FUNCTION MODEL OF THE TEICHMÜLLER SPACE OF A CLOSED HYPERBOLIC RIEMANN SURFACE

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Abstract. We introduce a function model for the Teichmüller space of a closed hyperbolic Riemann surface. Then we introduce a new metric by using the maximum norm on the function space on the Teichmüller space. We prove that the identity map from the Teichmüller space equipped with the usual Teichmüller metric to the Teichmüller space equipped with this new metric is uniformly continuous. Furthermore, we also prove that the inverse of the identity, that is, the identity map from the Teichmüller space equipped with this new metric to the Teichmüller space equipped with the usual Teichmüller metric, is continuous. Therefore, the topology induced by the new metric is just the same as the topology induced by the usual Teichmüller metric on the Teichmüller space. We give a remark about the pressure metric and the Weil-Petersson metric.

1. **Introduction**

A closed Riemann surface is a compact connected complex one-dimensional surface. We only consider an oriented surface. A topological characterization of a closed Riemann surface is its genus g. Riemann observed that all genus $g = 0$ closed Riemann surfaces are conformally equivalent to the standard Riemann sphere $\mathbb{P}^1$. However, this is not in general true for closed Riemann surfaces of positive genus. Suppose R is a closed Riemann surface of genus $g \geq 2$. Then it is hyperbolic and conformally equivalent to the open unit disk modulo a Fuchsian group. A marked Riemann surface by R is a pair (X, h) where $h : R \to X$ is an orientation-preserving homeomorphism. In the space of all marked Riemann surfaces (X, h) by R, one can introduces a conformal equivalence relation. This space modulo this equivalence relation is called the Techmüller space $T(R)$. In other words, $T(R)$ is the quotient space of all complex structures on R by those orientation-preserving diffeomorphisms which are isotopic to the identity. We know that

a) $T(R)$ is homeomorphic to $\mathbb{R}^{6g-6}$,

b) $T(R)$ admits a complex manifold structure of $3g - 3$,

2000 *Mathematics Subject Classification.* Primary 37F99, Secondary 32H02.

c) $T(R)$ can be embedded into $\mathbb{C}^{3g-3}$ as a contractible set, and
d) $T(R)$ is a pseudoconvex domain.

The Teichmüller space $T(R)$ is an important subject in the modern mathematics and physics. It is a cover of the moduli space which is the space of all complex structures on R modulo the action of orientation-preserving diffeomorphisms. The moduli space has the same dimension $6g-6$. The moduli space was first considered by Riemann and plays an important role in the modern string theory.

To understand the Teichmüller space $T(R)$, several models have been introduced. For examples, we have Bers' embedding and Thurston's embedding. Furthermore, several metrics have been introduced on the Teichmüller space $T(R)$. The first metric $d_T(\cdot,\cdot)$ is introduced by Teichmüller. There are other metrics, for examples, the Kobayashi metric and the Weil-Petersson metric. Royden proved that the Teichmüller metric is equal to the Kobayashi metric in this case and Gardiner even generalized this result for any Riemann surfaces of infinite analytic type (refer to [6] for a proof and furthermore references).

In this paper, we introduce a new model of the Teichmüller space from the dynamical system point of views. This new model is a space of functions defined on a Cantor set Σ_A^*. The graphs of these functions are in the infinite-dimenisonal cube $\prod_0^\infty (0,1)$ of $\mathbb{R}^\infty$. Therefore, we have the maximum norm on the function space. This maximum norm introduces a maximum metric $d_{max}(\cdot,\cdot)$ on the Teichmüller space $T(R)$. We prove that the identity map from the Teichmüller space equipped with the usual Teichmüller metric to the Teichmüller space equipped with this new metric is uniformly continuous (see Theorem 5). Furthermore, we also prove that the identity map from the Teichmüller space equipped with this new metric to the Teichmüller space equipped with the Teichmüller metric is continuous (see Theorem 6). Therefore, the topology induced by the new metric is just the same as the topology induced by the usual Teichmüller metric on the Teichmüller space.

The paper is organized as follows. In §2, we define and review the Teichmüller space $T(R)$ of a closed hyperbolic Riemann surface R and mention a theorem due to Earle and McMullen which we will used in this paper (see Theorem 1). In §3, we use the Nielsen development for Fuchsian groups to construct expanding transitive Markov maps for any marked Riemann surfaces by a standard closed hyperbolic Riemann surface. We use Bowen's paper [2] and Bowen and Series' paper [3] as two references. In §4, we define the symbolic space Σ_A and the dual symbolic space Σ_A^* for all marked Riemann surfaces by a fixed standard closed hyperbolic Riemann surface. The symbolic space Σ_A is treated as the topological model of all such marked Riemann surfaces. We define geometric models on the dual symbolic space Σ_A^* for all marked Riemann surfaces by a fixed standard closed hyperbolic Riemann surface in §4.2. The geometric models are Lipschitz continuous functions defined on Σ_A^*. Their graphs are contained in the infinite-dimensional cube $\prod_0^\infty (0,1)$. To prove these functions are geometric models, we mention Tukia's theorem which is a stronger version than Mostow's rigidity theorem in 2-dimensional case. For the sake of completeness of the paper, we give a proof of Tukia's theorem from the dynamical system point of views. We call each geometric model a scaling function. We use $\mathcal{F}$ to denote the space of all scaling functions. In §5, we prove that there is a one-to-one and onto maps between the Teichmüller space and the function space $\mathcal{F}$. In §6, we discuss Bers' embedding and the complex manifold structure on $\mathcal{F}$. Using the maximum norm on the function space $\mathcal{F}$, we define the maximum metric $d_{max}(\cdot,\cdot)$ on the Teichmüller space in §7. We prove, in the same section, Theorem 5, that is, the identity map from the Teichmüller space equipped with the usual Teichmüller metric to the Teichmüller space equipped with this new metric is uniformly continuous. Furthermore, we prove, in the same section, Theorem 6, that is, that the identity map from the Teichmüller space equipped with this new metric to the Teichmüller space equipped with the Teichmüller metric is continuous. Finally, in §8, we give a remark to compare our function model and McMullen's thermodynamical embedding in his recent paper [16]. Furthermore, by following McMullen's calculation of the Weil-Petersson metric on the tangent space of the Teichmüller space by the pressure metric, we show that the pressure metric of the tangent vector to any smooth curve at a point in our function model is a constant times the Weil-Petersson metric.

Acknowledgement: This work is partially done when I visited the Institut des Hautes Études Scientifiques in Bures-sur-Yvette, France and when I visited the Academy of Mathematics and System Science and the Morningside Center of Mathematics at the Chinese Academy of Sciences in Beijing, China. I would like to thank these institutions for their hospitality. This work is partially supported by NSF grants, PSC-CUNY awards, and the Bai Ren Ji Hua of Chinese Academy of Sciences. I would like to thank Professors Fred Gardiner and Bill Harvey for many interesting conversations. I would like to thank Professor Curt McuMullen for his useful comments and suggestions for the first version

of this paper and for pointing to me his recent paper [16] which leads to §8.

2. **Teichmüller Space of a closed hyperbolic Riemann surface**

We first discuss the Teichmüller space of a closed hyperbolic Riemann surface R. Suppose $\mathbb{D}$ is the unit disk and suppose $S^1 = \partial\mathbb{D}$ is the unit circle. Let $R = \mathbb{D}/\Gamma$ be a closed hyperbolic Riemann surface, presented as the quotient of the unit disk by a Fuchsian group whose limit set is the whole S^1. Any quasiconformal map $h : R \to R$ can be lift to a quasiconformal map $H : \mathbb{D} \to \mathbb{D}$. The map H can be extended to a homeomorphism of $\overline{\mathbb{D}} = \mathbb{D} \cup S^1$. The key in defining the Teichmüller space is to know which quasiconformal maps $h : R \to R$ are to be considered trivial. The following theorem gives an answer to this question.

Theorem 1 (Earle-McMullen [5])**.** *Suppose $h : R \to R$ is a quasiconformal map. The following are equivalent.*

1) *There is a lift of h to a map $H : \mathbb{D} \to \mathbb{D}$ that extends to the identity on S^1.*
2) *The map h is homotopic to the identity rel ideal boundary (in this case the ideal boundary is empty).*
3) *The map h is isotopic to the identity rel ideal boundary, through uniformly quasiconformal maps.*

A marked Riemann surface by R is a pair (X, h_X) where $h_X : R \to X$ is an orientation preserving quasiconformal homeomorphism. Two marked Riemann surfaces (X, h_X) and (Y, h_Y) are equivalent if there is a conformal isomorphism $\alpha : X \to Y$ such that

$$f = h_Y^{-1} \circ \alpha \circ h_X : R \to R$$

is isotopic to the identity. The Teichmüller space $T(R)$ of R is the space of equivalence classes $[(X, h_X)]$ of all marked Riemann surfaces (X, h_X) by R, that is,

$$T(R) = \{[(X, h_X)]\}.$$

3. **Nielsen development, Markov partition, and expanding transitive Markov map**

Suppose X is a closed hyperbolic Riemann surface. Let $g \geq 2$ be the genus of X. Since its universal cover is the unit disk $\mathbb{D}$, so through the universal cover, we can write $X = \mathbb{D}/\Gamma_X$ where Γ_X is a Fuchsian group whose limit set is the whole S^1.

Definition 1. A piecewise smooth map $f : S^1 \to S^1$ is called Markov for Γ_X if we can cut S^1 into finitely many intervals $I_1, \cdots, I_k$ such that

i) $S^1 = \cup_{i=1}^k I_i$,
ii) I_i and I_j have disjoint interiors for any $1 \leq i \neq j \leq k$,
iii) $f|I_i = \gamma_i|I_i$ for some $\gamma_i \in \Gamma_X$, and
iv) $f(I_j)$ is the union of some intervals of I_i's for each $1 \leq j \leq k$.

Let $W = \cup_{i=1}^n \partial I_i$. The iv) is equivalent to the statement that

$$f(W) \subset W.$$

For any $1 \leq i, j \leq k$, we write $i \to j$ if $f(I_i) \supset I_j$.

Definition 2. A Markov map $f : S^1 \to S^1$ for the surface group Γ_X is called transitive if for any $1 \leq i, j \leq k$, there are $1 \leq i_0, i_1, \cdots, i_n \leq k$ such that

$$i = i_0 \to i_1 \to \cdots \to i_n = j.$$

This is equivalent to say that $f^n(I_i) \supset I_j$

Definition 3. A Markov map f for Γ_X is called expanding if there are two constants $C > 0$ and $\lambda > 1$ such that

$$|(f^n)'(x)| \geq C\lambda^n$$

for $x \in I_i$ and $n \geq 0$.

Suppose $X_0 = \mathbb{D}/\Gamma_0$ is the closed Riemann surface of genus g such that the fundamental domain D of X_0 in $\mathbb{D}$ for $\Gamma_0 = \Gamma_{X_0}$ is a regular $4g$-sided non-Euclidean polygon. We call X_0 the standard Riemann surface of genus g.

Each angle of D is $\frac{\pi}{2g}$. Each vertex of D belongs to $4g$ distinct translations $\gamma(D)$ for $\gamma \in \Gamma_0$. All $\gamma(D)$ for $\gamma \in \Gamma_0$ form a net $\mathcal{R}$ in $\mathbb{D}$. The net $\mathcal{R}$ has the following property: the entire non-Euclidean geodesic passing through any edge in the net $\mathcal{R}$ is contained in the net $\mathcal{R}$. Let V_0 be the set of vertices of D. Let V be the set of vertices in the net $\mathcal{R}$ which are adjacent in $\mathcal{R}$ to V_0 but not in V_0. Consider all polygons $\tilde{D}$ adjacent to D. Then V are all vertices of $\tilde{D}$ minus V_0. For each vertex p of $\mathcal{R}$, there are $2g$ non-Euclidean geodesic passing through it. These $2g$ non-Euclidean geodesics have $4g$ endpoints at infinity. Let W_p be the set of these $4g$ points at infinity. Define

$$W = \cup_{p \in V} W_p.$$

Then $W_q \subset W$ for any vertex q of D.

The $4g$ sides of D give a set of generators for Γ_0 as follows. Divide the sides of D into g groups of 4 consecutive sides; label the j^{th} group

$a_j, b_j, a_j^{-1}, b_j^{-1}$. Call a_j and a_j^{-1} and b_j and b_j^{-1} corresponding sides. For each side s of D there is an element $\gamma_s \in \Gamma_0$ such that

$$\gamma_s(s) = D \cap \phi_s(D) = \text{side corresponding to } s.$$

The set $\{\gamma_s\}$ generates Γ_0. The non-Euclidean geodesic passing s cuts S^1 into two intervals. Let J_s be the smaller one. Then we have that

$$\gamma_s(J_s \cap W) \subset W.$$

Let $v \in V_0$. Then we have two sides s and s' belonging to v. Let β and β' are two non-Euclidean geodesic passing trough s and s', respectively. Consider the interval $J_s \cap J_{s'}$. Let $p \in \beta$, $p' \in \beta'$ be the vertices of the net $\mathcal{R}$ adjacent to v in the net. Let $\gamma(D)$ be the translation of D having v, p, and p' as its vertices for some $\gamma \in \Gamma_0$. Let q, q' are vertices of $\gamma(D)$ such that $q, p, v, p'q'$ are consecutive vertices of $\gamma(D)$. The non-Euclidean geodesics δ, δ' passing p, q and p', q', respectively, do not intersect and have points $w(v)$ and $w'(v)$ at infinity in the interior of $J_s \cap J_{s'}$. Let $J(v) = [w(v), w'(v)]$ in the interior of $J_s \cap J_{s'}$. Note that $w(v), w'(v) \in W$ and $J(v)$ does not contain any other points from W.

The set W cuts S^1 into finitely many intervals $I_{1,0}$, $\cdots$, $I_{k,0}$. Define the map $f_0 : S^1 \to S^1$ as

$$f_0 | I_{j,0} = \gamma_s | I_{j,0}, \quad J_s \supset I_{j,0}.$$

Then f_0 is a piecewise Möbius transformations and is Markov since $f_0(W) \subset W$.

Since some $I_{j,0}$'s belong to more than one J_s, there are a number of ways to define f_0. This flexibility allows us to eventually get an expanding Markov map. Given a vertex v of D. Let s be a side of D having v as their common vertex. Let v' be the other vertex of s. We assume that from v' to v are clockwise. Suppose $\tilde{J}_s$ is the maximal interval where

$$f_0 | \tilde{J}_s = \gamma_s | \tilde{J}_s.$$

We require that

a) $\mathrm{int} J_s \supset \tilde{J}_s \supset J_s \setminus (J(v) \cup J(v'))$, and
b) $J(v) \subset \tilde{J}_s$ and $\tilde{J}_s$ disjoint with the interior $\mathrm{int} J(v')$.

In other words, suppose J_s is cut by W into intervals clockwise $I_{1,0}$, $I_{2,0} = J(v')$, $I_{3,0}$, $\cdots$, $I_{l_s-1,0} = J(v)$, and $I_{l_s,0}$. So we define $\tilde{J}_s = \cup_{j=3}^{l_s-1} I_{j,0}$.

For each non-Eucildean geodesic β passing s, it is on the isometric circle C_s of γ_s, i.e., $|\gamma_s'(x)| = 1$ for $x \in \beta$. Inside this circle, $|\gamma_s'(x)| > 1$ and outside this circle, $|\gamma_s'(x)| < 1$. Since $\tilde{J}_s$ is inside this circle, so

$|f_0'(x)| > 1$ for $x \in \tilde{J}_s$. Since there are finitely many $\tilde{J}_s$ and each one is a compact interval, so there is a constant $\lambda_0 > 1$, such that

$$|f_0'(x)| \geq \lambda_0, \quad \forall x \in I_{j,0}, \quad 1 \leq j \leq k.$$

This implies that f_0 is an expanding Markov map for Γ_0 with the Markov partition

$$\eta_{0,0} = \{I_{1,0}, \cdots, I_{k,0}\}.$$

Now we prove that the Markov map f_0 is transitive. First, for each $1 \leq i \leq k$, there is some iterate $f^n(I_i)$ contains $J(v)$ for some vertex v of D. This is because that otherwise, f is continuous on $f^n(I_i)$ and $f^{n+1}(I_i)$ is an interval longer than $f^n(I_i)$ because of the expanding condition. But this can not continuous indefinitely.

From $\eta_{0,0}$, we can generate a sequence of Markov partitions

$$\eta_{n,0} = f_0^{-n}\eta_{0,0}$$

for $n = 0, 1, \cdots$. (See §4.1 for more detailed description about intervals in η_n.) Let

$$\nu_{n,0} = \max_{I \in \eta_{n,0}} |I|.$$

Since $|f_0'(x)| \geq \lambda_0$ for $x \in I_{j,0}$, $1 \leq j \leq k$, we have that

$$\nu_{n,0} \leq \lambda_0^{-n}$$

for $n = 0, 1, \cdots$.

Now we construct a transitive expanding Markov map for any closed Riemann surface $X = \mathbb{D}/\Gamma_X$ of genus g associated to an isomorphism ϕ. Suppose $\phi : \Gamma_0 \to \Gamma_X$ is an isomorphism. Then there is a unique homeomorphism $H : S^1 \to S^1$ such that

$$H(\gamma(x)) = \phi(\gamma)(H(x))$$

for any $x \in S^1$ and any $\gamma \in \Gamma_0$. Here H is called the boundary correspondence. Moreover, H is quasisymmetric.

Let $I_j = I_{j,X} = H(I_{j,0})$. Then $S^1 = \cup_{j=1}^k I_j$. Define

$$f_X = f_{X,\phi} : S^1 \to S^1$$

as

$$f_X | I_j = \phi(\gamma_s) | I_j, \quad J_s \supset I_{j,0}.$$

In other words,

$$f_X = H \circ f_0 \circ H^{-1}.$$

Then f_X is a transitive Markov map for Γ_X with the Markov partition

$$\eta_0 = \eta_{0,X,\phi} = \{I_1, \cdots, I_k\}.$$

Furthermore, we can generate a sequence of Markov partitions,

$$\eta_n = \eta_{n,X,\phi} = f_X^{-n}\eta_0$$

for $n = 0, 1, \cdots$. Let

$$\nu_n = \nu_{n,X,\phi} = \max_{I \in \eta_n} |I|$$

for $n \geq 0$. Since a quasisymmetric homeomorphism is Hölder continuous, so there are constants $A > 0$ and $0 < \mu < 1$ such that

$$\nu_n \leq A\mu^n, \quad \forall\, n \geq 0.$$

Furthermore, we have that

Lemma 1. *The Markov map f_X is expanding and the sequence of Markov partitions has bounded geometry, that is, there is a constant $C > 0$ such that*

$$\frac{|J|}{|I|} \geq C$$

for any $J \subset I$ with $J \in \eta_{n+1}$ and $I \in \eta_n$.

Proof. Since f_X is piecewise Möbius transformations, so it is a piecewise C^2 Markov map. That is

$$\Big| \log |f_X'(x)| - \log |f_X'(y)| \Big| \leq \frac{1}{m} |f_X'(x) - f_X'(y)| \leq \frac{M}{m} |x - y|$$

for any $x, y \in I \in \eta_0$, where

$$0 < m = \min_{x \in \in I \in \eta_0} |f_X'(x)|, \quad M = \max_{x \in I \in \eta_0} |f_X''(x)| < \infty.$$

Consider any $x, y \in I \in \eta_n$. Then $f_X^i(x), f_X^i(y) \in \eta_{n-i}$ and

$$\Big| \log \frac{|(f_X^n)'(x)|}{|(f_X^n)'(y)|} \Big| \leq \sum_{i=0}^{n-1} | \log f_X'(f_X^i(x)) - \log f_X'(f_X^i(y))|$$

$$\leq \frac{M}{m} \sum_{i=0}^{n-1} |f_X^i(x) - f_X^i(y)| \leq \frac{AM}{m} \sum_{i=0}^{n-1} \mu^{n-i} \leq \frac{AM}{m(1-\mu)}.$$

Thus we get a distortion result,

$$B^{-1} \leq \frac{|(f_X^n)'(x)|}{|(f_X^n)'(y)|} \leq B \tag{1}$$

for any $x, y \in I \in \eta_n$ and any $n > 0$, where

$$B = \exp\Big(\frac{AM}{m(1-\mu)}\Big).$$

Now for any $n > 0$ and any $J \subset I$ with $J \in \eta_{n+1}$ and $I \in \eta_n$,

$$\frac{|f_X^n(J)|}{|f_X^n(I)|} = \frac{|(f_X^n)'(x)|}{|(f_X^n)'(y)|} \frac{|J|}{|I|}$$

for some $x, y \in I$. Let

$$E = \min_{J\subset I, J\in\eta_1, I\in\eta_0} \frac{|J|}{|I|}.$$

Then

$$\frac{|J|}{|I|} \geq C = \frac{E}{B}.$$

This says that the sequence $\{\eta_n\}_{n=0}^{\infty}$ of Markov partitions has bounded geometry.

Now for any $n > 0$ and $x \in I \in \eta_n$, let $y \in I$ such that

$$|f_X^n(I)| = |(f_X^n)'(y)||I|.$$

then we have that

$$|(f_X^n)'(x)| = \frac{|(f_X^n)'(x)|}{|(f_X^n)'(y)|}\frac{|f_X^n(I)|}{|I|} \geq \frac{K}{BA}\mu^{-n} = C_0\lambda^n$$

where $K = \min_{I\in\eta_0} |I|$ and $C_0 = K/(BA)$ and $\lambda = \mu^{-1}$. So f_X is expanding. $\square$

4. Symbolic representation and dual symbolic representation

4.1. **Topological model.** Given any marked Riemann surface (X, h_X) by X_0. Since $h_X : X_0 \to X$ is an orientation preserving quasiconformal homeomorphism, it induces an isomorphism $\phi : \Gamma_0 \to \Gamma_X$. Let $f_X : S^1 \to S^1$ be the transitive expanding Markov map constructed in the previous section with the initial Markov partition $\eta_0 = \{I_1, \cdots, I_k\}$. Associating to η_0, we have a $k \times k$ 0-1 matrix $A = (a_{ij})_{k\times k}$, where $a_{ij} = 1$ if $f(I_i) \supset I_j$ and $a_{ij} = 0$ otherwise. Let

$$\Sigma_A = \{w = i_0 i_1 \cdots i_n i_{n+1} \cdots \mid i_n \in \{1, \cdots, k\}, a_{i_n} a_{i_{n+1}} = 1, n = 0, 1, \cdots\}.$$

The topology of Σ_A is given as follows. For any $n \geq 0$, let

$$w_n = i_0 i_1 \cdots i_n, \quad a_{i_l i_{l+1}} = 1,\ 0 \leq l \leq n-1.$$

Define the left cylinder

$$[w_n] = \{w' = w_n i'_{n+1} \cdots i'_{n+m} i'_{n+m+1} \cdots\}$$

where $i'_{n+m} \in \{1, \cdots, k\}$, $a_{i_n i'_{n+1}} = 1$ and $a_{i'_{n+m} i'_{n+m+1}} = 1, m = 0, 1, \cdots$. Then all these left cylinders form a topological basis. The space Σ_A with this topological basis is called the symbolic space for (X, h_X).

Let σ_A be the shift defined as

$$\sigma_A : w = i_0 i_1 \cdots i_n i_{n+1} \cdots \to \sigma_A(w) = i_1 \cdots i_n i_{n+1} \cdots .$$

Then (Σ_A, σ_A) is a sub-shift of finite type.

For any pair i and j such that $a_{ij} = 1$, there is an interval $I_{ij} \in \eta_1$ such that $f : I_{ij} \to I_j$ is a homeomorphism. Let $g_{ij} : I_j \to I_{ij}$ be its inverse. For each $w = i_0 i_1 \cdots i_n i_{n+1} \cdots \in \Sigma_A$, let $w_n = i_0 i_1 \cdots i_n$. Define

$$g_{w_n} = g_{i_0 i_1} \cdots g_{i_{n-1} i_n}$$

and

$$I_{w_n} = g_{w_n}(I_{i_n}).$$

Then $I_{w_n} \in \eta_{n+1}$ and $g_{w_n} : I_{i_n} \to I_{w_n}$ is a homeomorphism. One can check that

$$\cdots \subset I_{w_n} \subset I_{w_{n-1}} \subset I_{w_1} \subset I_{i_0}$$

Since the length of I_{w_n} tends to zero exponentially as n goes to infinity, the set $\cap_{n=0}^{\infty} I_{w_n}$ contains one point x_w. Define

$$\pi(w) = x_w.$$

Then we have that

$$\pi(\sigma_A(w)) = f_X(\pi(w)), \quad \forall\, w \in \Sigma_A.$$

Note that π is 1-1 except for countably many points which are endpoints of I_{w_n} for all w_n, $n \geq 0$.

Thus from the dynamical system point of views, the symbolic dynamical system (Σ_A, σ_A) is the topological model for all marked Riemann surfaces (X, h_X) by X_0.

4.2. **Geometric models.** Now we are going to define the dual symbolic space Σ_A^* and geometric models for all marked Riemann surfaces (X, h_X) by X_0. For any finite strings $w_n = i_0 i_1 \cdots i_{n-1} i_n$ with $a_{i_l i_{l+1}} = 1$, $0 \leq l \leq n-1$, we rewrite it as $w_n^* = j_n j_{n-1} \cdots j_1 j_0$, where $j_n = i_0$, $j_{n-1} = i_1$, $\cdots$, $j_1 = i_{n-1}$, $j_0 = i_n$. Then $a_{j_l j_{l-1}} = 1$. Define the right cylinder

$$[w_n^*] = \{\tilde{w}^* = \cdots j'_{n+m+1} \cdots j'_{n+m} \cdots j'_{n+1} w_n^*\}$$

for all $j'_{n+m} \in \{1, \cdots, k\}$ and $a_{j'_{n+m+1} j'_{n+m}} = 1$, $m = 1, \cdots$ and $a_{j'_{n+1} j_n} = 1$. All these right cylinders form a topological basis for the space

$$\Sigma_A^* = \{w^* = \cdots j_n \cdots j_1 j_0 \mid j_n \in \{1, \cdots, k\}\}.$$

We call this topological space the dual symbolic space. The left shift $\sigma_A^* : \Sigma_A^* \to \Sigma_A^*$ is defined as

$$\sigma_A^* : w^* = \cdots j_n \cdots j_1 j_0 \to \sigma_A^*(w^*) = \cdots j_n \cdots j_1.$$

A function

$$S(w^*) : \Sigma_A^* \to \mathbb{R}$$

is called Lipschitz if there are constants $C > 0$ and $0 < \mu < 1$ such that

$$|S(w^*) - S(\tilde{w}^*)| \leq C\mu^n$$

as long as the first n digits from the right of w^* and $\tilde{w}^*$ are the same. The geometric model for a marked Riemann surface (X, h_X) by X_0 is defined as a Lipschitz function as follows.

Let

$$f_X : S^1 \to S^1$$

be the transitive expanding Markov map with the Markov partition

$$\eta_0 = \{I_1, \cdots, I_k\}$$

for the marked Riemann surface (X, h_X) by X_0. Then we have a sequence of Markov partitions

$$\eta_n = f_X^{-n}\eta_0$$

for $n = 1, 2, \cdots$. Each interval in η_n has a unique labeling $w_n^* = j_n \cdots j_1 j_0$, which we denote as $I_{w_n^*}$. Then $I_{\sigma_A^*(w_n^*)}$ is an interval in η_{n-1}, where $\sigma_A^*(w_n^*) = j_n \cdots j_1$. We have that $I_{w_n^*} \subset I_{\sigma_A^*(w_n^*)}$. Define the pre-scaling function at w_n^* as

$$S_X(w_n^*) = \frac{|I_{w_n^*}|}{|I_{\sigma_A^*(w_n^*)}|}.$$

Lemma 2. *For any $w^* = \cdots j_n \cdots j_1 j_0$, let $w_n^* = j_n \cdots j_1 j_0$, then*

$$S_X(w^*) = \lim_{n\to\infty} S_X(w_n^*) = \lim_{n\to\infty} \frac{|I_{w_n^*}|}{|I_{\sigma_A^*(w_n^*)}|}$$

exists and converges uniformly on $w^* \in \Sigma_A^*$. Moreover,

$$S_X(w^*) : \Sigma_A^* \to (0, 1)$$

defines a Lipschitz continuous function.

Proof. For any $w^* = \cdots j_n \cdots j_1 j_0$, let $w_n^* = j_n \cdots j_1 j_0$, we consider the sequence $\{S_X(w_n^*)\}_{n=0}^{\infty}$. For any $m > n > 0$,

$$S_X(w_m^*) = \frac{|I_{w_m^*}|}{|I_{\sigma_A^*(w_m^*)}|} = \frac{|(f^{m-n})'(x)|}{|(f^{m-n})'(y)|}\frac{|I_{w_n^*}|}{|I_{\sigma_A^*(w_n^*)}|} = \frac{|(f^{m-n})'(x)|}{|(f^{m-n})'(y)|}S_X(w_n^*)$$

for some $x \in I_{w_n^*}$ and $y \in I_{\sigma_A^*(w_n^*)}$. Thus we get

$$|S_X(w_m^*) - S_X(w_n^*)| \leq \Big|\frac{|(f^{m-n})'(x)|}{|(f^{m-n})'(y)|} - 1\Big| S_X(w_n^*) \leq \Big|\frac{|(f^{m-n})'(x)|}{|(f^{m-n})'(y)|} - 1\Big|$$

From the calculation which we got (1), we have a constant $C > 0$ such that

$$\Big|\frac{|(f^{m-n})'(x)|}{|(f^{m-n})'(y)|} - 1\Big| \leq C\mu^n.$$

Thus,

$$|S_X(w_m^*) - S_X(w_n^*)| \leq C\mu^n.$$

This implies that $\{S_X(w_n^*)\}_{n=0}^\infty$ is a Cauchy sequence and

$$S_X(w^*) = \lim_{n\to\infty} S_X(w_n^*) = \lim_{n\to\infty} \frac{|I_{w_n^*}|}{|I_{\sigma_A^*(w_n^*)}|}$$

exists. Furthermore, the limit is uniformly on Σ_A^*.

Moreover, if we consider w^* and $\tilde{w}^*$ with the same first n digits from the right, then we can write them as $w^* = \cdots w_n^*$ and $\tilde{w}^* = \cdots w_n^*$. So we have that for any $m \geq n$,

$$|S_X(w_m^*) - S_X(\tilde{w}_m^*)| = \Big| \frac{|(f^{m-n})'(x)|}{|(f^{m-n})'(y)|} - \frac{|(f^{m-n})'(\tilde{x})|}{|(f^{m-n})'(\tilde{y})|} \Big| S_X(w_n^*) \leq 2C\mu^n$$

for some $x, y \in I_{\sigma_A^*(w_m^*)}$ and $\tilde{x}, \tilde{y} \in I_{\sigma_A^*(\tilde{w}_m^*)}$. By taking limit, we get

$$|S_X(w^*) - S_X(\tilde{w}^*)| \leq 2C\mu^n.$$

Therefore,

$$S_X(w^*) : \Sigma_A^* \to (0,1)$$

is a Lipschitz function. □

Definition 4. For a marked Riemann surface (X, h_X) by X_0, we call the Lipschitz function

$$S_X(w^*) : \Sigma_A^* \to (0,1)$$

its scaling function.

The scaling function is a geometric model because of the following theorem.

Theorem 2. *Suppose (X, h_X) and (Y, h_Y) are two marked Riemann surfaces by X_0. Then there is a conformal map $\alpha : X \to Y$ such that $h_Y^{-1} \circ \alpha \circ h_X : X_0 \to X_0$ is homotopic to the identity if and only if $S_X = S_Y$.*

Proof. We first prove the "only if" part. Suppose there is a conformal map $\alpha : X \to Y$ such that $h_Y^{-1} \circ \alpha \circ h_X : X_0 \to X_0$ is homotopic to the identity. Let

$$H_X, H_Y, \Psi : \mathbb{D} \to \mathbb{D}$$

be lifts of h_X, h_Y, and α. Then $H_Y^{-1} \circ \Psi \circ H_X : \mathbb{D} \to \mathbb{D}$ is a lift of $h_Y^{-1} \circ \alpha \circ h_X : X_0 \to X_0$. From Theorem 1, there is a lift $H_Y^{-1} \circ \Psi \circ H_X$ whose restriction to the unit circle S^1 is the identity. Thus

$$H_Y(x) = \Psi \circ H_X(x), \quad \forall\, x \in S^1.$$

Note that h_X and h_Y induce isomorphisms from Γ_0 to Γ_X and Γ_Y, respectively, and H_X and H_Y are the corresponding boundary correspondences. From the definition of the Markov maps f_X and f_Y, we have that

$$\Psi \circ f_X = f_Y \circ \Psi.$$

This further implies that $I_{w_n^*,Y} = \Psi(I_{w_n^*,X})$ for all w_n^*. By the mean value theorem, we have $\xi_n \in I_{w_n^*,Y}$ and $\eta_n \in I_{\sigma_A^*(w_n^*),Y}$ such that

$$S_Y(w^*) = \lim_{n\to\infty} \frac{|I_{w_n^*,Y}|}{|I_{\sigma_A^*(w_n^*),Y}|} = \lim_{n\to\infty} \frac{|\Psi(I_{w_n^*,X})|}{|\Psi(I_{\sigma_A^*(w_n^*),X})|}$$

$$= \lim_{n\to\infty} \frac{|\Psi'(\xi_n)|}{|\Psi'(\eta_n)|} \frac{|I_{w_n^*,X}|}{|I_{\sigma_A^*(w_n^*),X}|} = \lim_{n\to\infty} \frac{|I_{w_n^*,X}|}{|I_{\sigma_A^*(w_n^*),X}|} = S_X(w^*)$$

since $|\xi_n - \eta_n| \leq |I_{\sigma_A^*(w_n^*),X}| \to 0$.

Now we prove the "if" part. Consider $h = h_Y \circ h_X^{-1} : X \to Y$. Suppose $X = \mathbb{D}/\Gamma_X$ and $Y = \mathbb{D}/\Gamma_Y$. Let $\phi_{XY} : \Gamma_X \to \Gamma_Y$ be the isomorphism induced by h. Let $H : S^1 \to S^1$ be the boundary correspondence, that is,

$$H \circ \gamma(x) = \phi_{XY}(\gamma) \circ H(x)$$

for all $\gamma \in \Gamma_X$ and $x \in S^1$. From the definition of f_X and f_Y, H is a topological conjugacy from f_X to f_Y on S^1, that is,

$$f_Y \circ H = H \circ f_X.$$

From $S_Y = S_X$, we claim that H is a Lipschitz map from S^1 to S^1.

We prove this claim. For any interval $I_{w_n^*,X} \in \eta_{n,X}$, $I_{w_n^*,Y} = H(I_{w_n^*,X}) \in \eta_{n,Y}$. Then

$$\Big| \log \Big(\frac{|H(I_{w_n^*,X})|}{|I_{w_n^*,X}|} \Big) \Big| = \Big| \log \Big(\frac{|I_{w_n^*,Y}|}{|I_{w_n^*,X}|} \Big) \Big| = | \log |I_{w_n^*,Y}| - \log |I_{w_n^*,X}||$$

$$= \Big| \sum_{k=0}^{n-1} \Big(\log S_Y(w_{n-k}^*) - \log S_X(w_{n-k}^*) \Big) + \log |I_{j_0,Y}| - \log |I_{j_0,X}| \Big|$$

$$\leq \sum_{k=0}^{n-1} \Big| \log S_Y(w_{n-k}^*) - \log S_X(w_{n-k}^*) \Big| + \Big| \log |I_{j_0,Y}| - \log |I_{j_0,X}| \Big|$$

$$\leq \sum_{k=0}^{n-1} \Big| S_Y(w_{n-k}^*) - S_X(w_{n-k}^*) \Big| + C_1$$

where

$$C_1 = \sup_{j_0} \Big| \log |I_{j_0,Y}| - \log |I_{j_0,X}| \Big| < \infty.$$

There are constants $C_2 > 0$ and $0 < \mu < 1$ such that that

$$|S_X(w_{n-k}^*) - S_X(w^*)| \leq C_2 \mu^{n-k}$$

and

$$|S_Y(w_{n-k}^*) - S_Y(w^*)| \leq C_2 \mu^{n-k}.$$

But $S_X(w^*) = S_Y(w^*)$. So we have that

$$\Big| S_X(w^*_{n-k}) - S_Y(w^*_{n-k}) \Big| \leq 2C_2 \mu^{n-k}.$$

This implies that

$$C_3^{-1} \leq \frac{|H(I_{w_n^*, X})|}{|I_{w_n^*, X}|} \leq C_3$$

where $C_3 = \exp(2C_2/(1-\mu) + C_1)$.

Since the set of all endpoints of $\{H(I_{w_n^*, X})\}$ and the set of all endpoints $\{I_{w_n^*, X}\}$ are both dense in S^1, so the additive formula implies that for any $x, y \in S^1$,

$$C_3^{-1} \leq \frac{|H(x) - H(y)|}{|x-y|} \leq C_3.$$

That is H is a bi-Lipschitz map. Therefore, H is differentiable almost everywhere and has a point $x_0 \in S^1$ such that $H'(x_0) \neq 0$. According to the following Tukia's theorem, H is a Möbius transformation Ψ on S^1. This implies that

$$H_Y^{-1} \circ \Psi \circ H_X(x) = x$$

for all $x \in S^1$. From Theorem 1,

$$h_Y^{-1} \circ \alpha \circ h_X : X_0 \to X_0$$

is homotopic to the identity, where $\alpha : X \to Y$ is the conformal map which has a lift Ψ. ☐

The following theorem is first proved by Tukia in [17]. It is a stronger version of Mostow's rigidity theorem in the 2-dimensional case. We have proved a similar result for dynamical systems with possibly critical points presented in [7, 8, 9] (see also a survey article [14]). For the completeness of this paper, we give a proof of Tukia's theorem from the dynamical system point of views.

Theorem 3 (Tukia [17]). *Suppose $X = \mathbb{D}/\Gamma_X$ and $Y = \mathbb{D}/\Gamma_Y$ are two closed hyperbolic Riemann surface of genus $g \geq 2$. Suppose $\phi : \Gamma_X \to \Gamma_Y$ is an isomorphism. Let $H : S^1 \to S^1$ be the boundary correspondence. Then H is a Möbius transformation if and only if H is differentiable at one point with non-zero derivative.*

Proof. Since H is the boundary correspondence for $\phi : \Gamma_X \to \Gamma_Y$, we have that

$$H \circ \gamma(x) = \phi(\gamma) \circ H(x)$$

for any $\gamma \in \Gamma_X$ and $x \in S^1$.

Consider a marked Riemann surface (X, h_X) by $X_0 = \mathbb{D}/\Gamma_0$. Then h_X induces an isomorphism $\phi_X : \Gamma_0 \to \Gamma_X$. Consider the isomorphism $\phi_Y = \phi \circ \phi_X : \Gamma_0 \to \Gamma_Y$ and its boundary correspondence $H_Y : S^1 \to S^1$, that is

$$H_Y \circ \gamma(x) = \phi_Y(\gamma) \circ H_Y(x) \tag{2}$$

for any $\gamma \in \Gamma_0$ and $x \in S^1$. Then H_Y can be extended to $\mathbb{D}$ still satisfying the above equation (2) (this extension may not be unique but the boundary correspondence is unique). So it induces a quasiconformal homeomorphism $h_Y : X_0 \to Y$. Thus we get a marked Riemann surface (Y, h_Y). Suppose f_X and f_Y are the transitive expanding Markov maps corresponding to (X, h_X) and (Y, h_Y). Then

$$f_Y \circ H = H \circ f_X$$

on S^1.

If H is a Möbius transformation, since it is a diffeomorphism of S^1, $H'(x) \neq 0$. This is the "only if" part.

To prove the "if" part, suppose H is differentiable at x_0 with $H'(x_0) > 0$. Let $\{\eta_{n,X}\}_{n=0}^{\infty}$ be the sequence of Markov partitions for f_X. Then there is a sequence of nested intervals $I_{w_n} \in \eta_{n,X}$ such that

$$x_0 \in \cdots \subset I_{w_n} \subset I_{w_{n-1}} \subset \cdots I_{w_1}.$$

Without loss of generality, we assume that x_0 is an interior point of I_{w_n} for all $n \geq 0$.

Suppose H is differentiable at a point x_0 on the circle. Then

$$H(x) = H(x_0) + H'(x_0)(x - x_0) + o(|x - x_0|)$$

for x close to x_0.

Consider $\{x_n = f_X^n(x_0)\}_{n=0}^{\infty}$. Let $0 < a < 1$ be a real number. Consider the interval $I_n = (x_n, x_n + a)$. Let $J_n = (x_0, z_n)$ be an interval such that

$$f_X^n : J_n \to I_n$$

is a C^2 diffeomorphism. Let $f_X^{-n} : I_n \to J_n$ denote its inverse. Since f_X is expanding, the length $|J_n| \to 0$ as $n \to \infty$. Similarly, we have

$$f_Y^n : H(J_n) \to H(I_n)$$

is a C^2 diffeomorphism. Let $f_Y^{-n} : H(I_n) \to H(J_n)$ be its inverse. Then

$$H(x) = f_Y^n \circ H \circ f_X^{-n}(x), \quad x \in I_n.$$

Let

$$\alpha_n(x) = \frac{x - x_0}{x_n - x_0} : J_n \to (0, 1)$$

and

$$\beta_n(x) = \frac{x - H(x_0)}{H(x_n) - H(x_0)} : H(J_n) \to (0,1).$$

Then

$$H(x) = (f_Y^n \circ \beta_n^{-1}) \circ (\beta_n \circ H \circ \alpha_n^{-1}) \circ (\alpha_n \circ f_X^{-n})(x), \quad x \in I_n.$$

The key estimate comes from the following distortion result (the proof is similar to the proof of Lemma 1 or refer to [12, Chapter 1]): There is a constant $C > 0$ independent of n and any inverse branches of f_X^n and f_Y^n such that

$$\left| \log |\frac{(f_X^{-n})'(x))}{(f_X^{-n})'(y))}| \right| \leq C|x-y|, \text{ for all } x \text{ and } y \text{ in } I_n \tag{3}$$

and

$$\left| \log |\frac{(f_Y^{-n})'(x))}{(f_Y^{-n})'(y))}| \right| \leq C|x-y|, \text{ for all } x \text{ and } y \text{ in } H(I_n). \tag{4}$$

From this distortion property, one can conclude that $f_Y^n \circ \beta_n^{-1}$ and $\alpha_n \circ f_X^{-n}$ are sequences of bi-Lipschitz homeomorphisms with a uniform Lipschitz constant. Therefore, they have convergent subsequences. Without loss of generality, let us assume that these two sequences themselves are convergent. The map $\beta_n \circ H \circ \alpha_n^{-1}$ converges to a linear map.

Since the unit circle is compact and all I_n have fixed length a, there is a subsequence I_{n_i} of intervals such that $\cap_{i=1}^{\infty} I_{n_i}$ contains an interval I of positive length. Without loss of generality, let us assume that $\cap_{n=1}^{\infty} I_n$ contains an interval I of positive length. Thus H is a bi-Lipschitz homeomorphism on I.

Since $H|I$ is bi-Lipschitz, H' exists a.e. in I and is integrable. Since $(H|I)'(x)$ is measurable and $H|I$ is a homeomorphism, we can find a point y_0 in I and a subset E_0 containing y_0 such that

1) $H|I$ is differentiable at every point in E_0;
2) y_0 is a density point of E_0;
3) $H'(y_0) \neq 0$; and
4) the derivative $H'|E_0$ is continuous at y_0.
5) y_0 is not an endpoint of an interval in $\eta_{n,X}$ for all $n \geq 0$.

Since S^1 is compact, there is a subsequence $\{f_X^{n_k}(y_0)\}_{k=1}^{\infty}$ converging to a point z_0 in S^1. Let $I_0 = (b,c)$ be an open interval such that $z_0 \in \overline{I}_0$. There is a sequence of interval $\{I_k\}_{k=1}^{\infty}$ such that $y_0 \in I_k \subseteq I$ and $f_X^{n_k} : I_k \to I_0$ is a C^2 diffeomorphism. Then $|I_k|$ goes to zero as k tends to infinity.

From the distortion property (1), there is a constant $C_4 > 0$, such that

$$\Big| \log \Big(\frac{|(f_X^{n_k})'(w)|}{|(f_X^{n_k})'(z)|} \Big) \Big| \leq C_4, \quad \forall w, z \in I_k, \ \ \forall k \geq 1.$$

Since y_0 is a density point of E_0, for any integer $s > 0$, there is an integer $k_s > 0$ such that

$$\frac{|E_0 \cap I_k|}{|I_k|} \geq 1 - \frac{1}{s}, \quad \forall k \geq k_s.$$

Let $E_k = f_X^{n_k}(E_0 \cap I_k)$. Then H is differentiable at every point in E_k and, from the distortion property (1), there is a constant $C_5 > 0$ such that

$$\frac{|E_k \cap I_0|}{|I_0|} \geq 1 - \frac{C_5}{s}, \quad \forall k \geq k_s.$$

Let

$$E = \cap_{s=1}^{\infty} \cup_{k \geq k_s} E_k.$$

Then E has full measure in I_0 and H is differentiable at every point in E with non-zero derivative.

Next, we are going to prove that $H'|E$ is uniformly continuous. For any x and y in E, let z_k and w_k be the preimages of x and y under the diffeomorphism $f_X^{n_k} : I_k \longrightarrow I_0$. Then z_k and w_k are in E_0. From $H \circ f_X = f_Y \circ H$, we have that

$$H'(x) = \frac{(f_Y^{n_k})'(H(z_k))}{(f_X^{n_k})'(z_k)} H'(z_k)$$

and

$$H'(y) = \frac{(f_Y^{n_k})'(H(w_k))}{(f_X^{n_k})'(w_k)} H'(w_k).$$

So

$$\Big| \log \Big(\frac{H'(x)}{H'(y)} \Big) \Big| \leq \Big| \log \Big| \frac{(f_Y^{n_k})'(H(z_k))}{(f_Y^{n_k})'(H(w_k))} \Big| \Big| + \Big| \log \Big| \frac{(f_X^{n_k})'(w_k)}{(f_X^{n_k})'(z_k)} \Big| \Big| + \Big| \log \Big(\frac{H'(z_k)}{H'(w_k)} \Big) \Big|.$$

Since f_X and f_Y are both piecewise C^2. From the distortion property (3) and (4), there is a constant $C_6 > 0$ such that

$$\Big| \log \Big| \frac{(f_X^{n_k})'(w_k)}{(f_X^{n_k})'(z_k)} \Big| \Big| \leq C_6 |x - y|$$

and

$$\Big| \log \Big| \frac{(f_Y^{n_k})'(H(z_k))}{(f_Y^{n_k})'(H(w_k))} \Big| \Big| \leq C_6 |H(x) - H(y)|$$

for all $k \geq 1$. Therefore,

$$\Big| \log \Big(\frac{H'(x)}{H'(y)} \Big) \Big| \leq C_6 \Big(|x - y| + |H(x) - H(y)| \Big) + \Big| \log \Big(\frac{H'(z_k)}{H'(w_k)} \Big) \Big|$$

for all $k \geq 1$. Since $H'|E_0$ is continuous at y_0, the last term in the last inequality tends to zero as k goes to infinity. Hence

$$\Big| \log \Big(\frac{H'(x)}{H'(y)} \Big) \Big| \leq C_6 \Big(|x-y| + |H(x) - H(y)| \Big).$$

This means that $H'|E$ is uniformly continuous. So it can be extended to a continuous function ϕ on I_0. Because $H|I_0$ is absolutely continuous and E has full measure,

$$H(x) = H(a) + \int_a^x H'(x)dx = H(a) + \int_a^x \phi(x)dx$$

on I_0. This implies that $H|I_0$ is actually C^1. (This, furthermore, implies that $H|I_0$ is C^2).

Now for any $x \in S^1$, let J be an open interval about x. By the expansive and transitivity properties of f_X, there is an integer $n > 0$ and an open interval $J_0 \subset I_0$ such that $f_X^n : J_0 \to J$ is a C^1 diffeomorphism. By the equation $H \circ f_X = f_Y \circ H$, we have that $H|J$ is C^1. Therefore, H is C^1.

Since $H : S^1 \to S^1$ is the boundary correspondence for $\phi_{XY} : \Gamma_X \to \Gamma_Y$, we have that

$$H \circ \gamma(x) = \phi_{XY}(\gamma) \circ H(x)$$

for all $\gamma \in \Gamma_X$ and $x \in S^1$. By composition and post-composition Möbius transformations, we can assume that $\gamma(x) = \lambda x$ and $\phi_{XY}(\gamma)(x) = \lambda x$ (C^1-diffeomorphism preserves the eigenvalue at a periodic point), then we can get that $H(x) = ax$. Thus H is a Möbius transformation. We proved the theorem. □

5. Teichmüller space represented by the space of functions

Let $T(X_0)$ be the Teichmüller space of X_0. It is the space of all equivalence classes $\tau = [(X, h_X)]$ of all marked Riemann surfaces (X, h_X) by X_0. Let

$$\mathcal{F} = \{S_X\}$$

be the space of all scaling functions. The following result is a consequence of Theorem 2 now.

Theorem 4. *Any marked Riemann surfaces (X, h_X) and (Y, h_Y) by X_0 are in a same point $\tau \in T(X_0)$ if and only if they have the same scaling functions, that is, $S_X = S_Y$.*

Proof. Since $(X, h_X), (Y, h_Y) \in \tau$ if and only if there is a conformal map $\alpha : X \to Y$ such that $h = h_Y^{-1} \circ \alpha \circ h_X : X_0 \to X_0$ is homotopic to identity. □

Thus we can denote $S_\tau = S_X$ for any $(X, h_X) \in \tau \in T(X_0)$ and introduce a bijective map from the Teichmüller space $T(X_0)$ to the function space $\mathcal{F}$,

$$\iota : T(X_0) \to \mathcal{F}; \quad \iota(\tau) = S_\tau, \quad \tau \in T(X_0).$$

Now suppose R is any closed Riemann surface of genus g. Let $h_0 : X_0 \to R$ be a quasiconformal homeomorphism. For any marked Riemann surface (X, h_X) by R, we have a marked Riemann surface $(X, h_X \circ h_0)$ by X_0. For any two marked Riemann surfaces (X, h_X) and (Y, h_Y) by R, if there is a conformal map $\alpha : X \to Y$ such that

$$h_Y^{-1} \circ \alpha \circ h_X : R \to R$$

is homotopic to the identity, then

$$h_0^{-1} \circ h_Y^{-1} \circ \alpha \circ h_X \circ h_0 : X_0 \to X_0$$

is also homotopic to the identity. Thus this gives a bijective map $\vartheta : T(R) \to T(X_0)$. Therefore, we have a bijective map

$$\iota_R = \iota \circ \vartheta : T(R) \to \mathcal{F}.$$

6. **Bers' embedding for $\mathcal{F}$.**

Let $R = \mathbb{D}/\Gamma$ be a closed hyperbolic Riemann surface of genus $g \geq 2$. The group Γ acts on the whole Riemann sphere $\mathbb{P}^1$. The limit set of Γ is just the unit disk S^1. The quotient

$$R^* = (\mathbb{P}^1 \setminus \overline{\mathbb{D}})/\Gamma$$

of the outer of the closed unit disk is another closed hyperbolic Riemann surface complex conjugate to R. Let $T(R)$ be the Teichmüller space of R. Suppose $\tau = [(X, h_X)] \in T(R)$ and $X = \mathbb{D}/\Gamma_X$. Then h_X can be lift to a quasiconformal homeomorphism $H : \mathbb{D} \to \mathbb{D}$ conjugating the Fuchsian group Γ and Γ_X. Let $\mu_H = H_{\overline{z}}/H_z$ be the Belrtami coefficient of H. Extend it to the whole Riemann sphere $\mathbb{P}^1$ by

$$\mu(z) = \begin{cases} \mu_H(z), & z \in \mathbb{D} \\ 0, & z \in \mathbb{D}^* = \mathbb{P}^1 \setminus \overline{\mathbb{D}}. \end{cases}$$

Let $\Phi(z) : \mathbb{P}^1 \to \mathbb{P}^1$ be the normalized solution to the Beltrami equation

$$\Phi_{\overline{z}} = \mu(z)\Phi_z.$$

Since μ is Γ-invariant, we have that

$$\gamma_*(\mu_\Phi)(z) = \mu_\Phi(z)$$

for all $\gamma \in \Gamma$. Since the solution of the Beltrami equation is unique up to post-composition with a Möbius transformation, so there is an isomorphism

$$\phi : \Gamma \to \Gamma_X$$

such that

$$\Phi \circ \gamma = \phi(\gamma) \circ \Phi$$

for all $\gamma \in \Gamma$. Thus Φ conjugates Γ to Γ_X. By the construction of Φ, it is conformal in $\mathbb{D}^*$. This implies that

$$R^* = \Phi(\mathbb{D}^*)/\Gamma_X \quad \text{and} \quad X = \Phi(\mathbb{D})/\Gamma_X.$$

The Schwarzian derivative

$$s(\Phi)(z) = \Big(\frac{\Phi'''(z)}{\Phi'(z)} - \frac{3}{2} \Big(\frac{\Phi''(z)}{\Phi'(z)} \Big)^2 \Big) dz^2, \quad z \in \mathbb{D}^*,$$

is Γ-invariant. It induces a quadratic differential on R^* and is independent of the choice of $(X, h_X) \in \tau$. Thus we get a quadratic differential $s(\tau)$ on R^*. Let $Q(R^*)$ be the space of all quadratic differentials $q = q(z)dz^2$ on R^* with the norm

$$||q|| = \sup_{z \in R^*} (|q(z)| \rho^{-2}(z)).$$

where ρ is the hyperbolic metric on S^*. Then it is a complex dimension $3g-3$ linear space. Then we can embed $T(R)$ into $Q(R^*)$ by

$$s(\tau) : T(R) \to Q(R^*).$$

It is called Bers's embedding. So we can think $T(R)$ as an open domain of $Q(R^*)$.

Now take $R = X_0$ and consider

$$s \circ \iota^{-1} : \mathcal{F} \to Q(R^*).$$

Then we can embed $\mathcal{F}$ into a complex manifold $Q(R^*)$. Let $B(1/2)$ and $B(3/2)$ be the balls of radii $1/2$ and $3/2$ in $Q(R^*)$. Then we have that

$$B(1/2) \subset s(\iota^{-1}(\mathcal{F})) \subset B(3/2).$$

7. **Teichmüller metric and maximum metric.**

The Teichmüller metric on $T(R)$ is defined as follows. Given any two points $\tau, \tau' \in T(R)$. Let $(X, h_X) \in \tau$ and $(Y, h_Y) \in \tau'$. Then $h_{XY} = h_Y \circ h_X^{-1} : X \to Y$ is a quasiconformal homeomorphism. Define

$$d_T(\tau, \tau') = \inf \frac{1}{2} \inf \{ \log K(h) \mid h : X \to Y \text{ is homotopic to } h_{XY} \}$$

where the first inf takes over all marked Riemann surfaces $(X, h_X) \in \tau$ and $(Y, h_Y) \in \tau'$ by R and where

$$K(h) = \sup_{z \in X} \frac{|h_z| + |h_{\overline{z}}|}{|h_z| - |h_{\overline{z}}|}.$$

Since the map $\vartheta : T(R) \to T(X_0)$ is an isometry, we only need to consider the case $R = X_0$ for the purpose of the study of metrics.

Suppose $R = X_0$. Since $\iota : T(R) \to \mathcal{F}$ is one-to-one and onto, we can define the Teichmüller metric on $\mathcal{F}$ as

$$d_T(S, S') = d_T(\iota^{-1}(S), \iota^{-1}(S')).$$

Since $S = \iota(\tau)$ is a function on Σ_A^*, it has a natural maximum norm

$$||S|| = \sup_{w^* \in \Sigma_A^*} |S(w^*)|.$$

The distance is defined as

$$d(S, S') = ||S - S'||.$$

This introduces a metric $d_{max}(\cdot, \cdot)$ on the Teichmüller space $T(R)$,

$$d_{max}(\tau, \tau') = d(\iota(\tau), \iota(\tau'))$$

for any $\tau, \tau' \in T(R)$.

Theorem 5. *The identity map*

$$id_{TM} : (T(R), d_T) \to (T(R), d_{max})$$

is uniformly continuous.

To prove this theorem, we need several lemmas. Weierstrass' $\mathcal{P}$-function is defined by

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum \Big(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \Big).$$

It is a solution of the differential equation

$$\mathcal{P}'(z)^2 = 4(\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3)$$

where

$$e_1 = \mathcal{P}(\frac{\omega_1}{2}), \quad e_2 = \mathcal{P}(\frac{\omega_2}{2}), \quad e_3 = \mathcal{P}(\frac{\omega_1 + \omega_2}{2}).$$

They are distinct numbers.

Let $\kappa = \omega_2/\omega_1$ and consider only the half-plane $\Im \kappa > 0$. Then we have a function

$$\rho(\kappa) = \frac{e_3 - e_1}{e_2 - e_1}.$$

Lemma 3. *$\rho(\kappa)\neq 0,1$ is an analytic function and*

$$\rho(i)=\frac{1}{2}.$$

Suppose $\mathbb{H}$ is the upper half plane. It is another model of the hyperbolic disk $\mathbb{D}$. Suppose $\Phi:\mathbb{H}\to\mathbb{H}$ is a K-quasiconformal orientation-preserving homeomorphism. Then it can be extended to a homeomorphism, which we still denote as Φ, of $\mathbb{H}\cup\mathbb{R}$. Let $\psi:\mathbb{R}\to\mathbb{R}$ be the restriction of Φ to the boundary of $\mathbb{H}$. Then ψ is a quasisymmetric homeomorphism and $\psi(-\infty)=-\infty$ and $\phi(+\infty)=+\infty$. We have that

Lemma 4.

$$\lambda(K)^{-1}=\frac{1-\rho(iK)}{\rho(iK)}\leq\frac{|\psi(x+t)-\psi(x)|}{|\psi(x)-\psi(x-t)|}\leq\lambda(K)=\frac{\rho(iK)}{1-\rho(iK)}.$$

The proofs of the above two lemmas can be found in Ahlfors' book [1]. The further estimation of $\lambda(K)$ is that (refer to [15])

$$\lambda(K)\leq e^{5(K-1)}. \tag{5}$$

Let $\varphi:S^1\to S^1$ be an orientation preserving homeomorphism. Suppose $\epsilon>0$ and $M>1$ are two constants. We call it (ϵ,M)-quasisymmetric if

$$M^{-1}\leq\frac{|\varphi(x)-\varphi(\frac{x+y}{2})|}{|\varphi(\frac{x+y}{2})-\varphi(y)|}\leq M$$

for any $x,y\in S^1$ and $|x-y|\leq\epsilon$, where $|\cdot|$ means the Lebesgue metric on S^1.

By considering the equality (5) or Lemmas 3 and 4 and $\rho(\kappa)$ is continuous at i, we have that

Lemma 5. *There are two bounded functions $\epsilon(K)>0$ and $M(K)>1$ with $\epsilon(K)\to 0^+$ and $M(K)\to 1^+$ as $K\to 1^+$ such that if H is a K-quasiconformal homeomorphism of $\mathbb{D}$, then $\varphi=H|S^1$ is a $(\epsilon(K),M(K))$-quasisymmetric homeomorphism of S^1.*

Proof. Suppose Υ is a Möbius transformation mapping $\mathbb{R}$ to S^1. Then $\Upsilon^{-1}\circ H\circ\Upsilon$ is a K-quasiconformal homeomorphism of $\mathbb{H}$. From Lemma 4,

$$\frac{|\Upsilon'(\xi)|}{|\Upsilon'(\eta)|}\lambda(K)^{-1}\leq\frac{|\varphi(x)-\varphi(\frac{x+y}{2})|}{|\varphi(\frac{x+y}{2})-\varphi(y)|}\leq\frac{|\Upsilon'(\xi)|}{|\Upsilon'(\eta)|}\lambda(K)$$

where $\xi,\eta\in[x,y]$. Without loss of generality, we assume that $\Upsilon^{-1}([x,y])$ is in a fixed compact set of $\mathbb{R}$ (otherwise, we use a different Υ such that

$\Upsilon^{-1}([x,y])$ away from ∞). Thus we have a number $\epsilon(K) > 0$ such that

$$\frac{|\Upsilon'(\xi)|}{|\Upsilon'(\eta)|} \leq e^{5(K-1)}$$

for any $|y-x| \leq \epsilon(K)$. Thus we can take

$$M(K) = e^{10(K-1)} \to 1^+, \quad \text{as} \quad K \to 1^+.$$

$\square$

Suppose $[a,b]$ is an interval and $H : [a,b] \to H([a,b])$ is a homeomorphism. We say H is M-quasisymmetric on $[a,b]$ if

$$M^{-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq M$$

for any $x, x+t, x-t \in [a,b]$ and $t > 0$. We give a proof of the following lemma.

Lemma 6. *There is a bounded function $\zeta(M) > 0$ satisfying $\zeta(M) \to 0$ as $M \to 1^+$ such that for any M-quasisymmetric homeomorphism H of $[0,1]$ with $H(0) = 0$ and $H(1) = 1$,*

$$|H(x) - x| \leq \zeta(M), \quad \forall\, x \in [0,1].$$

Proof. Consider points $x_n = 1/2^n$, $n = 0, 1, \cdots$. The M-quasisymmetry condition implies that

$$M^{-1} \leq \frac{H(\frac{1}{2^{n-1}}) - H(\frac{1}{2^n})}{H(\frac{1}{2^n}) - H(0)} \leq M.$$

From this and the fact that $H(0) = 0$, we get

$$(1 + M^{-1})H(\frac{1}{2^n}) \leq H(\frac{1}{2^{n-1}}) \leq (1+M)H(\frac{1}{2^n}).$$

This gives

$$\frac{1}{1+M}H(\frac{1}{2^{n-1}}) \leq H(\frac{1}{2^n}) \leq \frac{1}{1+M^{-1}}H(\frac{1}{2^{n-1}}).$$

Using the fact that $H(1) = 1$, we further get

$$\Big(\frac{1}{1+M}\Big)^n \leq H(\frac{1}{2^n}) \leq \Big(\frac{1}{1+M^{-1}}\Big)^n, \quad \forall\, n \geq 1.$$

Furthermore, by M-quasisymmetry and induction on $n = 1, 2, \cdots$, yield

$$\Big(\frac{1}{1+M}\Big)^n \leq H(\frac{i}{2^n}) - H(\frac{i-1}{2^n}) \leq \Big(\frac{1}{1+M^{-1}}\Big)^n, \quad \forall\, n \geq 1, \ \ 1 \leq i \leq 2^n.$$

Let

$$\chi_n = \max\left\{ \left(\frac{M}{M+1} \right)^n - \frac{1}{2^n}, \frac{1}{2^n} - \left(\frac{1}{M+1} \right)^n \right\}, \quad n = 1, 2, \cdots .$$

Then for $n = 1$,

$$|H(\frac{1}{2}) - \frac{1}{2}| \leq \chi_1 = \frac{1}{2} \frac{M-1}{M+1},$$

and for any $n > 1$, we have

$$\max_{0 \leq i \leq 2^n} \left| H(\frac{i}{2^n}) - \frac{i}{2^n} \right| \leq \max_{0 \leq i \leq 2^{n-1}} \left| H(\frac{i}{2^{n-1}}) - \frac{i}{2^{n-1}} \right| + \chi_n$$

By summing over k for $1 \leq k \leq n$, we obtain

$$\max_{0 \leq i \leq 2^n} \left| H(\frac{i}{2^n}) - \frac{i}{2^n} \right| \leq \delta_n = \sum_{k=1}^{n} \chi_k.$$

If we put $\zeta(M) = \sup_{1 \leq n < \infty} \{\delta_n\}$, by summing geometric series, we obtain

$$\zeta(M) = \max_{1 \leq n < \infty} \left\{ M - 1 + \frac{1}{2^n} - M \left(\frac{M}{1+M} \right)^n, 1 - \frac{1}{M} + \frac{1}{M} \left(\frac{1}{M} \right)^n - \frac{1}{2^n} \right\}.$$

Clearly, $\zeta(M) \to 0$ as $M \to 1$, and since the dyadic points

$$\{ i/2^n \mid n = 1, 2, \cdots; 0 \leq i \leq 2^n \}$$

are dense in $[0, 1]$, we conclude

$$|H(x) - x| \leq \zeta(M) \quad \forall\, x \in [0, 1],$$

which proves the lemma. □

Concluding from the above four lemmas, we have that

Lemma 7. *There is a bounded function* $\varrho(\xi) > 0$ *with* $\varrho(\xi) \to 0$ *as* $\xi \to 0$ *such that*

$$d_{max}(\tau, \tau') \leq \varrho(d_T(\tau, \tau'))$$

for any two $\tau, \tau' \in T(X_0)$.

Proof. Suppose $K = \exp(2 d_T(\tau, \tau')) \geq 1$. Then we have two marked Riemann surfaces $(X, h_X) \in \tau$ and $(Y, h_Y) \in \tau'$ such that

$$h_{XY} = h_Y \circ h_X^{-1} : X = \mathbb{D}/\Gamma_X \to Y = \mathbb{D}/\Gamma_Y$$

is a K-quasisconformal homeomorphism. (We can pick h_{XY} as the Teichmüller map.) Then h_{XY} can be lift to a K-quasiconformal homeomorphism H of $\mathbb{D}$ such that $H|S^1$ is the boundary correspondence for the isomorphism from $\Gamma_X \to \Gamma_Y$ induced by h_{XY}. We still use H to denote its restriction to S^1. Then, from Lemma 6, it is $(\epsilon(K), M(K))$-quasisymmetric on S^1 and the conjugacy between the transitive expanding Markov maps f_X and f_Y, that is,

$$H \circ f_X = f_Y \circ H.$$

For any point $w^* = \cdots w_n^* \in \Sigma_A^*$, we have that

$$I_{w_n^*,X} \in \eta_{n,X} \quad \text{and} \quad I_{\sigma^*(w_n^*),X} \in \eta_{n-1,X}$$

and

$$I_{w_n^*,Y} = H(I_{w_n^*,X}) \in \eta_{n,Y} \quad \text{and} \quad I_{\sigma^*(w_n^*),Y} = H(I_{\sigma^*(w_n^*),X}) \in \eta_{n-1,Y}.$$

Note that

$$I_{w_n^*,X} \subset I_{\sigma^*(w_n^*),X} \quad \text{and} \quad I_{w_n^*,Y} \subset I_{\sigma^*(w_n^*),Y}.$$

Let $n_0 > 0$ be an integer such that

$$|I_{\sigma^*(w_n^*),X}| \leq \epsilon(K)$$

for all $n \geq n_0$. Then $H|I_{\sigma^*(w_n^*),X}$ is a $M(K)$-quasisymmetric homeomorphism.

By considering $[0,1]$ gluing 0 and 1 as a model of S^1, then by rescaling $I_{\sigma^*(w_n^*),X}$ and $I_{\sigma^*(w_n^*),Y}$ into the unit interval $[0,1]$ by linear maps, we can think $H|I_{\sigma^*(w_n^*),X}$ is a $M(K)$-quasisymmetric homeomorphism of $[0,1]$ and fixes 0 and 1. Then Lemma 6 implies that

$$|S_Y(w_n^*) - S_X(w_n^*)| = \Big| \frac{|H(I_{w_n^*,X})|}{|H(I_{\sigma^*(w_n^*),X})|} - \frac{|I_{w_n^*,X}|}{|I_{\sigma^*(w_n^*),X}|} \Big| \leq \zeta(M(K)).$$

This implies that

$$|S_Y(w^*) - S_X(w^*)| \leq \zeta(M(K)).$$

Therefore,

$$d_{max}(\tau, \tau') \leq \zeta(M(d_T(\tau, \tau')).$$

We take $\varrho(\xi) = \zeta(M(\xi))$. The bounded function $\varrho(\xi) \to 0$ as $\xi \to 0^+$. We completed the proof. □

Proof of Theorem 5. For any $\epsilon > 0$, there is a $\delta > 0$ such that $\varrho(\xi) < \epsilon$ for any $0 \leq \xi < \delta$. Thus for any $\tau, \tau' \in T(R)$ with $d_T(\tau, \tau') < \delta$, from Lemma 7, $d_{max}(\tau, \tau') \leq \varrho(d_T(\tau, \tau')) < \epsilon$. Thus

$$id : (T(R), d_T(\cdot, \cdot)) \to (T(R), d_{max}(\cdot, \cdot))$$

is uniformly continuous. We have proved the theorem. □

Theorem 6. *The identity map*

$$id_{MT} : (T(R), d_{max}) \to (T(R), d_T)$$

is continuous.

Proof. Suppose $id_{MT} : (T(R), d_{max}) \to (T(R), d_T)$ is not continuous. That is, we have a real number $\epsilon > 0$ and a point $S = \iota(\tau)$ and a sequence of points $\{S_m = \iota(\tau_m)\}_{m=1}^{\infty}$ in the Teichmüller space $T(R)$ such that

$$d_{\max}(\tau_m, \tau) = \|S_m - S\| \to 0 \quad \text{as} \quad m \to \infty$$

but

$$d_T(\tau_m, \tau) \geq \epsilon, \quad \forall\, m.$$

Let $(X, h_X) \in \tau$ be a fixed representation and $(X_m, h_{X_m}) \in \tau_m$ for each m be a representation such that

$$h_m = h_{X_m} \circ h_X^{-1} : X = \mathbb{D}/\Gamma_X \to X_m = \mathbb{D}/\Gamma_m$$

is a $K_m = \exp(2d_T(\tau_m, \tau))$-quasiconformal homeomorphism. (We can pick h_m as the Teichmüller map.) Then h_m can be lift to a K_m-quasiconformal map H_m of $\mathbb{D}$ such that $H_m|S^1$ is the boundary correspondence for the isomorphism from $\Gamma_m \to \Gamma$ induced by h_m. We still use H_m to denote this boundary correspondence. Let f_{X_m} and f_X are the corresponding Markov maps. Let $\{\eta_n\}_{n=0}^{\infty}$ and $\{\eta_{m,n}\}_{n=0}^{\infty}$ be the corresponding sequences of nested Markov partitions. Since $\|S_m - S\| \to 0$ as $m \to \infty$, we have a constant $a = a(S) > 0$ such that $S_m(w^*) \geq a$ for sufficient large m and all $w^* \in \Sigma_A^*$. Let us assume this true for all m. Since Σ_A^* is a compact set, we have that there is another constant $b = b(a) > 0$ such that $S_m(w_n^*) \geq b$ (pre-scaling functions in Lemma 2) for all m and all n. This says that the collection of the sequences $\{\eta_{m,n}\}_{n=0}^{\infty}$ of nested Markov partitions has uniformly bounded geometry. From a method in [10], which gives a calculation of quasisymmetric dilatation from bounded geometry, we have a constant $M > 0$ such that the quasisymmetric dilatations of all H_m are less than or equal to M. From [4], we know the quasiconformal dilatation of the Douady-Earle extension of H_m to $\mathbb{D}$ is controlled by the quasisymmetric dilatation of H_m. Thus we have a constant K such that all K_m is less than or equal to K. This says that the sequence $\{\tau_m\}_{m=1}^{\infty}$ is contained in the closed ball

$$B_K(S) = \{\eta \in T(R) \mid d_T(\eta, \tau) \leq \frac{1}{2} \log K\}$$

which is a compact set. So we have a convergent subsequence. Let us assume that $\{\tau_m\}_{m=1}^{\infty}$ itself is convergent and converges to $\tilde{\tau} = [(Y, h_Y)]$.

Let

$$h_{XY} = h_X \circ h_Y^{-1} : X = \mathbb{D}/\Gamma_X \to Y = \mathbb{D}/\Gamma_Y$$

be a $\tilde{K} = \exp(2d_T(\tau, \tilde{\tau}))$-quasiconformal homeomorphism. From our assumption, we know that $\tilde{K} > 1$. Let H_{XY} be the corresponding boundary correspondence. Then H_m converges to H_{XY} on S^1 modulo Möbius transformations as $m \to \infty$. Let us just assume that H_m converges to H_{XY} on S^1 as $m \to \infty$.

Let f_Y be the corresponding Markov map and let $\{\eta_{n,Y}\}_{n=0}^{\infty}$ be the sequence of nested Markov partitions. For any w_n^*, let $I_{w_n^*,X_m} \in \eta_{m,n}$ and $I_{w_m^*,Y} \in \eta_{n,Y}$. We have that $|I_{w_n^*,X_m}| \to |I_{w_m^*,Y}|$ as $m \to \infty$ for each fixed n and w_n^*.

Since the sequences $\{\eta_{m,n}\}_{n=0}^{\infty}$ of nested Markov partitions have uniformly bounded geometry, this again says that there are constants $C = C(S) > 0$ and $0 < \mu = \mu(S) < 1$ such that $\nu_{n,m} \leq C\mu^n$ for all n and m, where

$$\nu_{n,m} = \max_{I \in \eta_{n,m}} |I|.$$

This implies that $S_m(w_n^*) \to S_m(w^*)$ and $S_Y(w_n^*) \to S_Y(w^*)$ as $n \to \infty$ uniformly on $m \geq 1$ and $w^* \in \Sigma_A^*$. Thus we can change double limits for each $w^* \in \Sigma_A^*$,

$$S(w^*) = \lim_{m\to\infty} S_m(w^*) = \lim_{m\to\infty} \lim_{n\to\infty} S_m(w_n^*)$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} S_m(w_n^*) = \lim_{n\to\infty} S_Y(w_n^*) = S_Y(w^*).$$

From Theorem 4, this implies that $\tilde{\tau} = \tau$, therefore, $d_T(\tilde{\tau}, \tau) = 0$. This is a contradiction. The contradiction says that

$$id_{MT} : (T(R), d_{max}(\cdot,\cdot)) \to (T(R), d_T(\cdot,\cdot))$$

is continuous at each point S. We have completed the proof. $\square$

However, the map in the last theorem is general not uniformly continuous (actually all constants in the proof depend on S). This can be examined by the union of graphs of $S \in T(R)$ which is an open section in the open unit cube $\prod_0^\infty (0,1)$ and the maximum norm on this open unit cube is incomplete. However, from Theorems 5 and 6, we have that

Corollary 1. *The topology on $T(R)$ induced from the maximum metric d_{max} is the same as the topology on $T(R)$ induced from the usual Teichmüller metric d_T.*

8. Added Remark: pressure metric and WP metric

It is interesting to compare our function model and McMullen's thermodynamical embedding. More interestingly, from McMullen's calculation in [16], we have that the pressure metric for the tangent vector $d(\log S_t)/dt|_{t=0}$ of a smooth path $\iota(\tau_t) = S_t$ through S_0 is a constant times the Weil-Petersson metric of $d\tau_t/dt|_{t=0}$.

Consider the subshift of finite type (Σ_A, σ_A) in §4 associated to all Marked Riemann surfaces (X, h_X) by R. Let $C^H = C^H(\Sigma_A)$ be the space of all Hölder continuous functions on Σ_A. Two functions $\phi, \psi \in C^H(\Sigma_A)$ are said to be cohomologously equivalent, denoted as $\phi \sim_{co} \psi$ if there is a continuous function u on Σ_A such that

$$\phi - \psi = u \circ \sigma_A - u.$$

It is an equivalence relation. We say ϕ is a co-boundary if $\phi \sim_{co} 0$. We use

$$\mathcal{CC}^H = C^H(\Sigma_A)/\sim_{co}$$

to denote the space of all cohomologous equivalence classes. For each $\theta \in \mathcal{CC}^H$, there is an important thermodynamical quantity called the pressure $P(\theta) = P(\phi)$ for any $\phi \in \theta$ associated to it. It is a smooth concave function on $\mathcal{CC}^H$. Let

$$\mathcal{CC}_0^H = \{\theta \in \mathcal{CC}^H \mid P(\theta) = 0\}$$

be the subspace of all equivalence classes with zero pressure. In [16], McMullen embedded the Teichmüller space $T(R)$ into $\mathcal{CC}_0^H$ through cohomologous equivalence classes $\theta = [\phi_X]$ where

$$\phi_X = -\log f_X' \circ \pi_X$$

and where f_X are Markov maps associated to all marked Riemann surfaces (X, h_x) by R in §3. Just like we did in the circle expanding mappings case in [11] (also see [12]), the scaling function S_τ on the dual symbolic space Σ_A^* can be thought as a single function representation for the cohomologous equivalence class $\theta = [\phi_X]$ for any marked Riemann surface $(X, h_X) \in \tau$ but it is in the dual point of view. Therefore, our scaling function model $\mathcal{F}$ can be thought as a dual version of McMullen's thermodynamical embedding. However, our model gives a single function representation for each coholomogous equivalence class as well as for each Teichmüller equivalence class.

For every $\theta \in \mathcal{CC}_0^H$, there is a unique Gibbs measure m_θ for the system $(\Sigma_A, \sigma_A, \phi)$ where ϕ is any function in θ (see, for example, [13] and other references in it). For every $[\psi] \in \mathcal{CC}_0^H$ with zero mean, that is, $\int_{\Sigma_A} \psi dm_\phi = 0$, the variance is given by

$$Var([\psi], m_\phi) = \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A} \Big| \sum_{k=0}^{n-1} \psi \circ \sigma_A^k(w) \Big|^2 dm_\phi.$$

Then by convexity, the second derivative

$$D^2 P([\psi]) = Var([\psi], m_\theta).$$

The pressure metric of $[\psi]$, given by

$$||[\psi]||_P^2 = \frac{Var([\psi], m_\theta)}{-\int_{\Sigma_A} \phi dm_\theta},$$

is nondegenerate. Suppose τ_t is a smooth path in $T(R)$ through τ_0. The tangent vector $\dot{\tau_0} = d\tau_t/dt|_{t=0}$ can be represented uniquely by a harmonic Beltrami differential $\mu = \rho^{-2}\overline{\phi}$ where ρ is the hyperbolic metric and ϕ is a holomorphic quadratic differential. The Weil-Petersson metric on the tangent space $T_{\tau_0}T(R)$ is given by

$$||\dot{\tau_0}||_{WP}^2 = ||\mu||_{WP}^2 = \int \rho^2 |\mu|^2 = \int \rho^{-2} |\phi|^2.$$

Suppose τ_t is a smooth path in $T(R)$ through τ_0. There is a unique family of homeomorphisms H_t of S^1 such that it is the family of boundary correspondences from Γ_t to Γ_0 where $X_t = \mathbb{D}/\Gamma_t \in \tau_t$. Let Φ_t be the family of quasiconformal homeomorphisms from Bers' embedding in §6. Let Λ_t be the image of S^1 under Φ_t. Then Λ_t is a quasicircle and is the limit set of the quasi-Fuchsian group $\tilde{\Gamma}_t$ obtained by gluing the unit disk and the outer of unit disk by H_t. Let $a(t) = HD(\Lambda_t)$ be the Hausdorff dimension of Λ_t. Then $a(t)$ has the minimum value 1 at $t = 0$ since $\Lambda_0 = S^1$. Let $m_t = H_{t*}Leb$ be the pushforward measure of Lebesgue measure on S^1 by H_t. Then from Theorem 3, m_t is totally singular with respect to Lebesgue measure. Let $b(t) = HD(m_t)$ be the Hausdorff dimension of the measure m_t, that is,

$$b(t) = \inf\{HD(E) \mid m_t(E) = 1\}.$$

Then $b(t)$ has the maximum value 1 at $t = 0$. Let θ_t be the corresponding cohomologous equivalence classes to τ_t from McMullen's thermodynamical embedding. Using a key equality in thermodynamical formalism,

$$P([\phi] + t[\psi]) = P([\phi]) + \frac{t^2}{2} Var([\psi], m_{[\phi]}) + O(t^3),$$

where $m_{[\phi]}$ is the Gibbs measure for the system $(\Sigma_A, \sigma_A, \phi)$ and $[\psi]$ has zero mean and $Var([\psi], m_{[\phi]})$ is the variance, McMullen proved that

$$\frac{1}{4}||\dot{\theta}_0||_P^2 = \frac{d^2a(t)}{dt^2}|_{t=0} = -\frac{1}{4}\frac{d^2b(t)}{dt^2}|_{t=0} = \frac{1}{3}\frac{||\dot{\tau}_0||_{WP}^2}{area(\tau_0)}.$$

Now let us consider the dual symbolic dynamical system (Σ_A^*, σ_A^*) and the space of all functions $\log S_\tau$ for $\tau \in T(R)$ and $\iota(\tau) = S_\tau$. First we have that the pressure $P(\log S_\tau) = 0$ for every S_τ. Let $S_t = \iota(\tau_t)$ be the corresponding smooth path through S_0 in our function model. Let $m_0^* = m_{\log S_0}^*$ be the Gibbs measure for the system $(\Sigma_A^*, \sigma_A^*, \log S_0)$. From the fact that $P(\log S_t) = 0$, we have that

$$\frac{dP(\log S_t)}{dt}\Big|_{t=0} = \int_{\Sigma_A^*} \frac{d(\log S_t)}{dt}\Big|_{t=0} dm_0^* = 0.$$

Thus the vector $d\log S_t/dt|_{t=0}$ has zero mean. The variance is then can be calculated as

$$Var\Big(\frac{d(\log S_t)}{dt}\Big|_{t=0}, m_0^*\Big) = \lim_{n\to\infty} \frac{1}{n}\int_{\Sigma_A^*} \Big|\sum_{k=0}^{n-1} \frac{d(\log S_t)}{dt}\Big|_{t=0} \circ (\sigma_A^*)^k(w^*)\Big|^2 dm_0^*.$$

The pressure metric for $d(\log S_t)/dt|_{t=0}$ can be then defined and is given by

$$\Big|\Big|\frac{d(\log S_t)}{dt}\Big|_{t=0}\Big|\Big|_P^2 = \frac{Var\Big(\frac{d(\log S_t)}{dt}\Big|_{t=0}, m_0^*\Big)}{-\int_{\Sigma_A^*} \log S_0 dm_0^*}.$$

From [12, page 76-77], for each periodic point $w^* = (j_{n-1}\cdots j_0)^\infty$ of σ_A^*, we have a periodic point $w = (i_0 \cdots i_{n-1})^\infty$ of σ_A. This correspondence of periodic points is bijective. Moreover, from [12, Proposition 3.3],

$$\sum_{k=0}^{n-1} \log S_\tau((\sigma_A^*)^k(w^*)) = \sum_{k=0}^{n-1} \phi_X(\sigma_A^k(w)).$$

for any $(X, h_X) \in \tau \in T(R)$. Since the pressures $P([\phi_X])$ and $P(\log S_\tau)$ only depend on summations of values over periodic cycles, so they are equal, that is,

$$P(\log S_\tau) = P([\phi_X]).$$

Just like we did in [11], there is a one-to-one correspondence between Gibbs measures $m_{[\phi_X]}$ for systems $(\Sigma_A, \sigma_A, \phi_X)$ and Gibbs measure $m_{\log S_\tau}^*$ for systems $(\Sigma_A, \sigma_A, \log S_\tau)$. Thus for the smooth curve $\{\iota(\tau_t) = S_t\}$, the variance

$$Var\Big(\frac{d(\log S_t)}{dt}\Big|_{t=0}, m_{\log S_0}^*\Big) = Var(\dot{\theta}_0, m_{\theta_0}).$$

Moreover,

$$\int_{\Sigma_A^*} \log S_0 dm_0^* = \int_{\Sigma_A} \phi_0 dm_{\theta_0}$$

for any $\phi_0 \in \theta_0$. Thus we have that the pressure metric

$$\Big\| \frac{d(\log S_t)}{dt}\Big|_{t=0} \Big\|_P^2 = \frac{4}{3} \frac{||\dot{\tau_0}||_{WP}^2}{area(\tau_0)}.$$

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