

HOLOMORPHIC MOTIONS AND NORMAL FORMS IN COMPLEX ANALYSIS

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ABSTRACT. We give a brief review of holomorphic motions and its relation to quasiconformal mapping theory. Furthermore, we apply the holomorphic motions to give new proofs of the famous König's Theorem and Böttcher's Theorem in classical complex analysis.

1. INTRODUCTION

Two of the fundamental tools in the study of complex dynamical systems are König's Theorem and Böttcher's Theorem in classical complex analysis, which were proved back to 1884 [18] and 1904 [8], respectively, by using some well-known methods in complex analysis. These theorems say that an attractive or repelling or super-attractive fixed point of an analytic map can be written in a normal form under suitable conformal changes of coordinate.

During the study of complex dynamical systems, a subject called holomorphic motions becomes more and more interesting and useful. The subject of holomorphic motions over the open unit disk shows some interesting connections between classical complex analysis and problems on moduli. This subject even became an interesting branch in complex analysis [4, 6, 10, 13, 24, 27, 31, 30].

In this paper, we apply the holomorphic motions to show new proofs of König's Theorem and Böttcher's Theorem. From the technical point of view, our proofs are more complicated and use a sophisticated result in holomorphic motions. But from the conceptual point of view, our proofs give some insight into the mechanism of the normal forms for fixed points. The technique involved in the new proofs will yield some new results too (refer to [16]).

In our new proofs, we use quasiconformal theory, in particular, holomorphic motions over the open unit disk. Holomorphic motions over a simply connected complex Banach manifold are also interesting and useful. In a sequel, we will discuss them and show some applications [17].

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several points in the development of the measurable Riemann mapping theorem and holomorphic motions. I express my sincere thanks to them.

2. HOLOMORPHIC MOTIONS AND QUASICONFORMAL MAPS

In the study of complex analysis, the measurable Riemann mapping theorem plays an important role. Consider the Riemann sphere $\hat{\mathbb{C}}$. A measurable function μ on $\hat{\mathbb{C}}$ is called a Beltrami coefficient if there is a constant $0 \leq k < 1$ such that $\|\mu\|_\infty \leq k$, where $\|\cdot\|_\infty$ means the L^∞ -norm of μ on $\hat{\mathbb{C}}$. The equation

$$H_{\bar{z}} = \mu H_z$$

is called the Beltrami equation with the given Beltrami coefficient μ . The measurable Riemann mapping theorem says that the Beltrami equation has a solution H which is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ whose quasiconformal dilatation is less than or equal to $K = (1+k)/(1-k)$. The study of the measurable Riemann mapping theorem has a long history since Gauss considered in 1820's the connection with the problem of finding isothermal coordinates for a given surface. As early as 1938, Morrey [28] systematically studied homeomorphic L^2 -solutions of the Beltrami equation. But it took almost twenty years until in 1957 Bers [5] observed that these solutions are quasiconformal (refer to [19, pp. 24]). Finally the existence of a solution to the Beltrami equation under the most general possible circumstance, namely, for measurable μ with $\|\mu\|_\infty < 1$, was shown by Bojarski [7]. In this generality the existence theorem is sometimes called the measurable Riemann mapping theorem (refer to [15, pp. 10]).

If one only considers a normalized solution in the Beltrami equation (a solution fixes 0, 1, and ∞), then H is unique, which is denoted as H^μ . The solution H^μ is expressed as a power series made up of compositions of singular integral operators applied to the Beltrami equation on the Riemann sphere. In this expression, if one considers μ as a variable, then the solution H^μ depends on μ analytically. This analytic dependence was emphasized by Ahlfors and Bers in their 1960 paper [2] and is essential in determining a complex structure for Teichmüller space (refer to [1, 15, 19, 23, 29]). Note that when $\mu \equiv 0$, H^0 is the identity map. A 1-quasiconformal map is conformal. Twenty years later, due to the development of complex dynamics, this analytic dependence presents an even more interesting phenomenon called holomorphic motions as follows.

Let $\Delta_r = \{c \in \mathbb{C} \mid |c| < r\}$ be the disk centered at 0 and of radius $r > 0$. In particular, we use Δ to denote the unit disk. Given a Beltrami coefficient μ , consider a family of Beltrami coefficients $c\mu$ for $c \in \Delta$ and the family of normalized solutions $H^{c\mu}$. Note that $H^{c\mu}$ is a quasiconformal homeomorphism whose quasiconformal dilatation is less than or equal to $(1+|c|k)/(1-|c|k)$. Moreover, $H^{c\mu}$ is a family which is holomorphic on c . Consider a subset E of $\hat{\mathbb{C}}$ and its image $E_c = H^{c\mu}(E)$. One can see that E_c moves holomorphically in $\hat{\mathbb{C}}$ when c moves in Δ . That is, for any point $z \in E$, $z(c) = H^{c\mu}(z)$ traces a holomorphic path starting from z as c moves in the unit disk. Although E may start out as smooth as a circle and although the points of E move holomorphically, E_c can be an interesting fractal with fractional Hausdorff dimension for every $c \neq 0$ (see [14]).

Surprisingly, the converse of the above fact is true too. This starts from the famous λ -lemma of Mañé, Sad, and Sullivan [25] in complex dynamical systems. Let us start to understand this fact by first defining holomorphic motions.

Definition 1 (Holomorphic Motions). Let E be a subset of $\hat{\mathbb{C}}$. Let

$$h(c, z) : \Delta_r \times E \rightarrow \hat{\mathbb{C}}$$

be a map. Then h is called a holomorphic motion of E parametrized by Δ_r if

- (1) $h(0, z) = z$ for $z \in E$;
- (2) for any fixed $c \in \Delta_r$, $h(c, \cdot) : E \rightarrow \hat{\mathbb{C}}$ is injective;
- (3) for any fixed z , $h(\cdot, z) : \Delta_r \rightarrow \hat{\mathbb{C}}$ is holomorphic.

For example, for a given Beltrami coefficient μ ,

$$H(c, z) = H^{c\mu}(z) : \Delta \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$$

is a holomorphic motion of $\hat{\mathbb{C}}$ parametrized by Δ .

Note that even continuity does not directly enter into the definition; the only restriction is in the c direction. However, continuity is a consequence of the hypotheses from the proof of the λ -lemma of Mañé, Sad, and Sullivan [25, Theorem 2]. Moreover, Mañé, Sad, and Sullivan prove in [25] that

Lemma 1 (λ -Lemma). *A holomorphic motion of a set $E \subset \hat{\mathbb{C}}$ parametrized by Δ_r can be extended to a holomorphic motion of the closure of E parametrized by the same Δ_r .*

Furthermore, Mañé, Sad, and Sullivan show in [25] that $f(c, \cdot)$ satisfies the Pesin property. In particular, when the closure of E is a domain, this property can be described as the quasiconformal property. A further study of this quasiconformal property is given by Sullivan and Thurston [31] and Bers and Royden [6]. In [31], Sullivan and Thurston prove that there is a universal constant $a > 0$ such that any holomorphic motion of any set $E \subset \hat{\mathbb{C}}$ parametrized by the open unit disk Δ can be extended to a holomorphic motion of $\hat{\mathbb{C}}$ parametrized by Δ_a . In [6], Bers and Royden show, by using classical Teichmüller theory, that this constant actually can be taken to be $1/3$. Moreover, in the same paper, Bers and Royden show that in any holomorphic motion $H(c, z) : \Delta_r \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ for $0 < r \leq 1$, $H(c, \cdot) : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a quasiconformal homeomorphism whose quasiconformal dilatation less than or equal to $(1 + |c|)/(1 - |c|)$ for $c \in \Delta_r$. In the both papers [31, 6], they expect $a = 1$. This was eventually proved by Slodkowski in [30].

Theorem 1 (Slodkowski's Theorem). *Suppose*

$$h(c, z) : \Delta \times E \rightarrow \hat{\mathbb{C}}$$

is a holomorphic motion of a set $E \subset \hat{\mathbb{C}}$ parametrized by Δ . Then h can be extended to a holomorphic motion

$$H(c, z) : \Delta \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$$

of \mathbb{C} parametrized by also Δ . Moreover, following [6, Theorem 1], for every $c \in \Delta$, $H(c, \cdot) : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a quasiconformal homeomorphism whose quasiconformal dilatation

$$K(H(c, \cdot)) \leq \frac{1 + |c|}{1 - |c|}.$$

Holomorphic motions of a set $E \subset \hat{\mathbb{C}}$ parametrized by a connected complex manifold with a base point can be also defined. They have many interesting relationships with the Teichmüller space $T(E)$ of a closed set E (refer to [27]).

In addition to the references we mentioned above, there is a partial list of references [3, 11, 12, 20, 21] about holomorphic motions and quasiconformal mapping theory. The reader who is interested in holomorphic motions may refer to those papers and books.

3. A NEW PROOF OF KÖNIG'S THEOREM

We first give a new proof of König's Theorem. The idea of the new proof follows the viewpoint of holomorphic motions. For the classical proof of König's Theorem, the reader may refer to König's original paper [18] or most recent books [9, 26]. As we mentioned in the introduction, from the technique point of views, the new proof may be more complicate, but from the conceptual point of view, it gives some insight into the mechanism of the normal form for an attractive fixed point.

Theorem 2 (König's Theorem). *Let $f(z) = \lambda z + \sum_{j=2}^{\infty} a_j z^j$ be an analytic function defined on Δ_{r_0} , $r_0 > 0$. Suppose $0 < |\lambda| < 1$ or $|\lambda| > 1$. Then there is a conformal map $\phi : \Delta_\delta \rightarrow \phi(\Delta_\delta)$ for some $0 < \delta < r_0$ such that*

$$\phi^{-1} \circ f \circ \phi(z) = \lambda z.$$

The conjugacy ϕ^{-1} is unique up to multiplication of constants.

Proof. We only need to prove it for $1 < |\lambda| < 1$. In the case of $|\lambda| > 1$, we can consider f^{-1} .

First, we can find a $0 < \delta < r_0$ such that

$$|f(z)| < |z|, \quad z \in \bar{\Delta}_\delta$$

and f is injective on $\bar{\Delta}_\delta$. For every $0 < r \leq \delta$, let

$$S_r = \{z \in \mathbb{C} \mid |z| = r\}$$

and

$$T_r = |\lambda|S_r = \{z \in \mathbb{C} \mid |z| = |\lambda|r\}.$$

Denote $E = S_r \cup T_r$. Define

$$\phi_r(z) = \begin{cases} z, & z \in S_r; \\ f(\frac{z}{\lambda}), & z \in T_r. \end{cases}$$

It is clear that

$$\phi_r^{-1} \circ f \circ \phi_r(z) = \lambda z$$

for $z \in S_r$.

Now write $\phi_r(z) = z\psi_r(z)$ for $z \in T_r$, where

$$\psi_r(z) = 1 + \sum_{j=1}^{\infty} \frac{a_{j+1}}{\lambda^{j+1}} z^j.$$

Define

$$h_r(c, z) = \begin{cases} z, & z \in S_r \\ z\psi_r(\frac{\delta cz}{r}), & z \in T_r \end{cases} : \Delta \times E \rightarrow \hat{\mathbb{C}}.$$

Note that

$$h(c, z) = z\psi_r\left(\frac{cz\delta}{r}\right) = \frac{r}{c\delta}\phi\left(\frac{cz\delta}{r}\right) = \frac{r}{c\delta}f\left(\frac{cz\delta}{r\lambda}\right), \quad z \in T_r, c \neq 0.$$

For each fixed $z \in E$, it is clear that $h(c, z)$ is a holomorphic function of $c \in \Delta$. For each fixed $c \in \Delta$, the restriction $h(c, \cdot)$ to S_r and T_r , respectively, are injective. Now we claim that their images do not cross either. That is because for any $z \in T_r$, $|z| = |\lambda|r$ and $|cz\delta|/|r\lambda| \leq \delta$, so

$$|h(c, z)| = \left|\frac{r}{c\delta}\right| \left|f\left(\frac{cz\delta}{r\lambda}\right)\right| < \left|\frac{r}{c\delta}\right| \left|\frac{cz\delta}{r\lambda}\right| = r.$$

Therefore, $h(c, z) : \Delta \times E \rightarrow \hat{\mathbb{C}}$ is a holomorphic motion because we also have $h(0, z) = z$ for all $z \in E$. From Slodkowski's Theorem, h can be extended to a holomorphic motion $H(c, z) : \Delta \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, and moreover, for each fixed $c \in \Delta$, $H_c = h(c, \cdot) : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a quasiconformal homeomorphism whose quasiconformal dilatation is less than or equal to $(1 + |c|)/(1 - |c|)$. Now take $c_r = r/\delta$ and consider $H(c_r, \cdot)$. We have $H(c_r, \cdot)|_E = \phi_r$. Let

$$A_{r,j} = \{z \in \mathbb{C} \mid |\lambda|^{j+1}r \leq |z| \leq |\lambda|^j r\}.$$

We still use ϕ_r to denote $H(c_r, \cdot)|_{A_{r,0}}$.

For an integer $k > 0$, take $r = r_k = \delta|\lambda|^k$. Then

$$\bar{\Delta}_\delta = \cup_{j=-k}^{\infty} A_{r,j} \cup \{0\}.$$

Extend ϕ_r to $\bar{\Delta}_\delta$, which we still denote as ϕ_r , as follows.

$$\phi_r(z) = f^{-j}(\phi_r((\lambda^n z))), \quad z \in A_{r,j}, \quad j = -k, \dots, -1, 0, 1, \dots,$$

and $\phi_r(0) = 0$. Since $\phi_r|_E$ is a conjugacy from f to λz , ϕ_r is continuous on Δ_δ . Since f is conformal, ϕ_r is quasiconformal whose quasiconformal dilatation is the same as that of $H(c_r, \cdot)$ on $A_{r,0}$. So the quasiconformal dilatation of ϕ_r on Δ_δ is less than or equal to $(1 + r)/(1 - r)$. Furthermore,

$$f(\phi_r(z)) = \phi_r(\lambda z), \quad z \in \Delta_\delta.$$

Since $f(z) = \lambda z(1 + O(z))$, $f^k(z) = \lambda^k z \prod_{i=0}^{k-1} (1 + O(\lambda^i z))$. Because $|\lambda|^{-k} r_k = \delta$, the range of ϕ_{r_k} on Δ_δ is a Jordan domain bounded above and below uniformly on k . In addition, 0 is fixed by ϕ_k and the quasiconformal dilatations of the ϕ_k are uniformly bounded. Therefore, the sequence $\{\phi_{r_k}\}_{k=1}^{\infty}$ is a compact family (see [1]). Let ϕ be a limiting map of this family. Then we have

$$f(\phi(z)) = \phi(\lambda z), \quad z \in \Delta_\delta.$$

The quasiconformal dilatation of ϕ is less than or equal to $(1 + r_k)/(1 - r_k)$ for all $k > 0$. So ϕ is a 1-quasiconformal map, and thus is conformal. This is the proof of the existence.

For the sake of completeness, we also provide the proof of uniqueness but this is not new and the reader can find it on [9, 26]. Suppose ϕ_1 and ϕ_2 are two conjugacies such that

$$\phi_1^{-1} \circ f \circ \phi_1(z) = \lambda z \quad \text{and} \quad \phi_2^{-1} \circ f \circ \phi_2(z) = \lambda z, \quad z \in \Delta_\delta.$$

Then for $\Phi = \phi_2^{-1} \circ \phi_1$, we have $\Phi(\lambda z) = \lambda \Phi(z)$. This implies that $\Phi'(\lambda z) = \Phi'(z)$ for any $z \in \Delta_\delta$. Thus $\Phi'(z) = \Phi'(\lambda^n z) = \Phi'(0)$. So $\Phi(z) = \text{const}$ and $\phi_2^{-1} = \text{const} \cdot \phi_1^{-1}$. \square

4. A NEW PROOF OF BÖTTCHER'S THEOREM

In this section, we give a new proof of Böttcher's Theorem. The idea of the new proof follows the viewpoint of holomorphic motions. For the classical proof of Böttcher's Theorem, the reader may refer to Böttcher's original paper [8] or most recent books [9, 26]. The idea of the proof is basically the same as that in the previous section, but the actual proof is little bit different. The reason is that in the previous case, f is a homeomorphism so we can iterate both forward and backward, but in Böttcher's Theorem, g is not a homeomorphism.

Theorem 3 (Böttcher's Theorem). *Suppose $f(z) = \sum_{j=n}^{\infty} a_j z^j$, $a_n \neq 0$, $n \geq 2$, is analytic on a disk Δ_{δ_0} , $\delta_0 > 0$. Then there exists a conformal map $\phi : \Delta_{\delta} \rightarrow \phi(\Delta_{\delta})$ for some $\delta > 0$ such that*

$$\phi^{-1} \circ f \circ \phi(z) = z^n, \quad z \in \Delta_{\delta}.$$

The conjugacy ϕ^{-1} is unique up to multiplication by $(n-1)^{\text{th}}$ -roots of the unit.

Proof. Conjugating by $z \rightarrow bz$, we can assume $a_n = 1$, i.e.,

$$f(z) = z^n + \sum_{j=n+1}^{\infty} a_j z^j.$$

We use $\Delta_r^* = \Delta_r \setminus \{0\}$ to mean a punctured disk of radius $r > 0$. Write

$$f(z) = z^n \left(1 + \sum_{j=1}^{\infty} a_{j+n} z^j \right).$$

Assume $0 < \delta_1 < \min\{1/2, \delta_0/2\}$ is small enough such that

$$1 + \sum_{j=1}^{\infty} a_{j+n} z^j \neq 0 \quad \text{and} \quad \frac{1}{\sqrt[n]{|1 + \sum_{j=1}^{\infty} a_{j+n} z^j|}} \geq \frac{1}{2}, \quad z \in \Delta_{2\delta_1}.$$

Then $f : \Delta_{2\delta_1}^* \rightarrow f(\Delta_{2\delta_1}^*)$ is a covering map of degree n .

Let $0 < \delta < \delta_1$ be a fixed number such that $f^{-1}(\Delta_{\delta}) \subset \Delta_{\delta_1}$. Since

$$z \rightarrow z^n : \Delta_{\sqrt[n]{\delta}}^* \rightarrow \Delta_{\delta}^* \quad \text{and} \quad f : f^{-1}(\Delta_{\delta}^*) \rightarrow \Delta_{\delta}^*$$

are both of covering maps of degree n , the identity map of Δ_{δ} can be lifted to a holomorphic diffeomorphism

$$h : \Delta_{\sqrt[n]{\delta}}^* \rightarrow f^{-1}(\Delta_{\delta}^*),$$

i.e., h is a map such that the diagram

$$\begin{array}{ccc} \Delta_{\sqrt[n]{\delta}}^* & \xrightarrow{h} & f^{-1}(\Delta_{\delta}^*) \\ \downarrow z \rightarrow z^n & & \downarrow f \\ \Delta_{\delta}^* & \xrightarrow{\text{id}} & \Delta_{\delta}^* \end{array}$$

commutes. We pick the lift so that

$$h(z) = z \left(1 + \sum_{j=2}^{\infty} b_j z^{j-1} \right) = z\psi(z).$$

From

$$f(h(z)) = z^n, \quad z \in \Delta^*_{\sqrt[n]{\delta}},$$

we get

$$|h(z)| = \frac{|z|}{\sqrt[n]{|1 + \sum_{j=1}^{\infty} a_{n+j}(h(z))^j|}} \geq \frac{|z|}{2}.$$

For any

$$0 < r \leq \max \left\{ \left(\frac{1}{2} \right)^{\frac{n}{(n-1)}}, \delta^n \right\},$$

let $S_r = \{z \in \mathbb{C} \mid |z| = r\}$ and $T_r = \{z \in \mathbb{C} \mid |z| = \sqrt[n]{r}\}$. Consider the set $E = S_r \cup T_r$ and the map

$$\phi_r(z) = \begin{cases} z, & z \in S_r \\ z\psi(z), & z \in T_r. \end{cases}$$

Define

$$h_r(c, z) = \begin{cases} z, & z \in S_r \\ z\psi\left(\frac{cz}{\sqrt[n]{r}}\right), & z \in T_r \end{cases} : \Delta \times E \rightarrow \hat{\mathbb{C}}.$$

Note that

$$z\psi\left(\frac{cz}{\sqrt[n]{r}}\right) = \frac{\sqrt[n]{r}}{c} h\left(\frac{cz}{\sqrt[n]{r}}\right), \quad z \in T_r, c \neq 0.$$

This implies that

$$|h_r(c, z)| = \frac{\sqrt[n]{r}}{|c|} h\left(\frac{cz}{\sqrt[n]{r}}\right) \geq \frac{\sqrt[n]{r}}{|c|} \frac{|cz|}{2\sqrt[n]{r}} \geq \frac{\sqrt[n]{r}}{2} > r, \quad z \in T_r.$$

So images of S_r and T_r under $h_r(c, z)$ do not cross each other.

Now let us check that $h_r(c, z)$ is a holomorphic motion. First $h_r(0, z) = z$ for $z \in E$. For fixed $x \in E$, $h_r(c, z)$ is holomorphic on $c \in \Delta$. For fixed $c \in \Delta$, $h_r(c, z)$ restricted to S_r and T_r , respectively, are injective. But the images of S_r and T_r under $h_r(c, z)$ do not cross each other. So $h_r(c, z)$ is injective on E . Thus

$$h_r(c, z) : \Delta \times E \rightarrow \hat{\mathbb{C}}$$

is a holomorphic motion. By Slodkowski's Theorem, it can be extended to a holomorphic motion

$$H_r(c, z) : \Delta \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}.$$

And moreover, for each $c \in \Delta$, $H_r(c, \cdot)$ is a quasiconformal map whose quasiconformal dilatation satisfies

$$K(H_r(c, \cdot)) \leq \frac{1 + |c|}{1 - |c|}.$$

Now consider $H(\sqrt[n]{r}, \cdot)$. It is a quasiconformal map with quasiconformal constant

$$K_r \leq \frac{1 + \sqrt[n]{r}}{1 - \sqrt[n]{r}}.$$

Let

$$A_{r,j} = \{z \in \mathbb{C} \mid \sqrt[n]{r} \leq |z| \leq \sqrt[n]{r^{j+1}}\}, \quad j = 0, 1, 2, \dots$$

Consider the restriction $\phi_{r,0} = H(\sqrt[n]{r}, \cdot)|_{A_{r,0}}$. It is an extension of ϕ_r , i.e., $\phi_{r,0}|_E = \phi_r$.

Let $\tilde{A}_{r,0}$ be the annulus bounded by S_r and $f^{-1}(S_r)$ and define $\tilde{A}_{r,j} = f^{-j}(\tilde{A}_{r,0})$, $j \geq 0$. Since $z \rightarrow z^n : A_{r,1} \rightarrow A_{r,0}$ and $f : \tilde{A}_{r,1} \rightarrow \tilde{A}_{r,0}$ are both covering maps of

degree n , so $\phi_{r,0}$ can be lifted to a quasiconformal map $\phi_{r,1} : A_{r,1} \rightarrow \tilde{A}_{r,1}$, i.e., the following diagram

$$\begin{array}{ccc} A_{r,1} & \xrightarrow{\phi_{r,1}} & \tilde{A}_{r,1} \\ \downarrow z \rightarrow z^n & & \downarrow f \\ A_{r,0} & \xrightarrow{\phi_{r,0}} & \tilde{A}_{r,0} \end{array}$$

commutes. We pick the lift $\phi_{r,1}$ such that it agrees with $\phi_{r,0}$ on T_r . The quasiconformal dilatation of $\phi_{r,1}$ is less than or equal to K_r .

For an integer $k > 0$, take $r = r_k = \delta^{n^k}$. Inductively, we can define a sequence of K_r -quasiconformal maps $\{\phi_{r,j}\}_{j=0}^k$ such that

$$\begin{array}{ccc} A_{r,j} & \xrightarrow{\phi_{r,j}} & \tilde{A}_{r,j} \\ \downarrow z \rightarrow z^n & & \downarrow f \\ A_{r,j-1} & \xrightarrow{\phi_{r,j-1}} & \tilde{A}_{r,j-1} \end{array}$$

commutes and $\phi_{r,j}$ and $\phi_{r,j-1}$ agree on the common boundary of $A_{r,j}$ and $A_{r,j-1}$. Note that

$$\overline{\Delta}_\delta = \Delta_r \cup \bigcup_{j=0}^k A_{r,j}.$$

Now we can define a quasiconformal map, which we still denote by ϕ_r as follows.

$$\phi_r(z) = \begin{cases} z, & z \in \Delta_r; \\ \phi_{r,j}, & z \in A_{r,j}, \quad j = 0, 1, \dots, k. \end{cases}$$

The quasiconformal dilatation of ϕ_r on Δ_δ is less than or equal to K_r and

$$f(\phi_r(z)) = \phi_r(z^n), \quad z \in \bigcup_{j=1}^k A_{r,j}.$$

Since $f(z) = z^n(1 + O(z))$, $f^k(z) = z^{n^k} \prod_{i=0}^{k-1} (1 + O(z^{n^i}))$. Because $\sqrt[n^k]{r_k} = \delta$, the range of ϕ_{r_k} on Δ_δ is a Jordan domain bounded above and below uniformly in k . In addition, 0 is fixed by ϕ_k and the quasiconformal dilatations of the ϕ_k are uniformly bounded in k . Therefore, the sequence $\{\phi_{r_k}\}_{k=1}^\infty$ is a compact family (see [1]). Let ϕ be a limiting map of this family. Then we have

$$f(\phi(z)) = \phi(z^n), \quad z \in \Delta_\delta.$$

Since the quasiconformal dilatation of ϕ is less than or equal to $(1 + \sqrt[n]{r_k})/(1 - \sqrt[n]{r_k})$ for all $k > 0$, it follows that ϕ is a 1-quasiconformal map, and thus conformal. This is the proof of the existence.

Suppose ϕ_1 and ϕ_2 are two conjugacies such that

$$\phi_1^{-1} \circ f \circ \phi_1(z) = z^n \quad \text{and} \quad \phi_2^{-1} \circ f \circ \phi_2(z) = z^n, \quad z \in \Delta_\delta.$$

For

$$\Phi(z) = \phi_2^{-1} \circ \phi_1(z) = \sum_{j=1}^{\infty} a_j z^j,$$

we have $\Phi(z^n) = (\Phi(z))^n$. This implies $a_1^n = a_1$ and $a_j = 0$ for $j \geq 2$. Since $a_1 \neq 0$, we have $a_1^{n-1} = 1$ and $\phi_2^{-1} = a_1 \phi_1^{-1}$. This is the uniqueness. \square

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