formula (5.2), this map depends only on $\eta(t)$, since E_0 is dense in E. Hence there is a well-defined map $\tilde{\phi} : V \times E \to \hat{\mathbf{C}}$ such that

$$\tilde{\phi}(\eta(t), z) = \tilde{\psi}(t, z)$$

for all t in U and z in E. We claim that $\tilde{\phi}$ is the desired extension of ϕ.

Let $(x, z) \in V \times E_0$. Then, $\tilde{\phi}(x, z) = \tilde{\psi}(t, z)$, where $\eta(t) = x$ and $z \in E_0$. And, $\tilde{\psi}(t, z) = \psi(t, z) = \phi(\eta(t), z) = \phi(x, z)$. Hence, $\tilde{\phi}$ extends ϕ.

Also, for each x in V, the map $\tilde{\phi}(x, \cdot) : E \to \hat{\mathbf{C}}$ is injective, because $\tilde{\psi}(t, \cdot) : E \to \hat{\mathbf{C}}$ is injective.

We have $\eta(t_0) = x_0$ where x_0 is the basepoint of V, and t_0 is a basepoint of U. Then, $\tilde{\phi}(x_0, z) = \tilde{\psi}(t_0, z) = z$ for all z in E.

Finally, for each z in E, t in U, and x in V, where $\eta(t) = x$, set $\tilde{\psi}^z(t) = \tilde{\psi}(t, z)$ and $\tilde{\phi}^z(x) = \tilde{\phi}(x, z)$. Fix any z in E. Since $\tilde{\psi}$ is a holomorphic motion, the map $\tilde{\psi}^z : U \to \hat{\mathbf{C}}$ is holomorphic. Since $\tilde{\psi}^z = \tilde{\phi}^z \circ \eta$ and η is locally biholomorphic, it follows that $\tilde{\phi}^z : V \to \hat{\mathbf{C}}$ is holomorphic.

Hence, $\tilde{\phi} : V \times E \to \hat{\mathbf{C}}$ is a holomorphic motion that extends ϕ. □

6. Holomorphic family of hyperbolic holomorphic dynamical systems

Suppose Ω and Ω' are simply connected sets in the Riemann sphere $\hat{\mathbf{C}}$. We assume either (1) Ω and Ω' are homeomorphic to discs with Ω relatively compact in Ω', i.e., $\overline{\Omega} \subset \Omega'$ or (2) $\Omega = \Omega'$ is the whole Riemann sphere. Consider $R(z) : \Omega \to \Omega'$ a surjective holomorphic mapping, proper of degree d. In the first case, R is a polynomial-like map defined in Douady-Hubbard's paper [5]. In the other case, R is a rational map. Anyhow we call R a holomorphic dynamical system on Ω since in both cases, we can consider iterations of R and study the dynamical property of R. We will be only interested in the case $d \geq 2$, and assume this in the rest of the paper.

A point $z \in \Omega$ is called a *Fatou point* if either $R^n(z) \notin \bar{\Omega}$ for some integer $n \geq 1$ or there is a neighborhood $U \subset \Omega$ about z such that $R^n(U) \subseteq \bar{\Omega}$ for all $n \geq 0$ and $\{R^n|U\}$ is a normal family. All Fatou points form an open set F. This set is called the *Fatou set* of R. The set $J = \Omega \backslash F$ is called the *Julia set* of R. The Julia set J is a compact subset of Ω. (For some properties of the Julia set, the reader may refer to [3, 17, 5].)

Since $d \geq 2$, J is a perfect set (refer to [3, Chapter 3, Theorem 1.8] and [5, Chapter 1, Theorem 1]). Therefore, it consists of infinitely many points.

A point $z \in \Omega$ is called a periodic point of period $n \geq 1$ if $R^i(z) \neq z$ for $0 < i < n$ but $R^n(z) = z$. In particular, if the period is one then we call z a fixed point.

For a periodic point z of period n, let $\lambda = (R^n)'(z)$. The number λ is called the multiplier of R at z. According to the multiplier of R at a periodic point z, we can classify z into the following classes:

1) attractive if $|\lambda| < 1$;
2) repelling if $|\lambda| > 1$;
3) rational neutral if $\lambda^k = 1$ for some $k > 0$;
4) irrational neutral if $\lambda^k \neq 1$ for all $k > 0$ and $|\lambda| = 1$.

An attractive periodic point z is called super-attractive if $\lambda = 0$.

A holomorphic dynamical system R is called hyperbolic if there are constants $C > 0$ and $\mu > 1$ such that

$$|(R^n)'(z)| \geq C\mu^n, \quad z \in J, n \geq 1.$$

An equivalent statement is that R is hyperbolic if and only if all periodic points of R are either repelling or attractive (see [3], [7]).

Let E_0 be the set of all repelling periodic points of R.

Lemma 6.1. *The closure of E_0 is J, i.e., $\overline{E_0} = J$.*

For a proof see [3] or [17]. This also implies that E_0 must be an infinite set.

We consider a family of holomorphic dynamical systems defined on a fixed simply connected domain Ω. Let V be a simply connected complex Banach manifold with a basepoint x_0. Let $R(x,z)$ be a map from $V \times \Omega$ to $\hat{\mathbf{C}}$. We say that it is a family of holomorphic dynamical systems over V if

i) $R(x,z) : V \times \Omega \mapsto \hat{\mathbf{C}}$ is holomorphic;

ii) for each $x \in V$, $R_x = R(x,\cdot) : \Omega \to \hat{\mathbf{C}}$ is a holomorphic dynamical system.

We call $R(x,z)$ a family of hyperbolic holomorphic dynamical systems if in addition to (i) and (ii), we also have,

iii) for each $x \in V$, R_x is hyperbolic.

7. Proof of theorem 3

For each x in V, $R_x = R(x,\cdot)$ is a holomorphic dynamical system. Suppose $E_{x,0}$ is the set of all repelling periodic points of R_x for x in V. Consider a periodic point $z(x) \in E_{x,0}$ of period n and the equation,

$$F(y,z) = R_y^n(z) - z = 0, \quad y \in V, \text{ and } z \in \Omega. \tag{7.1}$$

Then $(x, z(x))$ is a solution of Equation (7.1) and, furthermore, $R_x^j(z(x)) - z(x) \neq 0$ for all $0 < j < n$.

Since the multiplier $(R_x^n)'(z(x))$ has absolute value greater than one, we have

$$\frac{\partial F}{\partial z}(x, z(x)) \neq 0.$$

By the implicit function theorem and the continuity of $(\partial F/\partial z)(y,z)$, there is a neighborhood $U(x)$ about x such that for each $y \in U(x)$, Equation (7.1) has a unique solution $z(y)$ with the initial value condition $z(x)$. Furthermore, since $(\partial F/\partial z)(y,z)$ is jointly holomorphic on (y,z), $z(y)$ depends holomorphically on y. Since $R_x^j(z(x)) - z(x) \neq 0$ for all $0 < j < n$, we can choose $U(x)$ so that $R_y^j(z(y)) - z(y) \neq 0$ for all $0 < j < n$. Therefore, $z(y)$, $y \in U(x)$, is a periodic point of R_y of period n.

Recall that x_0 is the basepoint of V. We now consider $E_0 = E_{x_0,0}$ the set of all repelling periodic points of R_{x_0}. Then E_0 is a subset of Ω. Since E_0 contains infinitely many points, we can assume, without loss of generality, that E_0 contains 0, 1, and ∞. We would like to construct a holomorphic motion of E_0 over V.

For each periodic point $z \in E_0$ of period $n \geq 1$ of R_{x_0}, consider the solution $z(y)$ of Equation (7.1) on $U(x_0)$. Then $z(x_0) = z$. Since V is simply connected, by definition, it is also path connected. For each x in V, consider a path $c(t)$ from x_0 to x, then the graph W of $c(t)$, $0 \leq t \leq 1$, is a compact subset in V. The neighborhood $\{U(x), x \in W\}$ is an open cover of W. So it has a subcover consisting of a finite number of neighborhoods $\{U(x_i)\}_{i=0}^k$ such that (a) x_0 is our basepoint and $x_k = x$ and (b) $x_{i+1} \in U(x_i)$. In each $U(x_i)$, we have a solution $z_i(y)$ of Equation (7.1). By uniqueness, $z_i(y) = z_{i+1}(y)$ for $y \in U(x_i) \cap U(x_{i+1})$. Therefore, we have a unique solution of $z(y)$ on $\bigcup_{i=0}^k U(x_i)$ depending on y holomorphically. Thus we can holomorphically extend the solution $z(y)$ in $U(x_0)$ to any point in V. Since V is simply connected, this extension does not depend on the choice of a path. Furthermore, $z(y)$ has the same period n as we proved in the second paragraph.

Now we define a map $\phi : V \times E_0 \mapsto \hat{\mathbf{C}}$ as $\phi(x,z) = z(x)$. Then

(a) $\phi(x_0, z) = z$ for $\forall z \in E_0$;

(b) for each fixed $z \in E_0$, $\phi(\cdot, z) : V \mapsto \hat{\mathbf{C}}$ is a holomorphic map on V.

Now we will check (c) that for each fixed x in V, $\phi(x, \cdot) : E_0 \mapsto E_{x,0} \subset \hat{\mathbf{C}}$ is injective. Let $c(t)$ be a path from x_0 to x, then the graph W of $c(t)$, $0 \leq t \leq 1$, is a compact subset in V. Let $\{U(x_i)\}_{i=0}^k$ be the finite subcover of W which we got in the previous paragraph.

Suppose $z \neq w$ are two points in E_0. If z and w have different periods, then $z(x) \neq w(x)$ since they must have different periods. So let us assume z and w have the same period n. Then $z(x)$ and $w(x)$ have the same period n. Both of them are solutions of Equation (7.1) for the same n. Then $z(c(t))$ and $w(c(t))$ are two paths in $\hat{\mathbf{C}}$. We now have a similar argument as above but reversely as follows: If $z(x) = w(x)$, since both $z(y)$ and $w(y)$ are solutions of Equation (7.1) on $U(x_k)$, by the uniqueness, $z(x_{k-1}) = w(x_{k-1})$. Inductively, we get $z = z(x_0) = w(x_0) = w$. We get a contradiction. Thus we have shown that $z(x) \neq w(x)$ and so $\phi(x, \cdot)$ is injective. Therefore,

$$\phi(x,z) = z(x) : V \times E_0 \mapsto \hat{\mathbf{C}}$$

is a holomorphic motion. By Theorem 2, $\phi(x,z)$ can be extended to a holomorphic motion

$$\hat{\phi}(x,z) = z(x) : V \times \overline{E_0} \mapsto \hat{\mathbf{C}}.$$

The closure of $E_{x,0}$ is the Julia set J_x of R_x. Since $\phi(x,z)$ is continuous, for each $x \in V$ (see Lemma 5.1), $\hat{\phi}(x, \overline{E_0}) = \overline{\phi(x, E_0)} = \overline{E_{x,0}} = J_x$ is the Julia set of R_x. We conclude that J_x depends holomorphically on x over V.

By Theorem 3.2, for each $x \in V$, $\hat{\phi}(x, \cdot)$ is a restriction of a quasiconformal self-map of $\hat{\mathbf{C}}$ with dilatation less than or equal to $\exp(2\rho_V(x, x_0))$. This completes our proof. □

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