

DISTANCE ENTROPY OF DYNAMICAL SYSTEMS ON NONCOMPACT-PHASE SPACES

XIONGPING DAI

Department of Mathematics, Nanjing University
Nanjing, 210093, P. R. CHINA

YUNPING JIANG

Department of Mathematics, Queens College of CUNY
Flushing, NY 11367, USA
Department of Mathematics, CUNY Graduate School
New York, NY 10016, USA
Academy of Mathematics and System Sciences
Chinese Academy of Sciences, Beijing 100080, P. R. CHINA

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ABSTRACT. Let X be a separable metric space not necessarily compact, and let $f: X \rightarrow X$ be a continuous transformation. From the viewpoint of Hausdorff dimension, the authors improve Bowen's method to introduce a dynamical quantity distance entropy, written as $ent_{\mathbb{H}}(f; Y)$, for f restricted on any given subset Y of X ; but it is essentially different from Bowen's entropy(1973). This quantity has some basic properties similar to Hausdorff dimension and is beneficial to estimating Hausdorff dimension of the dynamical system. The authors show that if f is a local lipschitzian map with a lipschitzian constant ℓ then $ent_{\mathbb{H}}(f; Y) \leq \max\{0, \mathbb{H}\mathbb{D}(Y) \log \ell\}$ for all $Y \subset X$; if f is locally expanding with skewness λ then $ent_{\mathbb{H}}(f; Y) \geq \mathbb{H}\mathbb{D}(Y) \log \lambda$ for any $Y \subset X$. Here $\mathbb{H}\mathbb{D}(-)$ denotes the Hausdorff dimension. The countable stability of the distance entropy $ent_{\mathbb{H}}$ proved in this paper, which generalizes the finite stability of Bowen's h -entropy (1971), implies that a continuous pointwise periodic map has the distance entropy zero. In addition, the authors show examples which demonstrate that this entropy describes the real complexity for dynamical systems over noncompact-phase space better than that of various other entropies.

1. Introduction. Rudolf Clausius created the thermodynamical concept of entropy in 1854; Shannon carried it over to information theory in 1948 [32], to describe the complexity of information. In 1958 Kolmogorov [23] introduced the concept of measure-theoretic entropy to ergodic theory. Kolmogorov's definition was improved by Sinai in 1959 [33]. In 1960's Adler, Konheim, and McAndrew [1] introduced the concept of topological entropy, written as $ent(f)$ in this paper, as an analogue of measure-theoretic entropy but for a continuous map $f: X \rightarrow X$ of a compact Hausdorff topological space X . In each setting entropy is a measure of uncertainty

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or randomness or disorder. Since then, the word, entropy, has become a routine name appearing in dynamical systems and ergodic theory (c.f. [36, 29]). The first importance of entropy arises from its invariance under equivalence. Then, it can be used to classify dynamical systems. For example, in this way, Kolmogorov and Sinai settled in the negative the old question of whether or not the Bernoulli shifts $\mathcal{B}(\frac{1}{2}, \frac{1}{2})$ and $\mathcal{B}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ are isomorphic; in fact, entropy is a complete invariant for Bernoulli shifts from a theorem due to Ornstein in 1970 [27]. Another importance of entropy is that it is a quantitative description of complexity of a dynamical system. Although there exist chaotic dynamical systems with zero-entropy [39], a theorem due to Blanchard *et al* in 2002 [3] says that for compact dynamical systems positive entropy implies chaos. According to Walters [36], the entropy has been the most successful invariant in the theory of dynamical systems and ergodic theory. Actually, the entropy is not only useful in dynamical systems and ergodic theory but also very important in many other branches of sciences.

In order to discover the relationships between topological entropy and measure theoretic entropy, Dinaburg [13] and Bowen [5] gave a new, but equivalent, definition in 1970's when the topological space X has a certain metric structure d . In their original definition, X is required to be a compact metric space. However, the definition given by Bowen [5] works for a uniformly continuous map when the metric space X is not compact, written as $h(f)$ in this paper. Using the ergodic theory, topological entropy is described very well by the Goodwyn-Dinaburg-Goodman variational principle [36]

$$\text{ent}(f) = \sup\{h_\mu(f) \mid \mu \in \mathcal{M}_{\text{inv}}(X, f)\}$$

for any dynamical system f on a compact metric space X . Other definitions of topological entropy for non-compact topological spaces have been introduced in literature, see [6, 21, 24, 20, 8].

However, our intention is to consider the following important problems which originate from [18, 9, 4, 25, 37, 17] amongst others.

Problem 1. *Let (X, f) be a topological dynamical system.*

- (1) *What is the global relation between the system f and the topological structure of the underlying space X ? For example, we ask what is the relation between $\text{ent}_{\text{top}}(f)$ and $\text{HD}(X)$ or $\text{dim}_{\text{top}}(X)$?*
- (2) *Suppose μ is an invariant Borel probability measure of (X, f) . What is the topological structure of μ ? Precisely, letting*

$$\text{HD}(\mu) = \inf\{\text{HD}(B) : B \in \mathfrak{B}_X, \mu(B) = 1\}$$

called the Hausdorff dimension of μ (see [14, 37]), and

$$\text{ent}_{\text{top}}(f; \mu) = \inf\{\text{ent}_{\text{top}}(f; B) : B \in \mathfrak{B}_X, \mu(B) = 1\}$$

called the topological entropy of μ , we ask

- (a) *what is the relation between $h_\mu(f)$ and $\text{ent}_{\text{top}}(f; \mu)$?*
- (b) *is $\text{ent}_{\text{top}}(f; \mu)$ equal to $\text{ent}_{\text{top}}(f; B)$ for some $B \in \mathfrak{B}_X$ with $\mu(B) = 1$?*
- (c) *do we have any relation between $\text{HD}(\mu)$ and $h_\mu(f)$ or $\text{ent}_{\text{top}}(f; \mu)$?*

Here, it is $\text{ent}_{\text{top}}(-)$ that we are looking for!

To this end, we first consider Bowen's dimension entropy defined in 1973. When X is purely a topological space, $Y \subset X$, and when $f: X \rightarrow X$ is just continuous, Bowen [6] defined a new topological entropy, denoted by $h_{\text{top}}(f; Y)$ in this paper, from the Hausdorff-dimension point of view (see Section 3). The new definition is

compatible with the old given by Adler, Konheim, and McAndrew in the sense of $ent(f) = h_{top}(f; X)$ when X is a compact topological space. Also the new definition for topological spaces matches with the definition for metric spaces in the sense of $h(f) = h_{top}(f; X)$ when X is a compact metric space.

With the development of ergodic theory and dynamical systems, specially the involution of multifractal analysis of measures, Bowen’s Hausdorff dimension entropy for non-compact sets has drawn more and more attention, see [28, 19, 2, 15, 16, 35, 26, 7] and others. Bowen’s viewpoint of Hausdorff dimension is very interesting, but for systems over noncompact-phase spaces, Bowen’s entropy h_{top} does not work well. Let us consider first the following example.

Example 1.1. *Consider the translation map*

$$T: \mathbb{R} \rightarrow \mathbb{R}$$

given by $x \mapsto x + 1$ for all $x \in \mathbb{R}$. Let $I = [0, 1]$. It is easily seen from [6] that

$$h_{top}(T; I) = h_{top}(T) = \infty.$$

One does not satisfy that such a simple system (\mathbb{R}, T) has infinite topological entropy. On the other hand, some interesting relations between h_{top} -entropy and the Hausdorff dimension for a compact system, can not be extended to a noncompact system. For example, let us see the following

Example 1.2. *Let (X, d) be a compact metric space and $T: X \rightarrow X$ be a Lipschitzian map with a Lipschitzian constant ℓ , that is, $d(Tx, Ty) \leq \ell d(x, y)$ for any $x, y \in X$. Then the following relation holds [12, 26]:*

$$h_{top}(T; Y) \leq \max\{0, \text{HD}(Y) \log \ell\} \quad (\forall Y \subset X).$$

If the state space X is not compact, the above formula does not necessarily hold. Let’s see the simple translation system stated in Example 1.1. It is easily seen that

$$\infty = h_{top}(T) \not\leq \max\{0, \text{HD}(\mathbb{R}) \log \ell\} = 0, \text{ where } \ell = 1,$$

which contradicts the desired formula.

For the $h(-)$ entropy [5], we have the following counterexample:

Example 1.3. *Let \mathbb{Q} be the rational number set as a subspace of \mathbb{R} and $f: \mathbb{Q} \rightarrow \mathbb{Q}$ be defined by $x \mapsto 2x \forall x \in \mathbb{Q}$. Then from the calculation in Example 2.2 below we have $h(f) = \log 2$. So*

$$\log 2 = h(f) \not\leq \max\{0, \text{HD}(\mathbb{Q}) \log 2\} = 0 \text{ where } \ell = 2.$$

In this paper, we improve Bowen’s method to introduce a new entropy. Consider a continuous transformation $f: X \rightarrow X$ on a separable metric space X not necessarily compact. Inspired by Bowen’s Hausdorff dimension entropy h_{top} [6] we define in Section 2.1 from the viewpoint of Hausdorff dimension an entropy, written $ent_H(f; Y)$, for f restricted on any subset Y of X , called *distance entropy*, which depends upon the metric of X . We in this paper study many basic properties of the distance entropy.

We show in Section 3 that if X is compact,

$$ent_H(f; Y) = h_{top}(f; Y) \quad (\forall Y \subset X).$$

Therefore, when X is a compact metric space, $ent_H(f; Y)$ is an invariant under topological conjugacy. When X is not compact, $ent_H(f; Y)$ may not be invariant under topological conjugacy. However, we prove that $ent_H(f; Y)$ is an invariant

under uniform topological conjugacy. Therefore, the distance entropy becomes a useful tool in the study of dynamical systems over noncompact-phase spaces.

When $f: X \rightarrow X$ is a continuous map of the separable metric space X , we then show in Section 3

$$\text{ent}_{\text{H}}(f; K) \leq h(f; K) \quad (\forall K \subseteq X \text{ if } K \text{ compact});$$

in particular, if X is locally compact, then $\text{ent}_{\text{H}}(f) \leq h(f)$.

Our definition is beneficial to the estimation of the Hausdorff dimension. When considering the Hausdorff dimension by Bowen's methods, one often needs to choose a Lebesgue number for a finite open cover of the phase space considered as in the proof of [26, Theorem 2.1], so one has to work on a compact state space. However, based on the definition of the distance entropy stated in Section 2, there is a "natural" Lebesgue number ε . In Section 4, we show that, if $f: X \rightarrow X$ is a Lipschitz map with Lipschitz constant ℓ then

$$\text{ent}_{\text{H}}(f; Y) \leq \max\{0, \text{HD}(Y) \log \ell\} \quad (\forall Y \subset X).$$

This result extends the formula provided in Example 1.2. On the other hand, if $f: X \rightarrow X$ is locally expanding with skewness $\lambda > 1$, then

$$\text{ent}_{\text{H}}(f; Y) \geq \text{HD}(Y) \log \lambda \quad (\forall Y \subset X).$$

It is interesting to point out that the Cánovas-Rodríguez entropy [8] also need not satisfy the above inequality from the following example.

Example 1.4. Let $f: \mathbb{R} \rightarrow \mathbb{R}; x \mapsto 2x$. Then, the Cánovas-Rodríguez entropy is defined by

$$\text{ent}_{\text{C-R}}(f) = \sup\{\text{ent}(f|_K) \mid K \subset \mathbb{R} \text{ compact and } f\text{-invariant}\} = 0.$$

Thus, we have

$$\text{ent}_{\text{C-R}}(f) \not\geq \text{HD}(\mathbb{R}) \log 2 \text{ where } \lambda = 2.$$

We generalize many basic but important properties for Bowen's h -entropy (1971) to our distance entropy ent_{H} . In particular, we prove in Section 2 the countable stability for the distance entropy which generalizes the finite stability for h -entropy. The countable stability for the distance entropy becomes a useful tool for us. For example, we use this countable stability to give a new topological proof of an old result as well as generalize this old result to a dynamical system supported on a noncompact metric space as follows. Consider a pointwise periodic continuous transformation $T: X \rightarrow X$ of a metric space X , this means that, for each $x \in X$ there is some $n(x) \in \mathbb{N}$ such that $T^{n(x)}(x) = x$. When X is compact, it is known that $\text{ent}(T) = 0$ from the variational principle of entropy [36, Theorem 8.6 and Corollary 8.6.1]. Z.-L. Zhou [38] asked if there exists a topological proof for this result. The difficulty for a topological proof is that $n(x)$ may not be a continuous map. But using the distance entropy, we successfully find a topological proof. Actually, we prove that $\text{ent}_{\text{H}}(T)$ vanishes even without assumption that X is compact. This is discussed in Section 5.

Examples 1.1, 1.3 and 2.2 show that the distance entropy ent_{H} is more approximate to the real complexity for noncompact dynamical systems than that of Bowen's h_{top} (1973) and h -entropies (1971). We call a topological dynamical system (X, f) compact if X is a compact metric space and $f: X \rightarrow X$ is continuous. It is known that for a compact system (X, f) that $h(f) > 0$ implies chaos in the sense of Li and Yorke [3]. However, Example 2.2 shows that this might be false for a noncompact

dynamical system. In light of Proposition 3 below, for some typical noncompact systems such as geodesic flows of punctured Riemann surfaces, the positivity of the distance entropy $ent_{\mathbb{H}}$ might be another useful method to observe chaotic phenomenon besides the mixing. That is another reason we would like to introduce the distance entropy. See [10, 11] for further applications.

2. Definition of distance entropy. Let (X, d) be a separable metric space, i.e., with a countable base or equivalently with a countable dense subset, but not necessarily compact. Let $f: X \rightarrow X$ be a continuous (not necessarily uniformly) transformation. In this section, we are going to define the distance entropy for f with respect to the distance function d on X .

2.1. Definition. For any $\varepsilon > 0$ and for any $E \subseteq X$, let $l_{\varepsilon}^f(E)$ be the biggest nonnegative integer, called the *step length* of E with respect to f , such that

$$|f^k(E)| < \varepsilon \quad \forall k \in [0, l_{\varepsilon}^f(E)); \tag{1}$$

$$l_{\varepsilon}^f(E) = 0 \text{ if } |E| \geq \varepsilon; \quad l_{\varepsilon}^f(E) = +\infty \text{ if } |f^k(E)| < \varepsilon \quad \forall k \in \mathbb{Z}_+. \tag{2}$$

Here for $A \subseteq X$

$$|A| = \text{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}. \tag{3}$$

Set

$$\text{diam}_{\varepsilon}^f(E) = \exp(-l_{\varepsilon}^f(E)) \text{ and } D_{\varepsilon}^f(\mathfrak{E}, \lambda) = \sum_{i=1}^{\infty} (\text{diam}_{\varepsilon}^f(E_i))^{\lambda} \tag{4}$$

for any $\mathfrak{E} = \{E_i\}_1^{\infty}$ and for any $\lambda \in \mathbb{R}_+$. For any given $\varepsilon > 0$ and $\lambda \geq 0$, we now define an outer measure $\mathfrak{M}_{\varepsilon}^{\lambda}$ by

$$\mathfrak{M}_{\varepsilon}^{\lambda}(Y) = \inf \left\{ D_{\varepsilon}^f(\mathfrak{E}, \lambda) : \bigcup \{E_i \mid E_i \in \mathfrak{E}\} \supseteq Y, l_{\varepsilon}^f(E_i) > -\log \varepsilon \right\} \tag{5}$$

for any $Y \subseteq X$. Define a λ -measure \mathfrak{M}^{λ} by

$$\mathfrak{M}^{\lambda}(Y) = \lim_{\varepsilon \rightarrow 0} \mathfrak{M}_{\varepsilon}^{\lambda}(Y) \quad (\forall Y \subseteq X). \tag{6}$$

Note that $\mathfrak{M}^{\lambda}(Y) \notin \{0, +\infty\}$ for at most one $\lambda \in \mathbb{R}_+$. In fact, for $0 \leq s < t < \infty$, the inequality

$$\mathfrak{M}_{\varepsilon}^s(Y) \geq \varepsilon^{s-t} \mathfrak{M}_{\varepsilon}^t(Y)$$

implies that $\mathfrak{M}^s(Y) = +\infty$ if $\mathfrak{M}^t(Y) > 0$, and $\mathfrak{M}^t(Y) = 0$ if $\mathfrak{M}^s(Y) = 0$.

For any $Y \subseteq X$, let

$$ent_{\mathbb{H}}(f; Y) = \inf\{\lambda \mid \mathfrak{M}^{\lambda}(Y) = 0\} \tag{7}$$

and we call it the *distance entropy* of f restricted on Y . The quantity is well defined because of the second axiom of countability. Finally, define the *distance entropy* of f with respect to d by

$$ent_{\mathbb{H}}(f) = ent_{\mathbb{H}}(f; X). \tag{8}$$

For the translation system (\mathbb{R}, T) in Example 1.1, we easily have $ent_{\mathbb{H}}(T) = 0$. From the general understanding, the entropy should be a quantity to describe the complexity of a dynamical system. A bigger entropy should imply more complicated dynamical behaviors. The translation $T(x) = x + 1$ on \mathbb{R} has a very simple dynamical behavior, however, Bowen's entropy $h_{top}(T)$ of T is ∞ . That means Bowen's h_{top} -entropy has a certain limitation to describe the complexity of the dynamical behavior of a system over a noncompact phase space. This is one of the reasons we

would like to introduce distance entropy $ent_{\mathbb{H}}$. This example shows that the distance entropy $ent_{\mathbb{H}}$ has a certain advantage over Bowen's h_{top} -entropy for systems over a noncompact phase space.

2.2. Countable stability. The distance entropy has some basic properties similar to the Hausdorff dimension (see [14]).

Next, we will show that the distance entropy has the *countable stability* property as the Hausdorff dimension which generalizes the *finite stability* of Bowen's h -entropy (1972) which asserts: Let (X, d) be a metric space and T be uniformly continuous on X ; if $K \subseteq K_1 \cup \cdots \cup K_n$ are all compact subsets of X , then $h(T; K) \leq \max_{1 \leq i \leq n} h(T; K_i)$ (see [36, Theorem 7.5]). This property is very useful for calculations.

Theorem 2.1. *Let $f: X \rightarrow X$ be a continuous map of a separable metric space X , then the distance entropy $ent_{\mathbb{H}}$ has the following properties.*

(1) *Monotonicity:*

$$ent_{\mathbb{H}}(f; E) \leq ent_{\mathbb{H}}(f; F) \quad \text{if } E \subset F.$$

(2) *Countable stability: for any sequence of sets F_1, F_2, \dots*

$$ent_{\mathbb{H}}\left(f; \bigcup_{i=1}^{\infty} F_i\right) = \sup_i \{ent_{\mathbb{H}}(f; F_i)\}.$$

Proof. The statement (1) easily follows from the definition. We next show the second statement.

First, by the monotonicity we have the inequality

$$ent_{\mathbb{H}}\left(f; \bigcup_i F_i\right) \geq ent_{\mathbb{H}}(f; F_i) \quad (i = 1, 2, \dots).$$

Next, we prove the statement by showing the other inequality

$$ent_{\mathbb{H}}\left(f; \bigcup_i F_i\right) \leq \sup_i \{ent_{\mathbb{H}}(f; F_i)\}.$$

In the case where $\sup_{1 \leq i < \infty} \{ent_{\mathbb{H}}(f; F_i)\} = +\infty$, there is nothing to prove. We now assume that $\sup_i \{ent_{\mathbb{H}}(f; F_i)\} = \lambda < +\infty$. Then, for any $\epsilon > 0$ we have

$$ent_{\mathbb{H}}(f; F_i) \leq \lambda < \lambda + \epsilon \quad (i = 1, 2, \dots).$$

From the choice of λ and the definition of $ent_{\mathbb{H}}(f; F_i)$, we obtain

$$\mathfrak{M}^{\lambda+\epsilon}(F_i) = 0 \quad (i = 1, 2, \dots).$$

By the subadditivity of the measure $\mathfrak{M}^{\lambda+\epsilon}$ we have

$$\mathfrak{M}^{\lambda+\epsilon}\left(\bigcup_i F_i\right) \leq \sum_i \mathfrak{M}^{\lambda+\epsilon}(F_i) = 0.$$

Hence $\lambda + \epsilon \geq ent_{\mathbb{H}}(f; \bigcup_i F_i)$ and so

$$ent_{\mathbb{H}}\left(f; \bigcup_i F_i\right) \leq \lambda + \epsilon.$$

This completes the proof of the statement. \square

Note that for the h_{top} -entropy defined by Bowen, Bowen stated without proof that [6, Proposition 2],

$$h_{top} \left(f; \bigcup_i Y_i \right) = \sup_i \{h_{top}(f; Y_i)\}. \tag{9}$$

We now turn to some more properties of distance entropy generalizing in part [36, Theorem 7.10].

Theorem 2.2. *Suppose X is a separable metric space and $T: X \rightarrow X$ is a continuous map. Then, for any given $m \in \mathbb{N}$,*

- (1) $ent_{\mathbb{H}}(T^m; Y) \leq m \ ent_{\mathbb{H}}(T; Y) \ \forall Y \subseteq X$;
- (2) $ent_{\mathbb{H}}(T^m; Y) = m \ ent_{\mathbb{H}}(T; Y) \ \forall Y \subseteq X$ if T is uniformly continuous.

Proof. For any $E \subseteq X$ and for any $\varepsilon > 0$ the inequality

$$l_{\varepsilon}^T(E) \leq m \ l_{\varepsilon}^{T^m}(E)$$

implies that

$$ent_{\mathbb{H}}(T^m; Y) \leq m \ ent_{\mathbb{H}}(T; Y) \quad (Y \subseteq X).$$

This implies that the statement (1) holds. On the other hand, since T is uniformly continuous, $\forall \varepsilon > 0 \ \exists \delta > 0 \ (\delta < \varepsilon)$ such that

$$d(x, y) < \delta \Rightarrow \max_{0 \leq j < m} d(T^j x, T^j y) < \varepsilon.$$

So, if $\mathfrak{E} = \{E_i\}_1^{\infty}$ is a countable cover of Y , then

$$m \ l_{\delta}^{T^m}(E_i) \leq l_{\varepsilon}^T(E_i) \quad (i = 1, 2, \dots).$$

This means that for any $\lambda \in \mathbb{R}_+$

$$\mathfrak{M}_{\delta}^{m\lambda}(Y; T^m) \geq \mathfrak{M}_{\varepsilon}^{\lambda}(Y; T).$$

Hence

$$m \ ent_{\mathbb{H}}(T; Y) \leq ent_{\mathbb{H}}(T^m; Y).$$

Thus we have proved the statement. □

About the second statement of [36, Theorem 7.10 (ii)], we have the following

Question 2.1. *Let (X, T) be a compact dynamical system over a compact metric space X ; let (\tilde{X}, \tilde{T}) be a dynamical system of the separable metric space \tilde{X} . If $\pi: \tilde{X} \rightarrow X$ is a semi-conjugacy from \tilde{T} to T , i.e., $\pi \circ \tilde{T} = T \circ \pi$, then does one have*

$$ent_{\mathbb{H}}(\tilde{T}) \leq ent_{\mathbb{H}}(T) + \sup_{x \in X} \{ent_{\mathbb{H}}(\tilde{T}; \pi^{-1}(x))\}?$$

About Bowen's h_{top} -entropy, this relation is not necessary to hold. For example, let $\pi: \mathbb{R} \rightarrow S^1$ be given by $t \mapsto e^{2\pi i t}$ for all $t \in \mathbb{R}$, let $\tilde{T}(t) = t + 1$ for $t \in \mathbb{R}$. Then \tilde{T} is semi-conjugate to $T = Id_{S^1}$ by π . For any $x \in S^1$, $h_{top}(\tilde{T}; \pi^{-1}(x)) = 0$ by Eq. (9). Thus

$$\infty = h_{top}(\tilde{T}) \not\leq h_{top}(T) + \sup\{h_{top}(\tilde{T}; \pi^{-1}(x)) \mid x \in S^1\} = 0,$$

which is a contradiction to the formula in Question 2.1.

From the definitions, it is easily seen that the quantities h_{top} and $ent_{\mathbb{H}}$ are analogues of the Hausdorff dimension; the quantities ent -entropy and h -entropy are analogues of the box dimension. So, in general ent -entropy and h -entropy have no countable stability. We have the following example:

Example 2.1. Let $X = \mathbb{R}$ with the euclidean metric. Define

$$f: X \rightarrow X$$

by

$$x \mapsto 2x.$$

Then $\text{ent}_{\mathbb{H}}(f) = \log 2$ by Corollary 7 proved in Section 4. Let

$$K = \{0\} \cup \{n^{-1} : n = 1, 2, 3, \dots\}.$$

Clearly, K is a compact subset of X and $\text{ent}_{\mathbb{H}}(f; K) = h_{\text{top}}(f; K) = 0$ from the countable stability. But $h(f; K) = \log 2$.

In fact, for small $\delta > 0$ let $r_n(\delta, K)$ be the smallest cardinality of (δ, n) -spanning sets of K . We have

$$\begin{aligned} r_n(\delta, K) &\geq \left\lceil \frac{1}{2} + \sqrt{2^{n-1}/\delta + \frac{1}{4}} \right\rceil \geq \left\lceil \sqrt{2^{n-1}/\delta} \right\rceil \\ &= \left\lceil \frac{2^{n-2}}{\sqrt{\delta}} \right\rceil \geq 2^{n-2} \end{aligned}$$

where $[x]$ denotes the integer part of x . This implies that $h(f; K) \geq \log 2$.

Moreover, the above example shows that in general the distance entropy $\text{ent}_{\mathbb{H}}$ and Bowen’s entropy h are different for noncompact dynamical systems.

Example 2.2. Let \mathbb{Q} be the rational number set of \mathbb{R} viewed as a subspace of \mathbb{R} , and let $f: \mathbb{Q} \rightarrow \mathbb{Q}$ be given by $x \mapsto 2x$. Then $\text{ent}_{\mathbb{H}}(f) = 0$ but $h(f) = \log 2$ by a calculation similar to that of Example 2.1.

Although the system (\mathbb{Q}, f) in the above example has a positive h -entropy, it is by no means chaotic in the sense of Li and Yorke since $\text{card}\mathbb{Q}$ is countable. It must be a simple system. Therefore, the example shows that the entropy $\text{ent}_{\mathbb{H}}$ is more a reasonable description of the complexity for noncompact dynamical systems than that of Bowen’s h -entropy.

2.3. Invariance. We say $f_1: X_1 \rightarrow X_1$ and $f_2: X_2 \rightarrow X_2$ are *uniformly topologically conjugate* if there is a homeomorphism $\pi: X_1 \rightarrow X_2$ with $\pi \circ f_1 = f_2 \circ \pi$ such that π and its inverse $\pi^{-1}: X_2 \rightarrow X_1$ are both uniformly continuous. The next result shows that the distance entropy is an invariant of the uniform topological conjugacy.

Theorem 2.3. If (X_1, d_1) and (X_2, d_2) are metric spaces satisfying the second countable axiom. If $f_i: X_i \rightarrow X_i$ are continuous for $i = 1, 2$ and if $\pi: X_1 \rightarrow X_2$ is a uniformly continuous map with $\pi \circ f_1 = f_2 \circ \pi$, then for any $Y \subseteq X_1$ one has

$$\text{ent}_{\mathbb{H}}(f_1; Y) \geq \text{ent}_{\mathbb{H}}(f_2; \pi(Y)).$$

Proof. Since $\pi: X_1 \rightarrow X_2$ is uniformly continuous, for any $\varepsilon_2 > 0$ there is some $\varepsilon_1 > 0$ such that $\varepsilon_1 < \varepsilon_2$ and $d_2(\pi(x_1), \pi(x_2)) < \varepsilon_2$ whenever $d_1(x_1, x_2) < \varepsilon_1$ for any pair $x_1, x_2 \in X_1$. Then, for any $Y \subseteq X_1$, we have

$$D_{\varepsilon_1}^{f_1}(\mathfrak{E}, \lambda) \geq D_{\varepsilon_2}^{f_2}(\pi(\mathfrak{E}), \lambda)$$

for any $\lambda \in \mathbb{R}_+$ and for any countable cover \mathfrak{E} of Y . This implies

$$\mathfrak{M}_{\varepsilon_1}^{\lambda}(Y; f_1) \geq \mathfrak{M}_{\varepsilon_2}^{\lambda}(\pi(Y); f_2).$$

Hence

$$\text{ent}_{\mathbb{H}}(f_1; Y) \geq \text{ent}_{\mathbb{H}}(f_2; \pi(Y)).$$

The proof is thus completed. □

As Hausdorff dimension, the distance entropy depends strictly on the choice of the metric d of the state space X . Two metrics d and d' on X are *uniformly equivalent* if

$$Id: (X, d) \rightarrow (X, d') \text{ and } Id: (X, d') \rightarrow (X, d)$$

are both uniformly continuous. The following result may be obtained easily.

Corollary 1. *If d and d' are uniformly equivalent metrics on X and $f: X \rightarrow X$ is continuous, then*

$$ent_{H,d}(f; Y) = ent_{H,d'}(f; Y)$$

for any $Y \subseteq X$; in particular, $ent_{H,d}(f) = ent_{H,d'}(f)$.

Proof. The statement follows easily from Theorem 2.3. □

Notice that Corollary 1 above generalizes the corresponding result of h -entropy [36, Theorem 7.4]. The following is an example of two equivalent, not uniformly equivalent metrics which give different values of distance entropy for some transformation, borrowed from [36].

Example 2.3. *Let $X = (0, \infty)$. Define*

$$f: (0, \infty) \rightarrow (0, \infty)$$

by $f(x) = 2x$. Let d be the usual euclidean metric on $(0, \infty)$. By Corollary 7 in Section 4 we have $ent_{H,d}(f) = \log 2$. Let d' be the metric which coincides with d on $[1, 2]$ but is so that f is an isometry for d' , i.e., use the fact that the intervals $(2^{n-1}, 2^n], n \in \mathbb{Z}$, partition X and $f((2^{n-1}, 2^n]) = (2^n, 2^{n+1}]$. Then $ent_{H,d'}(f) = 0$. The metrics d, d' are equivalent but not uniformly equivalent.

Regarding the Hausdorff dimension, if $F \subset \mathbb{R}^n$ and $f: F \rightarrow \mathbb{R}^m$ is of Lipschitz, i.e.,

$$|f(x) - f(y)| \leq c|x - y| \quad (\forall x, y \in F)$$

where c is a constant, then

$$HD(f(F)) \leq HD(F).$$

In particular, the Hausdorff dimension is preserved by a bi-lipschitzian mapping. For the distance entropy $ent_H(f; Y)$ there is a similar property:

Corollary 2. *Let X be a separable metric space. If $f: X \rightarrow X$ is uniformly continuous, then*

$$ent_H(f; f(Y)) \leq ent_H(f; Y) \quad (\forall Y \subseteq X).$$

In particular, if f is bi-uniformly continuous, i.e., f and f^{-1} are both uniformly continuous, then

$$ent_H(f; f(Y)) = ent_H(f; Y) \quad (\forall Y \subseteq X).$$

Proof. Considering the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ f \downarrow & & \downarrow f \\ X & \xrightarrow{f} & X \end{array}$$

the statements easily follow from Theorem 2.3 before. □

2.4. A more general case. The distance entropy $ent_{\mathbb{H}}$ can be defined more generally as well. Let $u: X \rightarrow \mathbb{R}$ be a bounded strictly positive continuous function. For $\varepsilon > 0$ and $E \subseteq X$, let

$$u_{\varepsilon}^f(E) = \sup_{x \in E} \left\{ \sum_{i=0}^{l_{\varepsilon}^f(E)-1} u(f^i x) \right\}.$$

For each $Y \subseteq X$ and each $\lambda \in \mathbb{R}_+$, we define

$$M(Y; \lambda, u) = \liminf_{\varepsilon \rightarrow 0} \left\{ \sum_{i=1}^{\infty} \exp(-\lambda u_{\varepsilon}^f(E_i)) \mid \cup E_i \supseteq Y, l_{\varepsilon}^f(E_i) > -\log \varepsilon \right\},$$

and further

$$ent_{\mathbb{H}}^u(f; Y) = \inf\{\lambda \mid M(Y; \lambda, u) = 0\}.$$

In the case $u \equiv 1$, $ent_{\mathbb{H}}^u(f; Y) = ent_{\mathbb{H}}(f; Y)$.

For every Borel probability measure μ on X , let

$$ent_{\mathbb{H}}^u(f; \mu) = \inf\{ent_{\mathbb{H}}^u(f; Z) \mid Z \in \mathfrak{B}_X \text{ with } \mu(Z) = 1\}.$$

When X is compact, $ent_{\mathbb{H}}^u(f; Y)$ is independent of the choice of compatible metric d ; moreover, it coincides with the u -dimension $\dim_u Y$ of Y introduced by Barreira and Schmeling in [2] (see also [31]). For a Borel probability measure μ on X , $ent_{\mathbb{H}}^u(f; \mu)$ is the u -dimension of μ , written as $\dim_u \mu$ in [2] (see also [31]). Furthermore, the u -dimension of an ergodic μ and the measure-theoretic entropy has the following relation.

Theorem 2.4 ([2, Theorem 6.3]). *Let X be a compact metric space and $f: X \rightarrow X$ be a continuous map. When $\mu \in \mathcal{M}_{\text{erg}}(X, f)$ and $u: X \rightarrow \mathbb{R}$ is a strictly positive continuous function, one has*

$$\dim_u \mu = \frac{h_{\mu}(f)}{\int_X u d\mu}.$$

This theorem confirms Problem 1(2)(a) raised in the introduction in the case where (X, f) is a compact dynamical system. For this case, Problem 1(2)(b) will be positively confirmed in [11].

The positivity answer of the following question would be useful for multi-fractal analysis.

Question 2.2. *Let $u: X \rightarrow \mathbb{R}$ be a strictly positive continuous function.*

- (1) *Let X be a totally bounded metric space and $f: X \rightarrow X$ be a continuous map. When $\mu \in \mathcal{M}_{\text{inv}}(X, f)$, if one has*

$$ent_{\mathbb{H}}^u(f; \mu) \geq \frac{h_{\mu}(f)}{\int_X u d\mu}?$$

- (2) *Let X be a punctured compact Riemannian manifold, and $f: X \rightarrow X$ be a continuous map. When $\mu \in \mathcal{M}_{\text{erg}}(X, f)$, if one has*

$$ent_{\mathbb{H}}^u(f; \mu) = \frac{h_{\mu}(f)}{\int_X u d\mu}?$$

In the case $u \equiv 1$ the answer of the above question 2.2(1) is positive [11]; for the general cases, we expect answers to these questions are positive.

3. Some relations between various entropies. In this section, we will consider some relations between various topological entropies.

Let X be a topological space, not necessarily compact. Let $f: X \rightarrow X$ be a continuous map and $Y \subseteq X$. Bowen [6] defined the topological entropy $h_{top}(f, Y)$ much like the Hausdorff dimension, with the “size” of a set reflecting how f acts on it rather than its diameter. Let \mathfrak{U} be a finite open cover of X . We write $E \prec \mathfrak{U}$ if E is contained in some member of \mathfrak{U} and $\{E_i\} \prec \mathfrak{U}$ if every $E_i \prec \mathfrak{U}$. Note that in this paper the symbol “ \prec ” does not mean the “refine”. Let $l_{f, \mathfrak{U}}(E)$ be the biggest nonnegative integer such that

$$f^k(E) \prec \mathfrak{U} \quad \forall k \in [0, l_{f, \mathfrak{U}}(E));$$

$l_{f, \mathfrak{U}}(E) = 0$ if $E \not\prec \mathfrak{U}$ and $l_{f, \mathfrak{U}}(E) = +\infty$ if all $f^k(E) \prec \mathfrak{U}$. Now set

$$\text{diam}_{\mathfrak{U}}^f(E) = \exp(-l_{f, \mathfrak{U}}(E)),$$

and then

$$D_{\mathfrak{U}}^f(\mathfrak{E}, \lambda) = \sum_{i=1}^{\infty} (\text{diam}_{\mathfrak{U}}^f(E_i))^\lambda$$

for any $\mathfrak{E} = \{E_i\}_1^\infty$ and for any $\lambda \in \mathbb{R}_+$. Define a measure $M_{\mathfrak{U}}^\lambda$ by

$$M_{\mathfrak{U}}^\lambda(Y) = \liminf_{\varepsilon \rightarrow 0} \{D_{\mathfrak{U}}^f(\mathfrak{E}, \lambda) \mid \cup E_i \supseteq Y, \text{diam}_{\mathfrak{U}}^f(E_i) < \varepsilon\}. \tag{10}$$

Define

$$h_{H, \mathfrak{U}}(f; Y) = \inf\{\lambda \in \mathbb{R}_+ \mid M_{\mathfrak{U}}^\lambda(Y) = 0\}, \tag{11}$$

and then Bowen’s *dimension entropy* of f restricted on $Y \subset X$ is given by

$$h_{top}(f; Y) = \sup_{\mathfrak{U}} \{h_{H, \mathfrak{U}}(f; Y)\} \tag{12}$$

where \mathfrak{U} ranges over all finite open covers of X . For $Y = X$ we write

$$h_{top}(f) = h_{top}(f; X). \tag{13}$$

Note that, one of the differences between the definitions of Bowen’s entropy h_{top} and distance entropy ent_H is that Bowen uses all finite open covers \mathfrak{U} of X , and in our definition covers are only by open ε -balls. Another difference is that in Eqs. (5) and (10). $\mathfrak{M}_\varepsilon^\lambda(Y)$ has the same ε used on the right side; $M_{\mathfrak{U}}^\lambda$ has no epsilon.

By the definition, if Y is a forward f -invariant closed subset of X , i.e., $f(Y) \subseteq Y$, then

$$h_{top}(f; Y) = h_{top}(f|_Y).$$

If X is compact, Bowen [6] proved that $h_{top}(f)$ equals the usual topological entropy $ent(f)$ defined by Adler-Konheim-McAndrew [1].

A metric space Z is said to satisfy *Lebesgue* (respectively, *finite covering property*) provided that for any (finite) open cover \mathfrak{U} of Z there is a *Lebesgue number* δ such that each subset of Z of diameter less than or equal to δ lies in some member of \mathfrak{U} . When Z is a compact metric space, it satisfies the Lebesgue covering property from the Lebesgue Covering Lemma. But the converse is not necessarily true. For example, let Z be an infinitely countable metric space with metric $d(x, y) = 0$ if $x = y$, 1 if $x \neq y$. Clearly, this space has the Lebesgue covering property but not compact. Note that this property is conceptually weaker than the compactness, but the question if there exists a non-discrete noncompact metric space which has the property is still open. A metric space is called *totally bounded* (or *precompact*) iff for any $\varepsilon > 0$ there is a finite cover which consists of Borel sets of diameter less than or equal to ε , see [22]. A space which can be isometrically embedded into

a compact metric space is totally bounded, such as an open rectangle in \mathbb{R}^k or a punctured Riemann surface. Clearly every totally bounded space is bounded, but it may not be true conversely, see [34, Example 134]. It is easily seen that a metric space is compact if and only if it is totally bounded and has the Lebesgue covering property.

We now consider the relation between h_{top} and ent_H .

Proposition 1. *Let X be a separable metric space and let $f: X \rightarrow X$ be a continuous map. Then the following statements hold.*

(1) *If X satisfies the Lebesgue finite covering property, then*

$$h_{top}(f; Y) \leq ent_H(f; Y) \quad (\forall Y \subseteq X).$$

(2) *If X is totally bounded, then*

$$h_{top}(f; Y) \geq ent_H(f; Y) \quad (\forall Y \subseteq X).$$

(3) *In particular, if X is compact, then*

$$h_{top}(f; Y) = ent_H(f; Y) \quad (\forall Y \subseteq X).$$

Proof. Let X satisfy the Lebesgue finite covering property. For any finite open cover \mathfrak{U} of X , let $\delta > 0$ be a Lebesgue number of \mathfrak{U} . Then for any $\varepsilon \leq \delta$ and any cover $\mathcal{E} = \{E_i\}_1^\infty$ of Y with $l_\varepsilon^f(E_i) > -\log \varepsilon$, we have

$$l_\varepsilon^f(E_i) \leq l_{f, \mathfrak{U}}(E_i).$$

Hence for any $\lambda \in \mathbb{R}_+$ we have

$$D_\varepsilon^f(\mathcal{E}, \lambda) \geq D_{\mathfrak{U}}^f(\mathcal{E}, \lambda).$$

This implies that

$$\mathfrak{M}_\varepsilon^\lambda(Y) \geq \inf\{D_{\mathfrak{U}}^f(\mathcal{E}, \lambda) \mid \cup E_i \supseteq Y, \text{diam}_{\mathfrak{U}}^f(E_i) < \varepsilon\}.$$

Letting $\varepsilon \rightarrow 0$ we obtain

$$\mathfrak{M}^\lambda(Y) \geq M_{\mathfrak{U}}^\lambda(Y).$$

This implies that $ent_H(f; Y) \geq h_{H, \mathfrak{U}}(f; Y)$ and so $h_{top}(f; Y) \leq ent_H(f; Y)$ from the arbitrary choice of \mathfrak{U} . This shows the statement (1).

We now assume that X is totally bounded. In the case where $h_{top}(f; Y) = \infty$ the statement (2) trivially holds. Now we assume $h_{top}(f; Y) < \infty$. Fix $\lambda > h_{top}(f; Y)$. For any given $\varepsilon > 0$, take finitely many balls $B_\varepsilon(x_1), \dots, B_\varepsilon(x_r)$ covering X . Write

$$\mathcal{A} = \{B_\varepsilon(x_1), \dots, B_\varepsilon(x_r)\}.$$

Since $\lambda > h_{H, \mathcal{A}}(f; Y)$, we have $M_{\mathcal{A}}^\lambda(Y) = 0$. From the fact $l_{f, \mathcal{A}}(E) \leq l_{2\varepsilon}^f(E)$ for $E \subset X$, we easily obtain

$$\lim_{\varepsilon \rightarrow 0} \mathfrak{M}_{2\varepsilon}^\lambda(Y) = 0. \tag{14}$$

In fact, for any $\varepsilon' < \varepsilon$, if $\mathcal{E} = \{E_i\}$ covering Y satisfies $\text{diam}_{\mathcal{A}}^f(E_i) < \varepsilon'$, then

$$D_{\mathcal{A}}^f(\mathcal{E}, \lambda') \geq D_{2\varepsilon}^f(\mathcal{E}, \lambda') \text{ and } l_{2\varepsilon}^f(E_i) \geq l_{f, \mathcal{A}}(E_i) > -\log \varepsilon' > -\log \varepsilon$$

for any $\lambda' \in \mathbb{R}_+$. This implies that

$$\inf\{D_{\mathcal{A}}^f(\mathcal{E}, \lambda) \mid \text{diam}_{\mathcal{A}}^f(E_i) < \varepsilon'\} \geq \mathfrak{M}_{2\varepsilon}^\lambda(Y)$$

and hence

$$\begin{aligned} 0 &= M_{\mathcal{A}}^\lambda(Y) = \liminf_{\varepsilon' \rightarrow 0} \{D_{\mathcal{A}}^f(\mathcal{E}, \lambda) \mid \text{diam}_{\mathcal{A}}^f(E_i) < \varepsilon'\} \\ &\geq \mathfrak{M}_{2\varepsilon}^\lambda(Y). \end{aligned}$$

Equation (14) implies that $\lambda > \text{ent}_{\mathbb{H}}(f; Y)$. So $h_{\mathbb{H}, \mathcal{A}}(f; Y) \geq \text{ent}_{\mathbb{H}}(f; Y)$ as desired.

The statement (3) follows from the statements (1) and (2).

The proof is thus finished. □

Corollary 3. *If X is a compact metrizable space then*

$$\text{ent}(f) = h_{\text{top}}(f) = \text{ent}_{\mathbb{H}}(f) = h(f).$$

Proof. The statement follows easily from Proposition 1, Corollary 1, and [6, Proposition 1]. □

From the proof of Proposition 1, one can see that the compactness condition is sharp for the equality. For a noncompact system such as the translation system in Example 1.1, the equality may not be true.

Let $\rho^{-1}: (0, 1) \rightarrow \mathbb{R}$ be given by

$$\rho^{-1}(x) = \begin{cases} \frac{x-1/2}{x}, & \text{if } 0 < x \leq 1/2, \\ \frac{x-1/2}{1-x}, & \text{if } 1/2 \leq x < 1. \end{cases}$$

Define $\hat{d}: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ in the following way

$$\hat{d}(x, y) = |\rho(x) - \rho(y)| \quad (\forall (x, y) \in \mathbb{R} \times \mathbb{R}).$$

It is clear that \hat{d} is a distance function which is equivalent to the usual euclidean metric on \mathbb{R} . We have the following

Proposition 2. *(\mathbb{R}_+, \hat{d}) is totally bounded. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the translation transformation given by $x \mapsto x + 1$ for all $x \in \mathbb{R}_+$. Then*

$$\infty = h_{\text{top}}(f) \not\cong \text{ent}_{\mathbb{H}, \hat{d}}(f) = 0.$$

Proof. Since \hat{d} is equivalent to the euclidean metric, $h_{\text{top}}(f) = \infty$. From Definition 2.1 it easily follows that $\text{ent}_{\mathbb{H}, \hat{d}}(f) = 0$, for f is contracting. □

Next we will show that the relation between $\text{ent}_{\mathbb{H}}(f; K)$ and $h(f; K)$ for compact $K \subset X$. The proof is almost identical with the first part of [6, Proposition 1].

Proposition 3. *Let X be a metric space with metric d , not necessarily compact, and let $f: X \rightarrow X$ be a continuous map, not necessarily uniformly. For any compact subset K of X , one has*

$$\text{ent}_{\mathbb{H}}(f; K) \leq h(f; K).$$

In particular, $\text{ent}_{\mathbb{H}}(f) \leq h(f)$ if X is locally compact.

Proof. For $\varepsilon > 0$ and $n \in \mathbb{N}$, write

$$B_\varepsilon(x; n) = \{y \in X \mid d(f^k x, f^k y) < \varepsilon, 0 \leq k < n\}.$$

For any given compact subset K of X , let $\hat{r}_n(\varepsilon, K)$ denote the smallest number of $B_\varepsilon(x; n)$ -balls with $x \in K$ needed to cover K . We set

$$\hat{r}(\varepsilon, K) = \limsup_{n \rightarrow \infty} n^{-1} \log \hat{r}_n(\varepsilon, K).$$

Furthermore by [30, Lemma 2.1] we have

$$h(f; K) = \lim_{\varepsilon \rightarrow 0} \hat{r}(\varepsilon, K).$$

Letting $\mathfrak{C}_n^\varepsilon$ be a subcover with $\hat{r}_n(\varepsilon, K)$ members, then

$$D_{2\varepsilon}^f(\mathfrak{C}_n^\varepsilon, \lambda) \leq \hat{r}_n(\varepsilon, K) \exp(-\lambda n)$$

and

$$\mathfrak{M}_{2\varepsilon}^\lambda(K) \leq [\exp(-\lambda + n^{-1} \log \hat{r}_n(\varepsilon, K))]^n.$$

Furthermore,

$$\mathfrak{M}_{2\varepsilon}^\lambda(K) \leq \limsup_{n \rightarrow \infty} [\exp(-\lambda + n^{-1} \log \hat{r}_n(\varepsilon, K))]^n.$$

For $\lambda > \hat{r}(\varepsilon, K)$ we get $\mathfrak{M}_{2\varepsilon}^\lambda(K) = 0$. Hence if $\lambda > \lim_{\varepsilon \rightarrow 0} \hat{r}(\varepsilon, K)$, then

$$\mathfrak{M}^\lambda(K) = \lim_{\varepsilon \rightarrow 0} \mathfrak{M}_{2\varepsilon}^\lambda(K) = 0.$$

Thus,

$$\text{ent}_{\mathbb{H}}(f; K) \leq h(f; K).$$

The second part comes immediately from the first part and the countable stability of $\text{ent}_{\mathbb{H}}$ -entropy.

The proof is thus completed. \square

Proposition 4. *Let X be a metric space and $f: X \rightarrow X$ be a continuous transformation. For any forwardly f -invariant compact subset K of X , one has*

$$\text{ent}_{\mathbb{H}}(f; K) = \text{ent}_{\mathbb{H}}(f|_K).$$

Proof. The result follows directly from the definition of the distance entropy. \square

4. Distance entropy and the Hausdorff dimension. As the general philosophy, an entropy should be a quantity to describe the complexity of a dynamical system (X, f) . Therefore, in the definition, one has to involve the iterations f^k of the continuous transformation f . This causes an essential difficulty for the estimation and computation of the entropy for a given dynamical system. Finding some simple but essential relationship of the entropy with other quantities, which are relatively easier for computation or estimation like dimension, is an interesting and significant problem. In this section, we will concentrate in this problem and consider some relations between the distance entropy and the Hausdorff dimension of the state space.

4.1. Lipschitz maps. In this subsection, we will consider Lipschitz systems. Let $f: X \rightarrow X$ be a continuous map of a metric space (X, d) . For any given subset $Y \subseteq X$, we say $f|_Y$ is of Lipschitz with a Lipschitzian constant L_Y if

$$d(f^{n+1}x, f^{n+1}y) \leq L_Y d(f^n x, f^n y) \quad (\forall n \geq 0, \forall x, y \in Y) \quad (15)$$

holds.

Then, we obtain the following

Theorem 4.1. *Let X be a separable metric space, not necessarily compact, and let $f: X \rightarrow X$ be a continuous transformation satisfying that $f|_Y$ is of Lipschitz with a Lipschitzian constant L_Y for $Y \subset X$. Then*

$$\text{ent}_{\mathbb{H}}(f; Y) \leq \max\{0, \text{HD}(Y) \log L_Y\}.$$

Proof. Let $Y \subset X$ and $L > 0$ be given as in the assumptions of the statement. In the case where $L \leq 1$, $ent_{\mathbb{H}}(f; Y) = 0$ by the definition, so there is nothing needed to prove. We now assume $L > 1$. Let $|B| = \text{diam}(B)$ for any $B \subset Y$.

Let $\varepsilon > 0$. If $B \subset X$ and

$$\varepsilon/L^n \leq |B| < \varepsilon/L^{n-1} \tag{16}$$

then $|f^k(B)| < \varepsilon$ for $k = 0, 1, \dots, n - 1$ since $L > 1$, so $l_{\varepsilon}^f(B) \geq n$. Hence from Eq. (16) we have

$$\frac{\log \varepsilon - \log |B|}{\log L} \leq n \leq l_{\varepsilon}^f(B). \tag{17}$$

Rewrite the above inequality as

$$\text{diam}_{\varepsilon}^f(B) \leq e^{-c}|B|^{1/\log L}, \tag{18}$$

where $c = \log \varepsilon / \log L$. Therefore if $\mathcal{B} = \{B_i\}_1^{\infty}$ is a cover of Y with $B_i \subset Y$ then for any $\lambda \geq 0$ we have

$$D_{\varepsilon}^f(\mathcal{B}, \lambda) \leq \exp(-c\lambda) \sum_i |B_i|^{\lambda/\log L}. \tag{19}$$

Recall that for $s \geq 0$ the s -Hausdorff measure of Y is given by

$$\mathcal{H}^s(Y) = \liminf_{\delta \rightarrow 0} \left\{ \sum_i |B_i|^s : \bigcup_i B_i \supseteq Y \text{ and } \sup_i \{|B_i|\} < \delta \right\}. \tag{20}$$

Fix arbitrarily $\lambda > \text{HD}(Y) \log L$, namely $\lambda/\log L > \text{HD}(Y)$. Then the $\lambda/\log L$ -Hausdorff measure of Y is $\mathcal{H}^{\lambda/\log L}(Y) = 0$. So for every $\varepsilon > 0$ with $\varepsilon \leq \varepsilon^{\log L \varepsilon}$ there is a cover $\mathcal{B} = \{B_i\}_1^{\infty}$ of Y with $B_i \subseteq Y$ such that

$$\sup_i \{|B_i|\} < \frac{\varepsilon}{L^{-\log \varepsilon - 1}} \tag{21}$$

and

$$\exp(-c\lambda) \sum_i |B_i|^{\lambda/\log L} < \varepsilon, \tag{22}$$

since $\inf\{\cdot\} \uparrow 0$ as $\delta \downarrow 0$ in Eq. (20). For this \mathcal{B} , we then get $l_{\varepsilon}^f(B_i) \geq -\log \varepsilon$ by Eqs. (21, 17), and $D_{\varepsilon}^f(\mathcal{B}, \lambda) < \varepsilon$ by Eq. (19). Hence $\mathfrak{M}_{\varepsilon}^{\lambda}(Y) = 0$ as $\varepsilon \rightarrow 0$, and moreover letting $\varepsilon \rightarrow 0$ we obtain $ent_{\mathbb{H}}(f; Y) \leq \lambda$ whenever $\lambda > \text{HD}(Y) \log L$, so $ent_{\mathbb{H}}(f; Y) \leq \text{HD}(Y) \log L$.

The proof of Theorem 4.1 is thus complete. □

In particular, we have from Theorem 4.1

Corollary 4. *Let X be a separable metric space not necessarily compact, and let $f: X \rightarrow X$ be of Lipschitz with a Lipschitzian constant L . Then*

$$ent_{\mathbb{H}}(f; Y) \leq \max\{0, \text{HD}(Y) \log L\} \quad (\forall Y \subseteq X).$$

Remark 1. Let $f: X \rightarrow X$ be a continuous transformation on the metric space X .

- (1) As we have seen from Proposition 1(3), when X is compact we have $h_{top}(f; Y) = ent_{\mathbb{H}}(f; Y)$. So [26, Theorem 2.1] now follows from Theorem 4.1.
- (2) Since a Lipschitz map is uniformly continuous, from Theorem 2.2(1) it follows that [26, Remark 2.4] is still valid in our cases now.

The inequality of Theorem 4.1 can give us following result.

Corollary 5. *Let $f: M^m \rightarrow M^m$ be a differentiable map of an m -dimensional riemannian manifold. Then*

$$\text{ent}_{\text{H}}(f) \leq \max \left\{ 0, m \log \sup_{x \in M} \{ \|df|T_x M\| \} \right\}.$$

Proof. Take $L = \sup_{x \in M} \|df|T_x M\|$ and since $m = \text{HD}(M)$, we obtain the statement from Theorem 4.1. \square

4.2. Expanding systems. Next we will consider the upper bound of the Hausdorff dimension of the phase space by the distance entropy.

Theorem 4.2. *Let (X, d) be a separable metric space not necessarily compact, and let $f: X \rightarrow X$ be expanding with skewness $\lambda > 1$; namely $d(fx, fy) \geq \lambda d(x, y)$ for any $x, y \in X$. Then*

$$\text{HD}(Y) \log \lambda \leq \text{ent}_{\text{H}}(f; Y) \quad (\forall Y \subset X).$$

Proof. Let $Y \subset X$ be given and $\varepsilon > 0$. If $B \subset X$ and $l_{\varepsilon}^f(B) > n$, then $|f^k(B)| < \varepsilon$ for $k = 0, 1, \dots, n$. This implies that $d(f^n x, f^n y) \leq \varepsilon$ for any pair $x, y \in B$ and $\lambda^n d(x, y) \leq 2\varepsilon$ for any $x, y \in B$. We then obtain $|B| \leq \varepsilon/\lambda^n$ if $l_{\varepsilon}^f(B) > n$. Hence

$$\lambda^{l_{\varepsilon}^f(B)-1} \leq \frac{\varepsilon}{|B|}. \quad (23)$$

Equivalently,

$$l_{\varepsilon}^f(B) \leq b - \frac{\log |B|}{\log \lambda} \quad (24)$$

where $b = 1 + \frac{\log 2\varepsilon}{\log \lambda}$. Therefore, if $\mathcal{B} = \{B_i\}_1^{\infty}$ is a cover of Y then for any $\delta \geq 0$ we have

$$e^{b\delta} D_{\varepsilon}^f(\mathcal{B}, \delta) \geq \sum_i |B_i|^{\delta/\log \lambda}. \quad (25)$$

If $\text{ent}_{\text{H}}(f; Y) < \infty$, we take $\delta \log \lambda > \text{ent}_{\text{H}}(f; Y)$, i.e., $\delta > \text{ent}_{\text{H}}(f; Y)/\log \lambda$. From the definition of distance entropy, we can, for any small $\epsilon > 0$ ($\epsilon \ll \varepsilon$), choose a cover $\mathcal{B} = \{B_i\}$ of Y such that

$$\sup_i \{ |B_i| \} < \epsilon \quad \text{and} \quad \exp(b\delta \log \lambda) D_{\varepsilon}^f(\mathcal{B}, \delta \log \lambda) < \epsilon. \quad (26)$$

Letting $\varepsilon \rightarrow 0$ we obtain $\mathcal{H}^{\delta}(Y) = 0$ and hence $\text{HD}(Y) \leq \text{ent}_{\text{H}}(f; Y)/\log \lambda$, as desired.

We thus prove the statement. \square

Corollary 6. *If $f: M^m \rightarrow M^m$ is an expanding differentiable map of a riemannian manifold of dimension m , then*

$$\text{ent}_{\text{H}}(f) \geq m \log \inf_{x \in M} \{ \|df|T_x M\|_{\text{co}} \}.$$

Proof. Take $\lambda = \inf_{x \in M} \{ \|df|T_x M\|_{\text{co}} \}$. Since $\text{HD}(M) = m$ (see [14]), the statement follows immediately from Theorem 4.2. \square

Remark 2. Since $\text{ent}_{\text{H}}(f^n; Y) \leq n \cdot \text{ent}_{\text{H}}(f; Y)$ (see Theorem 2.2), as in the case of Lipschitz maps, one can improve Theorem 4.2 by replacing $\log \lambda$ by $\sup\{n^{-1} \log \lambda_n : n = 1, 2, \dots\}$ where λ_n is the expanding coefficient for f^n . Moreover, we can replace λ by the local expanding skewness too.

4.3. Applications. We now conclude this section with the following simple applications.

Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be given by $f(z) = e^z$, where \mathbb{C} is the complex plane with the standard metric d . It is well known that the exponential map $f(z)$ has positive-measure Julia set and very complex dynamical behavior.

Corollary 7. *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be given by $f(z) = e^z$. Then, $ent_{\mathbb{H}}(f) = \infty$.*

Proof. In fact, the subsystem $f|_{[t, \infty)}: [t, \infty) \rightarrow [t, \infty)$, $t \geq 0$ satisfies from Corollary 6 the following inequality

$$t = \text{HD}([t, \infty)) \log e^t \leq ent_{\mathbb{H}}(f; [t, \infty)) \leq ent_{\mathbb{H}}(f).$$

As $t \rightarrow \infty$ we obtain $ent_{\mathbb{H}}(f) = \infty$. □

Corollary 8. *Let $\alpha > 1$. Define*

$$f: \mathbb{R}^k \rightarrow \mathbb{R}^k$$

by $(x_1, \dots, x_k) \mapsto \alpha(x_1, \dots, x_k)$, where $k \in \mathbb{N}$ and \mathbb{R}^k with the usual euclidean metric. Then $ent_{\mathbb{H}}(f) = k \log \alpha$.

Proof. The statement follows easily from Theorems 4.1 and 4.2. □

Let $\Sigma_k^+ = \{0, 1, \dots, k - 1\}^{\mathbb{N}}$ be the one-sided symbolic space of k letters, $k \geq 2$, with the standard metric

$$d(x, x') = \alpha^{-n(x, x')} \quad (\forall x = (i_1, i_2, \dots), x' = (i'_1, i'_2, \dots) \in \Sigma_k^+), \tag{27}$$

where $\alpha > 1$ is a given constant and

$$n(x, x') = \begin{cases} \infty & \text{if } x = x', \\ \min\{\ell: i_\ell \neq i'_\ell\} & \text{if } x \neq x'. \end{cases} \tag{28}$$

Let

$$\sigma: \Sigma_k^+ \rightarrow \Sigma_k^+ \tag{29}$$

be the one-sided shift given by $(i_1, i_2, i_3, \dots) \mapsto (i_2, i_3, \dots) \forall x = (i_1, i_2, \dots) \in \Sigma_k^+$.

Corollary 9. *Let $\sigma: \Sigma_k^+ \rightarrow \Sigma_k^+$ be the one-sided shift. Then, for any $Y \subset \Sigma_k^+$ one has*

$$ent_{\mathbb{H}}(\sigma; Y) = h_{top}(\sigma; Y) = \text{HD}(Y) \log \alpha$$

Proof. The statement follows easily from Theorems 4.1 and 4.2 and Remark 2. □

Corollary 10. *Let $\sigma: \Sigma_k^+ \rightarrow \Sigma_k^+$ be the one-sided shift. Let*

$$G = \{x \in \Sigma_k^+ \mid \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\sigma^i(x)) = \varphi^*(x) \quad \forall \varphi \in C^0(\Sigma_k^+, \mathbb{R})\}.$$

Then $\text{HD}(G) = \text{HD}(\Sigma_k^+) = \log k / \log \alpha$.

Proof. As $ent_{\mathbb{H}}(\sigma; G) = h_{top}(\sigma; G) = h(\sigma)$ by [6, Theorem 2], the statement comes from Corollary 9. □

More generally, we have the following

Corollary 11. *Let $\Sigma_\infty^+ = \{x = (i_1, i_2, \dots, i_\ell, \dots) \mid i_\ell \in \mathbb{N}\}$ be the symbolic space of infinite letters with the metric as in Eq. (27). Let*

$$\sigma: \Sigma_\infty^+ \rightarrow \Sigma_\infty^+$$

be the one-sided shift as in Eq. (29). Then

$$\text{ent}_H(\sigma; Y) = \text{HD}(Y) \log \alpha \quad \forall Y \subset \Sigma_\infty^+$$

Proof. The statement follows easily from Theorems 4.1 and 4.2. □

The following simple result is related to [40, Open problem 3]:

Proposition 5. *Let $\sigma: \Sigma_\ell^+ \rightarrow \Sigma_\ell^+$ be the one-sided shift of the symbolic space of ℓ letters, where $2 \leq \ell < \infty$ or $\ell = \infty$. Then for any subset $Y \subset \Sigma_\ell^+$,*

$$\text{ent}_H(\sigma; Y) = 0 \Leftrightarrow \text{HD}(Y) = 0.$$

Proof. The statement immediately follows from Corollaries 9 and 11. □

Let (X, T) be a compact system on a compact metric space (X, d) . Let $\varphi: X \rightarrow \mathbb{R}^d$ be an arbitrarily given vector-valued measurable function, $d \geq 1$, called a displacement function.

A point $x \in X$ is called φ -directional of direction $\vec{v} \in \mathbb{R}^d$ (see [19]), provided that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i(y)) = \vec{v} \quad (\forall y \in \overline{\text{Orb}_T^+(x)}).$$

Write

$$\begin{aligned} D_\varphi(\vec{v}) &= \{x \in X \mid x \text{ is } \varphi\text{-directional of direction } \vec{v}\} \quad (\vec{v} \in \mathbb{R}^d), \\ D_\varphi &= \{x \in X \mid x \text{ is } \varphi\text{-directional}\}, \end{aligned}$$

called the *directional set* of (X, T, φ, \vec{v}) and (X, T, φ) respectively. For an ergodic probability measure μ of (X, T) , if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i(x)) = \int_X \varphi d\mu (= \vec{v}) \quad (\forall x \in \text{supp}(\mu)),$$

it is called φ -directional of direction \vec{v} [19]. For any $\vec{v} \in \mathbb{R}^d$ let

$$M_\varphi(\vec{v}) = \{\mu \text{ is a } \varphi\text{-directional ergodic measure of direction } \vec{v}\}.$$

Clearly, if μ is φ -directional of direction \vec{v} , we have $\text{supp}(\mu) \subseteq D_\varphi(\vec{v})$.

We next give a geometric description of the directional sets.

Proposition 6. *Let (X, T) be a compact subsystem of (Σ_k^+, σ) , not necessarily subshift of finite type, where $2 \leq k < \infty$. Then, for any given measurable function $\varphi: X \rightarrow \mathbb{R}^d$ the following equality holds.*

$$\text{HD}(D_\varphi(\vec{v})) = \sup\{\text{HD}(\mu) \mid \mu \in M_\varphi(\vec{v})\} \quad (\forall \vec{v} \in \mathbb{R}^d).$$

Proof. From an argument similar to that of [19, Proposition 4.1], it follows that

$$\text{ent}_H(T, D_\varphi(\vec{v})) = \sup_{\mu \in M_\varphi(\vec{v})} h_\mu(T). \tag{30}$$

As μ is ergodic, it follows by [6, Theorems 1 and 3] that

$$\text{ent}_H(T, Y) = h_\mu(T) \quad (\forall Y \subseteq G(\mu), \mu(Y) = 1),$$

where $G(\mu)$ is generic point set of μ . Thus $\text{HD}(\mu) = h_\mu(T)/\log \alpha$ and further by Eq. (30) we have $\text{HD}(D_\varphi(\vec{v})) = \sup\{\text{HD}(\mu) \mid \mu \in M_\varphi(\vec{v})\}$. This shows the statement. \square

Proposition 7. *If (X, T) is a subshift of finite type and $\varphi \in C^0(X, \mathbb{R}^d)$, then*

$$\text{HD}(D_\varphi) = \text{HD}(\Sigma_k^+).$$

Proof. For any φ -directional ergodic measure μ , by [6, Theorem 1] we have

$$\text{ent}_H(T; D_\varphi) \geq h_\mu(T). \tag{31}$$

On the other hand, from [19, Corollary 5.3] it follows that

$$h(T) = \sup\{h_\mu(T) \mid \mu \text{ is } \varphi\text{-directional ergodic}\}. \tag{32}$$

Equation (31) together with Eq. (32) implies that $\text{ent}_H(T; D_\varphi) = h(T)$.

Therefore, by Corollary 9 the statement holds. \square

5. Entropy of pointwise periodic maps. Let X be a topological space. A continuous map $T: X \rightarrow X$ is said to be *pointwise periodic* if for each $x \in X$, there is a positive integer $n(x)$ such that $T^{n(x)}(x) = x$. Clearly, every such a map T is bijective.

If X is a compact metric space, then $\text{ent}(T) = h(T) = 0$ by the well-known Dinaburg-Goodwyn-Goodman variational principle between the topological entropy and the measure-theoretic entropy [36, Theorem 8.6]. Z.-L. Zhou [38] has asked if there exists a topological proof for the statement. The difficult point by the topological method to prove is that the function $x \mapsto n(x)$ is not continuous. By using the distance entropy and its countable stability, we can provide a topological proof. Actually, we prove a more general result that is even true for a pointwise periodic continuous map on a noncompact metric space.

Proposition 8. *Let X be a separable metric space not necessarily compact. If the map $T: X \rightarrow X$ is continuous and pointwise periodic then we get $\text{ent}_H(T) = 0$.*

Proof. Write

$$X_i = \{x \in X \mid T^i(x) = x, T^k(x) \neq x \quad \forall k \in [1, i)\} \quad (i = 1, 2, \dots).$$

Clearly, $X = \bigcup_i X_i$. From Theorem 2.1, we have

$$\text{ent}_H(T) = \sup_i \{\text{ent}_H(T; X_i)\}.$$

Next we need only to prove $\text{ent}_H(T; X_i) = 0$ for all i .

If $T: X \rightarrow X$ is uniformly continuous, Theorem 2.2 implies

$$0 = \text{ent}_H(T^i; X_i) = i \cdot \text{ent}_H(T; X_i).$$

Otherwise, we take $\varepsilon > 0$ so that $\varepsilon < \exp(-i - 1)$. If $\mathcal{E} = \{E_k\}_1^\infty$ is a cover of X_i such that $E_k \subset X_i$ and $\text{diam}_\varepsilon^T(E_k) < \varepsilon$ for any k , then $l_\varepsilon^T(E_k) \geq i + 1$. So by the periodic property of T , $l_\varepsilon^T(E_k) = \infty$ and $\text{ent}_H(T; X_i) = 0$. \square

Proposition 9. *Let X be a topological space, not necessarily compact. If the continuous map $T: X \rightarrow X$ is pointwise periodic then $h_{\text{top}}(T) = 0$.*

Proof. The result follows easily from the statements (c) and (d) of [6, Proposition 2] (even [6] contains no proof, but by an argument similar to that of ent_H , we can prove it). \square

Note that in general $h_{\text{top}}(T) \neq \text{ent}_{\text{H}}(T)$ for a noncompact topological system (X, T) , so Proposition 8 cannot be deduced from Proposition 9. We now conclude this section with the following question.

Question 5.1. *Let X be a metric space, not necessarily satisfying the second countable axiom. If $T: X \rightarrow X$ is a uniformly continuous, pointwise periodic map, is $h(T) = 0$?*

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E-mail address: xpdai@nju.edu.cn

E-mail address: Yunping.Jiang@qc.cuny.edu