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# ON A QUESTION OF KATOK IN ONE-DIMENSIONAL CASE

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Abstract. We prove that a $C^1$ orientation-preserving circle endomorphism which is Hölder conjugate to a $C^1$ circle expanding endomorphism is itself expanding.

1. **Introduction and the main theorem.** Suppose $M$ is a $C^2$ dimension $n \geq 1$ Riemanian manifold. Suppose $f : M \to M$ is an orientation-preserving $C^{1+\alpha}$ endomorphism (or diffeomorphism) for some $0 < \alpha \leq 1$. Here the term endomorphism means that $f$ is locally diffeomorphic. Katok asked the following interesting question.

**Question 1.** *Suppose $f$ is $\beta$-Hölder conjugate to a $C^1$ expanding endomorphism (or a $C^1$ Anosov diffeomorphism) for some $0 < \beta \leq 1$. Is $f$ itself expanding (or Anosov)?*

In this paper we will give an affirmative answer to the question of Katok in the one-dimensional case.

**Theorem 1** (Main Theorem)**.** *Suppose $M$ is the unit circle. Let $f : M \to M$ be an orientation-preserving $C^{1+\alpha}$ endomorphism for some $0 < \alpha \leq 1$. Suppose $f$ is $\beta$-Hölder conjugate to a $C^1$ expanding endomorphism $g : M \to M$ for some $0 < \beta \leq 1$. Then $f$ itself is expanding.*

**Remark 1.** The condition $C^{1+\alpha}$ can be weaken to $C^1$; see Added Remarked 2.

In the theorem, $f$ is $\beta$-Hölder conjugate to $g$, we mean that there is a homeomorphism $h$ of the unit circle such that

$$f \circ h = h \circ g$$

and $h$ is $\beta$-Hölder continuous. (We do not need any additional condition on $h^{-1}$.) We would like to note that Höldericity is used only for the exponential decay property for Markov partitions from $g$ to $f$.

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The theorem is true for any one-dimensional Markov map as follows. Suppose $M$ is a closed interval or the unit circle. We call a piecewise $C^1$ map $f : M \to M$ is a Markov map if $M$ is cut into finitely many closed intervals $M = I_1 \cup \cdots \cup I_k$ such that (1) every pair of intervals have disjoint interiors; (2) $f|\overset{\circ}{\mathrm{I}}_i$ is a $C^1$-diffeomorphism for every $1 \leq i \leq k$; (3) $f(I_i)$ is a union of some intervals in $\{I_j\}_{1 \leq j \leq k}$. If (2) is replaced by (2′) $f|I_i$ is a $C^{1+\alpha}$-diffeomorphism for every $1 \leq i \leq k$, for some fixed $0 < \alpha \leq 1$, then we call it $C^{1+\alpha}$ Markov map. We have a more general theorem but the proof is similar to the proof of Theorem 1. So we omit the proof of Theorem 2.

**Theorem 2.** *Suppose $f : M \to M$ is a $C^{1+\alpha}$ Markov map for some $0 < \alpha \leq 1$. Suppose $f$ is $\beta$-Hölder conjugate to a $C^1$ expanding Markov map $g : M \to M$ for some $0 < \beta \leq 1$. Then $f$ itself is expanding.*

**Remark 2.** The condition $C^{1+\alpha}$ can be weaken to $C^1$; see Added Remarked 2.

## 2. **Exponential decay property.**

The universal cover of the unit circle

$$M = \{z \in \mathbb{C} \mid |z| = 1\}$$

is the real line $\mathbb{R}$. Let

$$\pi : \mathbb{R} \to M$$

be the universal cover such that $\pi(0) = 1$.

Suppose $f : M \to M$ is a covering map of degree $d > 1$. Then $f$ has a fixed point, which we always assume as 1. Let $F : \mathbb{R} \to \mathbb{R}$ be the lift of $f$ such that $F(0) = 0$. Then $F$ is an orientation-preserving diffeomorphism such that

$$\pi \circ F = f \circ \pi$$

and

$$F(x + 1) = F(x) + d, \ \forall x \in \mathbb{R}.$$

**Definition 1.** The map $f$ is called expanding if there are two constants $C > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C\lambda^n$$

for all $x \in \mathbb{R}$ and all $n \geq 1$.

The preimage $S = f^{-1}(1)$ contains $d$ points. It cuts $M$ into $d$ closed intervals $\eta_0 = \eta_{0,f} = \{I_1, \cdots, I_d\}$. Actually $\eta_0$ is a Markov partition in the meaning that

1. $M = \cup_{k=1}^d I_k$,
2. $I_i$ and $I_j$ have disjoint interiors for $1 \leq i \neq j \leq d$,
3. the restriction of $f$ on the interior of $I_i$ is one to one for every $1 \leq i \leq d$,
4. $f(I_i) = M$ for every $1 \leq i \leq d$.

Thus we can generate a sequence of Markov partitions

$$\eta_n = \eta_{n,f} = f^{-n}\eta_0$$

for $n = 1, 2, \cdots$. The set $\eta_n$ contains all intervals $I$ such that $f^n : I \to I_k$ for some $1 \leq k \leq d$ is a homeomorphism.

**Definition 2.** We say that $f$ has exponential decay property if the sequence $\{\eta_n\}_{n=0}^\infty$ is exponentially decaying to zero, that is, there are two constants $C > 0$ and $0 < \mu < 1$ such that

$$\max_{I \in \eta_n} |I| \leq C\mu^n$$

for all $n \geq 0$.

3. **Proof of the main theorem.** The idea of the proof is the following. First we need to show $f$ has the exponential decay property. Then from the exponential decay property we promote $f$ to have bounded distortion property. By using the bounded distortion property we further promote $f$ to be expanding. (See [1] for a similar discussion of calculating distortions for one-dimensional Markov maps with possible critical points.) The detailed proof is given as follows.

Since $g$ is expanding, by considering the sequence of Markov partitions $\{\eta_{n,g}\}_{n=0}^{\infty}$, we know that $g$ has the exponential decay property. That is there is a constant $C_0 > 0$ and $0 < \mu_0 < 1$ such that

$$|I| \leq C_0 \mu_0^n,$$

for all $I \in \eta_{n,g}$ and all $n \geq 0$.

Since $f$ is $\beta$-Hölder conjugate to $g$, we have a homeomorphism $h$ such that

$$f \circ h = h \circ g.$$

Without loss of generality, we assume that $h(1) = 1$. Let $H$ be the lift of $h$ such that $H(0) = 0$. Then $H(x+1) = H(x) + 1$ for all $x \in \mathbb{R}$. And there is a constant $C_1 > 0$ such that

$$\sup_{x \neq y \in \mathbb{R}, |x-y| \leq 1} \frac{|H(x) - H(y)|}{|x-y|^{\beta}} \leq C_1.$$

This implies that

$$|\pi(H(\tilde{I}))| \leq C_1 |\pi(\tilde{I})|^{\beta}$$

for all intervals $\tilde{I}$ such that $\pi(\tilde{I}) \in \eta_{n,g}$ for all $n \geq 0$. Note that $\pi(H(\tilde{I})) \in \eta_{n,f}$. Thus there are two constants $C_2 = C_0^{\beta} C_1 > 0$ and $0 < \mu_1 = \mu_0^{\beta} < 1$ such that

$$|I| \leq C_2 \mu_1^n,$$

for all $I \in \eta_{n,f}$ and all $n \geq 0$.

Given any interval $I \in \eta_{n,f}$, $f^n(I) = M$. Let $\tilde{I} \subset [0,1]$ be the lift of $I$. Then $F^n(\tilde{I}) = [0,1] \pmod 1$. For any $x, y \in \tilde{I}$, consider

$$\mathcal{A}_n(x,y) = log \frac{(F^n)'(x)}{(F^n)'(y)} = \sum_{i=0}^{n-1} \Big( \log F'(F^i(x)) - \log F'(F^i(y)) \Big) \tag{1}$$

Since $f$ is a $C^{1+\alpha}$ endomorphism and since $M$ is compact, we have a constant $C_3 > 0$ such that $F'(x) \geq C_3$. So

$$|\mathcal{A}_n(x,y)| \leq \frac{1}{C_3} \sum_{i=0}^{n-1} |F'(F^i(x)) - F'(F^i(y))|.$$

Since $f$ is $C^{1+\alpha}$, there is a constant $C_4 > 0$ such that

$$\sup_{x \neq y \in \mathbb{R}, |x-y| \leq 1} \frac{|F'(x) - F'(y)|}{|x-y|^{\alpha}} \leq C_4.$$

So

$$|\mathcal{A}_n(x,y)| \leq \frac{C_4}{C_3} \sum_{i=0}^{n-1} |\pi(F^i(x)) - \pi(F^i(y))|^{\alpha}.$$

Since $x, y \in \tilde{I}$ and $\pi(\tilde{I}) \in \eta_{n,f}$, $\pi(F^i(x)), \pi(F^i(y)) \in \pi(F^i(\tilde{I})) \in \eta_{n-i,f}$. So

$$|\pi(F^i(x)) - \pi(F^i(y))| \leq C_2 \mu_1^{n-i}.$$

Thus we get

$$|\mathcal{A}_n(x,y)| \leq \frac{C_4 C_2^{\alpha}}{C_3} \sum_{i=0}^{n-1} \mu_1^{\alpha(n-i)} \leq C_5 = \frac{C_4 C_2^{\alpha}}{C_3(1-\mu_1)} \tag{2}$$

Now for any point $x \in [0,1]$, $\pi(x) \in M$, there is a sequence of nested intervals $I_n \in \eta_{n,f}$ such that

$$\pi(x) \in \cdots \subset I_{n+1} \subset I_n \subset \cdots \subset I_1 \subset M.$$

For any $n \geq 1$, since $f^n(I_{n-1}) = M$. By the Mean Value Theorem, there is a $y \in \tilde{I}_{n-1}$ such that $(F^n)'(y)|\tilde{I}_{n-1}| = 1$, where $\tilde{I}_{n-1} \subset [0,1]$ is the lift of $I_{n-1}$. This implies that

$$(F^n)'(x)|\tilde{I}_{n-1}| = \frac{(F^n)'(x)}{(F^n)'(y)} \geq e^{-C_5}.$$

So we get that

$$(F^n)'(x) \geq \frac{e^{-C_5}}{|\tilde{I}_{n-1}|} \geq \frac{e^{-C_5}}{C_2 \mu_1^n} = C_6 \lambda^n \tag{3}$$

where $C_6 = 1/(e^{C_5} C_2)$ and $\lambda = 1/\mu_1$. This means that $f$ is expanding. We completed the proof of the Main Theorem.

**Added Remark 1.** In a recent communication with Professor Chengbo Yue and Professor Anatole Katok, I just learned that the answer to Question 1 is negative in the 2-dimensional case. Andrey Gogolev [3] constructed a concrete counterexample of a $C^{1+Lip}$ diffeomorphism of the 2-torus $T^2$ Hölder conjugate to an Anosov one but not Anosov itself. In fact, his counterexample can be constructed to be $C^r$ for any $r \in (1,3)$. It is interesting to note that in his 2006 Ph.D Thesis [2], Travis Fisher proved that a $C^{1+Lip}$-diffeomorphism that is conjugate to an Anosov one via a Hölder conjugacy $h$ is Anosov itself provided that the product of Hölder exponents for $h$ and $h^{-1}$ is greater than 1/2. In additional to this added remark, I would like to thank the referee for his comments and his correction of typos to the original version.

**Added Remark 2.** In a recent discussion with Professor Yongluo Cao, the condition that $f$ is $C^{1+\alpha}$ can be weaken to that $f$ is $C^1$ in Theorem 1 and in Theorem 2. The point is to have a weaker estimation for Inequality (2):

$$|\mathcal{A}_n(x,y)| \leq D_n \tag{4}$$

where $D_n/n \to 0$ as $n \to \infty$. One can take

$$a_n = \max_{I \in \eta_n} \left\{ \max_{x \in \tilde{I}} \log F'(x) - \min_{x \in \tilde{I}} \log F'(x) \right\}$$

and $D_n = \sum_1^n a_k$ since $\log F'$ is uniformly continuous on $\mathbb{R}$. This gives a weaker estimation for Inequality (3),

$$(F^n)'(x) \geq E_n \lambda^n = (\sqrt[n]{E_n}\lambda)^n \tag{5}$$

where $\sqrt[n]{E_n} \to 1$ as $n \to \infty$. Therefore, we have constants $C_7 > 0$ and $\lambda > \tilde{\lambda} > 1$ such that

$$(F^n)'(x) \geq C_7 \tilde{\lambda}^n \tag{6}$$

for all $x \in \mathbb{R}$ and $n \geq 1$. In additional to this added remark, I would like to thank Professor Yongluo Cao for pointing this to me.

## REFERENCES

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