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Combinatorial characterization of sub-hyperbolic rational maps

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Abstract

In 1980's, Thurston established a combinatorial characterization for post-critically finite rational maps among post-critically finite branched coverings of the two sphere to itself. A completed proof was written by Douady and Hubbard in their paper [A. Douady, J.H. Hubbard, A proof of Thurston's topological characterization of rational functions, Acta Math. 171 (1993) 263–297]. This criterion was then extended by Cui, Jiang, and Sullivan to sub-hyperbolic rational maps among sub-hyperbolic semi-rational branched coverings of the two sphere to itself. The goal of this paper is to present a new but simpler proof for the combinatorial characterization of sub-hyperbolic rational maps by adapting some arguments in the proof in Douady and Hubbard's paper.

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1. Introduction

We assume that the reader is familiar with the paper [6]. Let S^2 be the topological two sphere and $f : S^2 \to S^2$ be an orientation-preserving branched covering of degree $d \geqslant 2$. We denote by $\deg_x f$ the local degree of f at x. We will call

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$$\Omega_f = \{x \in S^2 \mid \deg_f(x) \geqslant 2\}$$

the critical set of f and

$$P_f = \overline{\bigcup_{k \geqslant 1} f^k(\Omega_f)}$$

the post-critical set. We say f is post-critically finite if P_f is a finite set.

In 1980's, Thurston established a combinatorial characterization for post-critically finite rational maps among post-critically finite branched coverings of S^2 to itself. A completed proof was written by Douady and Hubbard in their paper [6]. The theorem says that if the associated orbifold $\mathcal{O}_f$ for a post-critically finite f is hyperbolic, then f is combinatorially equivalent to a rational map if and only if it has no Thurston obstructions. The basic idea of the proof is as follows. Let $\widehat{\mathbb{C}}$ be the Riemann sphere. Consider the Teichmüller space T_f modeled on $(\widehat{\mathbb{C}}, P_f)$. Then f induces an analytic operator $\sigma_f : T_f \to T_f$. It turns out that the existence of a rational map which realizes f is equivalent to the existence of a fixed point of σ_f. The proof is then reduced to showing that σ_f has a unique attracting fixed point. The reader may refer to [6] for a detailed proof of this theorem.

A natural question is that to what extent, Thurston's theorem can be extended to rational maps with infinitely many post-critical points. It was proved by McMullen that having no Thurston obstruction is essentially true for any rational map with a hyperbolic orbifold — only trivial Thurston obstructions inside Siegel disks or Herman rings may occur for a rational map with a hyperbolic orbifold [10]. In a manuscript first circulated in 1994, Cui, Jiang, and Sullivan studied the combinatorial characterization problem for geometrically finite rational maps. They introduced sub-hyperbolic semi-rational branched coverings of S^2 to itself and established a combinatorial characterization of sub-hyperbolic rational maps among all sub-hyperbolic semi-rational branched coverings. The papers were published in [4,5]. Some relative details in [5] may also be found in [11]. After that, in the same spirit as [5], Cui and Tan presented in [3] an improved version. Both papers [3,5] are quite involved — a combinatorially complex and expositionally formidable surgery argument is used to reduce the problem to that of Thurston's original postcritically finite setup, together with checking that certain gluing data are analytically realizable. The goal of this paper is to give a new but simpler proof for the combinatorial characterization of sub-hyperbolic rational maps among sub-hyperbolic semi-rational branched coverings of S^2 to itself by adapting some arguments used in the proof of Thurston's theorem in Douady and Hubbard's paper. A combinatorial characterization of rational maps or entire functions is important and interesting in complex dynamical systems. Besides the work mentioned above, we would like also to mention Brown's thesis [2] on the unicritical polynomial case and Hubbard, Schleicher, and Shishikura's paper [9] on the exponential family.

Before we present the Main Theorem in this paper, let us introduce some definitions first. We say f is geometrically finite if P_f is an infinite set but with finitely many accumulation points. Suppose that f is geometrically finite. Then it is not difficult to see that the accumulation set of P_f consists of finitely many periodic cycles. We leave this to the reader as an exercise. Let P'_f denote the set of all the accumulation points of P_f. Throughout the whole paper, we use $\widehat{\mathbb{C}}$ to denote the Riemann sphere which is the two sphere endowed with the standard complex structure.

Definition 1.1. Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a *geometrically finite* branched covering of degree $d\geqslant 2$. We say f is a *sub-hyperbolic semi-rational* branched covering if for any $a\in P'_f$ of period $p\geqslant 1$, there is an open neighborhood U of a, such that f is holomorphic in U, and moreover, if $\deg_a f^p=1$, then

$$f^p(z)=a+\lambda(z-a)+o\big(|z-a|\big)\quad \text{for } z\in U$$

where $0<|\lambda|<1$ is some constant, and if $\deg_a f^p=k>1$, then

$$f^p(z)=a+\alpha(z-a)^k+o\big(|z-a|^k\big)\quad \text{for } z\in U$$

where $\alpha\neq 0$ is some constant.

As in the post-critically finite case, one can define Thurston obstructions for a sub-hyperbolic semi-rational branched covering f in a similar way. If γ is a simple closed curve in $S^2\setminus P_f$, then the set $f^{-1}(\gamma)$ is a union of disjoint simple closed curves. If γ moves continuously, so does each component of $f^{-1}(\gamma)$. A simple closed curve γ is non-peripheral if each component of $S^2\setminus\gamma$ contains at least two points of P_f. Consider a multi-curve

$$\Gamma=\{\gamma_1,\dots,\gamma_n\}$$

of simple, closed, disjoint, no two homotopic, and non-peripheral curves in $S^2\setminus P_f$. We say that Γ is f-stable if for any $\gamma\in\Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is homotopic in $S^2\setminus P_f$ to an element of Γ.

For each f-stable multi-curve Γ, define a linear transformation,

$$f_\Gamma:\mathbb{R}^\Gamma\to\mathbb{R}^\Gamma$$

as follows: let $\gamma_{i,j,\alpha}$ denote the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\widehat{\mathbb{C}}\setminus P_f$ and $d_{i,j,\alpha}$ be the degree of $f|_{\gamma_{i,j,\alpha}}:\gamma_{i,j,\alpha}\to\gamma_j$. Define

$$f_\Gamma(\gamma_j)=\sum_i\bigg(\sum_\alpha\frac{1}{d_{i,j,\alpha}}\bigg)\gamma_i.$$

Since the matrix of f_Γ is non-negative, there exists a largest eigenvalue $\lambda(\Gamma,f)\in\mathbb{R}_+$. We say that a multi-curve Γ is a Thurston obstruction of f if $\lambda(\Gamma,f)\geqslant 1$.

Definition 1.2. Suppose f and g are two sub-hyperbolic semi-rational branched coverings. We say that they are *CLH-equivalent* (combinatorially and locally holomorphically equivalent) if there exist a pair of homeomorphisms $\phi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ and $\psi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that

- ψ is isotopic to ϕ rel P_f,
- $\phi f=g\psi$,
- $\phi|U_f=\psi|U_f$ is holomorphic on some open set $U_f\supset P'_f$.

Now let us state the Main Theorem in this paper.

Main Theorem. *Suppose f is a sub-hyperbolic semi-rational branched covering. Then f is CLH-equivalent to a rational map R if and only if f has no Thurston obstructions. In this case, the rational map R is unique up to a Möbius conjugation of the Riemann sphere.*

Remark 1.1. There are branched coverings of S^2 to itself which are geometrically finite and having no Thurston obstructions but are not combinatorially equivalent to rational maps. For the construction of such maps, see [4].

The proof of the necessity part follows from a theorem of McMullen (see Appendix B of [10]). The main task of this paper is to prove the sufficiency part.

The essential difference between the post-critically finite case and the sub-hyperbolic case is that in the first case, the post-critical set is a finite set and the Thurston pull back induces an analytic operator defined on a finite-dimensional Teichmüller space, while in the latter case, the post-critical set is an infinite set and therefore, the induced operator is defined on an infinite-dimensional Teichmüller space. However, we observe in this paper that, in both cases, the following bounded geometry properties are similar. This allows us to prove the latter case by adapting the argument in the proof of the first case.

In the post-critically finite case, the basepoint of the Teichmüller space is the Riemann sphere minus the set of finite number of post-critical points. The branched covering induces a pull-back operator on this Teichmüller space. Iterations of this operator produce a sequence of sets of finite number of points in the Riemann sphere. The bounded geometry in this case means that there is a positive constant such that any two points in any element of this sequence have spherical distance greater than or equal to this constant.

In the sub-hyperbolic case, the basepoint of the Teichmüller space is the Riemann sphere minus the union of finitely many points and topological disks. Iterations of the pull-back operator produce a sequence of sets of finite number of points plus finite number of disks in the Riemann sphere. The bounded geometry in this case means that there is a positive constant such that in any element of this sequence, the spherical distance between any two points, any point and any disk, or any two disks is greater than or equal to this constant; moreover, any disk in any element of this sequence contains another round disk of radius greater than or equal to this constant.

The paper is organized as follows. In Section 2, we state the Shielding Ring Lemma which is crucial for our construction of the Teichmüller space. In Section 3, we construct the Teichmüller space T_f. In Section 4, we introduce the pull back operator $\sigma_f : T_f \to T_f$. In Section 5, we introduce the concept of bounded geometry. In Section 6, we prove that bounded geometry implies the strictly contracting property of σ_f. In Section 7, we prove that no Thurston obstruction implies the bounded geometry. This completes the proof of the Main Theorem.

2. Shielding Ring Lemma

We say an open annulus A is attached to an open topological disk D from the outside if A and D are disjoint but ∂D is one of the boundary components of the annulus A. Then $\overline{D \cup A}$ is a larger closed disk.

Suppose that f is a sub-hyperbolic semi-rational branched covering. Let $P'_f = \{a_i\}$.

Lemma 2.1 *(Shielding Ring Lemma). There is a finite collection $\{D_i\}$ of open disks and a finite collection of open annuli $\{A_i\}$ such that*

- *$a_i \in D_i$,*
- *every ∂D_i is a real analytic curve,*

- $\overline{D_i} \cap \overline{D_j} = \emptyset$ *for* $i \neq j$,
- *for each* i, A_i *is an annulus attaching* D_i *from the outside such that* $\overline{A_i} \cap P_f = \emptyset$,
- f *is holomorphic on* $\overline{D_i} \cup A_i$,
- *every* $f(\overline{D_i} \cup A_i)$ *is contained in some* D_j.

Lemma 2.1 follows easily from the fact that f is holomorphic attracting in an open neighborhood of every periodic cycle in P'_f. The reader shall easily supply the details of the proof. From now on we call the A_i and D_i respectively the shielding rings and holomorphic disks.

3. The Teichmüller space T_f

Let us now fix a collection of holomorphic disks $\{D_i\}$ and a collection of shielding rings $\{A_i\}$ for f. Let

$$D = \bigcup_i D_i \quad \text{and} \quad P_1 = P_f \setminus D.$$

By taking D_i smaller, we may assume that $\#(P_1) \geqslant 3$. We may further assume that $\{0, 1, \infty\} \subset P_1$. Define

$$Q = P_1 \cup \overline{D} \quad \text{and} \quad X = \partial Q = P_1 \cup \partial D.$$

Definition 3.1. The Teichmüller space T_f is the Teichmüller space modeled on $(\widehat{\mathbb{C}} \setminus Q, X)$.

The Teichmüller space T_f can be constructed as the space of all the Beltrami coefficients defined on $\widehat{\mathbb{C}} \setminus Q$ module the following equivalent relation: let μ and ν be two Beltrami coefficients defined on $\widehat{\mathbb{C}} \setminus Q$ and let

$$\phi_\mu : \widehat{\mathbb{C}} \setminus Q \to S \quad \text{and} \quad \phi_\nu : \widehat{\mathbb{C}} \setminus Q \to R$$

be two quasiconformal homeomorphisms which solve the Beltrami equations given by μ and ν, respectively. We say μ and ν are equivalent to each other if there exists a holomorphic isomorphism $h : R \to S$ such that the map ϕ_μ and $h \circ \phi_\nu$ are isotopic to each other rel X, that is, there is a continuous family of quasiconformal homeomorphisms $g_t : \widehat{\mathbb{C}} \setminus Q \to S$, $0 \leqslant t \leqslant 1$, such that

1. $g_0 = \phi_\mu$,
2. $g_1 = h \circ \phi_\nu$,
3. $g_t(z) = \phi_\mu(z) = (h \circ \phi_\nu)(z)$ for all $0 \leqslant t \leqslant 1$ and $z \in X$.

In the following we use $[\mu]$ to denote the element in T_f represented by μ.

Now let us give a brief description of the relative background about the Teichmüller space T_f. The reader may refer to [8] and [7] for more knowledge in this aspect.

Let $M(\widehat{\mathbb{C}} \setminus Q)$ denote the space of all the measurable Beltrami differentials $\mu(z)\frac{d\bar{z}}{dz}$ on $\widehat{\mathbb{C}} \setminus Q$ with $\|\mu\|_\infty < \infty$. Then $M(\widehat{\mathbb{C}} \setminus Q)$ has a natural complex analytic structure, and moreover, it is a Banach analytic manifold with respect to the norm $\|\cdot\|_\infty$. Let $B(\widehat{\mathbb{C}} \setminus Q) \subset M(\widehat{\mathbb{C}} \setminus Q)$ denote the unit ball. Then $B(\widehat{\mathbb{C}} \setminus Q)$ consists of all the Beltrami coefficients on $\widehat{\mathbb{C}} \setminus Q$. Let

$$P : B(\widehat{\mathbb{C}} \setminus Q) \to T_f$$

be the projection map given by $\mu \mapsto [\mu]$.

Lemma 3.1 *(See Chapter 6 of [8]). There exists a unique complex analytic structure on T_f such that with respect to this structure, the map P is complex analytic, and moreover, the map P is a holomorphic split submersion.*

Let μ be a Beltrami coefficient defined on $\widehat{\mathbb{C}} \setminus Q$. Let

$$\phi_\mu : \widehat{\mathbb{C}} \setminus Q \to \phi_\mu(\widehat{\mathbb{C}} \setminus Q)$$

be a quasiconformal homeomorphism which solves the Beltrami equation given by μ. Let

$$\mathrm{M}_\mu = \left\{ \xi(z) \frac{d\bar{z}}{dz} \;\middle|\; \xi(z) \text{ is measurable and } \|\xi\|_\infty < \infty \right\}$$

be the linear space of all the Beltrami differentials defined on $\phi_\mu(\widehat{\mathbb{C}} \setminus Q)$. Let

$$\mathrm{A}_\mu = \left\{ q(z)\, dz^2 \;\middle|\; q(z) \text{ is holomorphic and } \int\limits_{\phi_\mu(\widehat{\mathbb{C}} \setminus Q)} \bigl|q(z)\bigr|\, dz \wedge d\bar{z} < \infty \right\}$$

be the linear space of all the integrable holomorphic quadratic differentials defined on $\phi_\mu(\widehat{\mathbb{C}} \setminus Q)$.

A Beltrami differential $\xi(z)\frac{d\bar{z}}{dz} \in \mathrm{M}_\mu$ is called *infinitesimally trivial* if

$$\int\limits_{\phi_\mu(\widehat{\mathbb{C}} \setminus Q)} \xi(z) q(z)\, |dz|^2 = 0$$

holds for all $q(z)\, dz^2 \in \mathrm{A}_\mu$.

Let $\mathrm{N}_\mu \subset \mathrm{M}_\mu$ be the subspace of all the *infinitesimally trivial* Beltrami differentials. Then the tangent space of T_f at $[\mu]$ is isomorphic to the quotient space $\mathrm{M}_\mu/\mathrm{N}_\mu$.

Let μ be a Beltrami coefficient defined on $\widehat{\mathbb{C}} \setminus Q$. Let ξ be a tangent vector of T_f at $[\mu]$ which is identified with a Beltrami differential $\xi(z)\frac{d\bar{z}}{dz}$ defined on $\phi_\mu(\widehat{\mathbb{C}} \setminus Q)$.

Definition 3.2. The Teichmüller norm of the tangent vector ξ is defined to be

$$\|\xi\| = \sup \left| \int\limits_{\phi_\mu(\widehat{\mathbb{C}} \setminus Q)} q(z)\xi(z)\, |dz|^2 \right|,$$

where the sup is taken over all $q(z)\, dz^2 \in \mathrm{A}_\mu$ with $\int_{\phi_\mu(\widehat{\mathbb{C}} \setminus Q)} |q(z)||dz|^2 = 1$.

Definition 3.3. Let $[\mu], [\nu] \in T_f$. The Teichmüller distance $d_T([\mu], [\nu])$ is defined to be

$$\frac{1}{2} \inf \log K\bigl(\phi_{\mu'} \circ \phi_{\nu'}^{-1}\bigr)$$

where $\phi_{\mu'}$ and $\phi_{\nu'}$ are quasiconformal mappings with Beltrami coefficients μ' and ν' and the inf is taken over all μ' and ν' in the same Teichmüller classes as μ and ν, respectively.

Lemma 3.2. *Let μ and ν be two Beltrami coefficients defined on $\widehat{\mathbb{C}} \setminus Q$. Then*

$$d_T\big([\mu],[\nu]\big) = \inf \int_0^1 \big\| \tau'(t) \big\| \, dt$$

where inf *is taken over all the piecewise smooth curves $\tau(t)$ in T_f such that $\tau(0) = [\mu]$ and $\tau(1) = [\nu]$.*

4. The pull-back operator

As in the post-critically finite case, we may assume that f is a quasiregular map. (This is because except the finite holomorphic disks, there are only finitely many points in P_f, and therefore, the CLH-equivalent class of f must contain a quasiregular branched covering of the Riemann sphere $\widehat{\mathbb{C}}$.)

Remind that for a Beltrami coefficient μ defined on $\widehat{\mathbb{C}}$, the pull back of μ by f, which is denoted by $f^*(\mu)$, is defined to be

$$(f^*\mu)(z) = \frac{\mu_f(z) + \mu(f(z))\theta(z)}{1 + \overline{\mu_f(z)}\mu(f(z))\theta(z)} \tag{1}$$

where $\theta(z) = \overline{f_z}/f_z$ and $\mu_f(z) = f_{\bar{z}}/f_z$. It is important to note that if μ depends complex analytically on t, then so does $f^*(\mu)$.

Now let $\mu(z)$ be a Beltrami coefficient defined on $\widehat{\mathbb{C}} \setminus Q$. Define the Beltrami coefficient $\mathrm{Ext}(\mu)(z)$ on $\widehat{\mathbb{C}}$ by setting

$$\mathrm{Ext}(\mu)(z) = \begin{cases} \mu(z) & \text{for } z \in \widehat{\mathbb{C}} \setminus Q, \\ 0 & \text{for otherwise.} \end{cases} \tag{2}$$

By (1), $f^*(\mathrm{Ext}(\mu))$ is a Beltrami coefficient on $\widehat{\mathbb{C}}$. Let us simply use $f^*(\mu)$ to denote the restriction of $f^*(\mathrm{Ext}(\mu))$ on $\widehat{\mathbb{C}} \setminus Q$. Thus we can define a map

$$f^* : B(\widehat{\mathbb{C}} \setminus Q) \to B(\widehat{\mathbb{C}} \setminus Q)$$

by $\mu \mapsto f^*(\mu)$.

Lemma 4.1. *The map f^* induces a complex analytic operator $\sigma_f : T_f \to T_f$.*

Proof. Let us first show that the map σ_f is well defined. Suppose μ and ν are two Beltrami coefficients defined on $\widehat{\mathbb{C}} \setminus Q$ which are equivalent to each other. Let $\mathrm{Ext}(\mu)$ and $\mathrm{Ext}(\nu)$ be their extensions to $\widehat{\mathbb{C}}$. Let $\phi_{\mathrm{Ext}(\mu)}$ and $\phi_{\mathrm{Ext}(\nu)}$ be the corresponding quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$ which fix 0, 1, and the infinity. Let ϕ_μ and ϕ_ν denote their restrictions to $\widehat{\mathbb{C}} \setminus Q$, respectively. Since μ is equivalent to ν, we have a holomorphic isomorphism

$$h : \widehat{\mathbb{C}} \setminus \phi_{\mathrm{Ext}(\nu)}(Q) \to \widehat{\mathbb{C}} \setminus \phi_{\mathrm{Ext}(\mu)}(Q)$$

such that ϕ_μ is isotopic to $h \circ \phi_\nu$ rel X. Now define a homeomorphism $\mathrm{Ext}(h) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by setting

$$\mathrm{Ext}(h)(z) = \begin{cases} h(z) & \text{for } z \in \widehat{\mathbb{C}} \setminus \phi_{\mathrm{Ext}(\nu)}(Q), \\ \phi_{\mathrm{Ext}(\mu)} \circ \phi_{\mathrm{Ext}(\nu)}^{-1}(z) & \text{for otherwise.} \end{cases} \tag{3}$$

It is clear that $\mathrm{Ext}(h)$ is holomorphic everywhere except those points in $\phi_{\mathrm{Ext}(\nu)}(X)$. Since $\phi_{\mathrm{Ext}(\nu)}(X)$ is the union of finitely many points and finitely many quasicircles by Lemma 2.1, it follows that $\mathrm{Ext}(h)$ is a holomorphic homeomorphism of $\widehat{\mathbb{C}}$ to itself, and therefore a Möbius map. By the normalization condition, $\mathrm{Ext}(h)$ fixes 0, 1, and ∞ also. So $\mathrm{Ext}(h) = \mathrm{id}$. This implies that ϕ_μ and ϕ_ν are isotopic to each other rel X, and in particular, $\phi_\mu = \phi_\nu$ on X. Since $\phi_{\mathrm{Ext}(\mu)}$ and $\phi_{\mathrm{Ext}(\nu)}$ are holomorphic on D, it follows that $\phi_{\mathrm{Ext}(\mu)} = \phi_{\mathrm{Ext}(\nu)}$ on Q and therefore are isotopic to each other rel Q. Since $f(Q) \subset Q$, we can therefore lift this isotopy and get an isotopy between $\phi_{f^*(\mathrm{Ext}(\mu))}$ and $\phi_{f^*(\mathrm{Ext}(\nu))}$ rel Q. It follows that $\phi_{f^*(\mu)}$ and $\phi_{f^*(\nu)}$, which are respectively the restrictions of $\phi_{f^*(\mathrm{Ext}(\mu))}$ and $\phi_{f^*(\mathrm{Ext}(\nu))}$ on $\widehat{\mathbb{C}} \setminus Q$, are isotopic to each other rel X. This implies that $[f^*(\mu)] = [f^*(\nu)]$. We can thus define a map

$$\sigma_f : T_f \to T_f$$

by $\sigma_f([\mu]) = [f^*(\mu)]$.

Now it remains to proves that σ_f is complex analytic. To see this, note that by (1) the map

$$f^* : B(\widehat{\mathbb{C}} \setminus Q) \to B(\widehat{\mathbb{C}} \setminus Q)$$

is complex analytic. Since by Lemma 3.1 the projection map

$$P : B(\widehat{\mathbb{C}} \setminus Q) \to T_f$$

is a holomorphic split submersion, it follows that σ_f is analytic also. This completes the proof of the lemma. $\square$

Once no confusion is caused, let us simply use μ to denote either $\mathrm{Ext}(\mu)$ or μ. Let $\tilde{\mu}(z) = f^*(\mu)$.

Let $\phi_\mu, \phi_{\tilde{\mu}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the quasiconformal homeomorphisms which fix 0, 1, and the infinity and which solve the Beltrami equations given by μ and $\tilde{\mu}$, respectively. Let

$$g = \phi_\mu \circ f \circ \phi_{\tilde{\mu}}^{-1}.$$

It is clear that g is a rational map and the following diagram commutes.

$$\begin{array}{ccc} (\widehat{\mathbb{C}}, Q) & \xrightarrow{\phi_{\tilde{\mu}}} & (\widehat{\mathbb{C}}, \phi_{\tilde{\mu}}(Q)) \\ \Big\downarrow f & & \Big\downarrow g \\ (\widehat{\mathbb{C}}, Q) & \xrightarrow{\phi_{\mu}} & (\widehat{\mathbb{C}}, \phi_{\mu}(Q)) \end{array}$$

Now suppose that ξ is a tangent vector of T_f at $\tau = [\mu]$. This means that there is a smooth curve of Beltrami coefficients $\gamma(t)$ defined on $\widehat{\mathbb{C}} \setminus Q$, such that $\gamma(0) = \mu$ and

$$\xi = \frac{d}{dt}\bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1}}. \tag{4}$$

Let $d\sigma_f|_\tau$ denote the tangent map of σ_f at τ. Let $\tilde{\xi} = d\sigma_f|_\tau(\xi)$.

Lemma 4.2. *Let ξ and $\tilde{\xi}$ be as above. Then*

$$\tilde{\xi}(w) = \xi\big(g(w)\big)\frac{\overline{g'(w)}}{g'(w)}. \tag{5}$$

Proof. Note that

$$\tilde{\xi} = \frac{d}{dt}\bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ f \circ \phi_{\tilde{\mu}}^{-1}} = \frac{d}{dt}\bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1} \circ \phi_\mu \circ f \circ \phi_{\tilde{\mu}}^{-1}} = \frac{d}{dt}\bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1} \circ g}.$$

Since g is a rational map, by (1) we have

$$\mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1} \circ g}(w) = \mu_{\phi_{\gamma(t)} \circ \phi_\mu^{-1}}\big(g(w)\big)\frac{\overline{g'(w)}}{g'(w)}.$$

The lemma then follows from (4). □

Let $\tilde{q} = \tilde{q}(w)\,dw^2$ be a non-zero integrable holomorphic quadratic differential defined on $\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}}(Q)$. Define

$$q(z) = \sum_{g(w)=z} \frac{\tilde{q}(w)}{[g'(w)]^2}. \tag{6}$$

It is easy to see that $q = q(z)\,dz^2$ is a holomorphic quadratic differential defined on $\widehat{\mathbb{C}} \setminus \phi_\mu(Q)$.

Proposition 4.1. *For q and $\tilde{q}$ given as above, we have*

$$\int\limits_{\widehat{\mathbb{C}} \setminus \phi_\mu(Q)} \big|q(z)\big||dz|^2 \leqslant \int\limits_{\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}}(Q)} \big|\tilde{q}(w)\big||dw|^2 - \int\limits_{\bigcup_i \phi_{\tilde{\mu}}(A_i)} \big|\tilde{q}(w)\big||dw|^2.$$

Proof. By the definition of $\tilde{q}$, we have

$$\int\limits_{\widehat{\mathbb{C}} \setminus \phi_\mu(Q)} \big|q(z)\big||dz|^2 = \int\limits_{\widehat{\mathbb{C}} \setminus \phi_\mu(Q)} \bigg|\sum_{g(w)=z} \frac{\tilde{q}(w)}{[g'(w)]^2}\bigg||dz|^2.$$

Since $f(\bigcup A_i) \subset \bigcup D_i$ and $|dz|^2 = |g'(w)|^2\,|dw|^2$, we have

$$\int_{\widehat{\mathbb{C}}\setminus\phi_\mu(Q)} \left|\sum_{g(w)=z} \frac{\tilde{q}(w)}{[g'(w)]^2}\right| |dz|^2 \leqslant \int_{(\widehat{\mathbb{C}}\setminus\phi_{\tilde{\mu}}(Q))\setminus(\bigcup_i \phi_{\tilde{\mu}}(A_i))} \left|\tilde{q}(w)\right| |dw|^2.$$

Proposition 4.1 then follows since the right hand of the above inequality is equal to

$$\int_{\widehat{\mathbb{C}}\setminus\phi_{\tilde{\mu}}(Q)} \left|\tilde{q}(w)\right| |dw|^2 - \int_{\bigcup_i \phi_{\tilde{\mu}}(A_i)} \left|\tilde{q}(w)\right| |dw|^2. \qquad \square$$

Proposition 4.2. *We have the following duality of the pairing,*

$$\int_{\widehat{\mathbb{C}}\setminus\phi_{\tilde{\mu}}(Q)} \tilde{\xi}(w)\tilde{q}(w)\,|dw|^2 = \int_{\widehat{\mathbb{C}}\setminus\phi_\mu(Q)} \xi(z)q(z)\,|dz|^2.$$

Proof. Note that $\phi_{\tilde{\mu}}(Q) \subset g^{-1}(\phi_\mu(Q))$ and by (5) $\tilde{\xi}(w) = 0$ for all $w \in g^{-1}(\phi_\mu(Q)) \setminus \phi_{\tilde{\mu}}(Q)$. We thus have

$$\int_{\widehat{\mathbb{C}}\setminus\phi_{\tilde{\mu}}(Q)} \tilde{\xi}(w)\tilde{q}(w)\,|dw|^2 = \int_{\widehat{\mathbb{C}}\setminus g^{-1}(\phi_\mu(Q))} \tilde{\xi}(w)\tilde{q}(w)\,|dw|^2.$$

Now Proposition 4.2 follows from (5), (6) and the fact that $|dz|^2 = |g'(w)|^2\,|dw|^2$. $\square$

As a direct consequence of Propositions 4.1 and 4.2, we have

Corollary 4.1. *Let $\tau \in T_f$. Then $\|d\sigma_f|_\tau\| \leqslant 1$.*

Remark 4.1. Corollary 4.1 also follows from a general fact that a complex analytic operator does not increase Kobayashi's metric which is equal to Teichmüller metric in this case (refer to, for examples, [8] and [7]). But our particular argument used here will be improved in the latter sections to prove a strict inequality.

The next lemma reduces the proof of the Main Theorem to showing that the pull back operator σ_f has a unique fixed point in T_f.

Lemma 4.3. *The map f is CLH-equivalent to a unique rational map (up to Möbius conjugations) if and only if σ_f has a unique fixed point in T_f.*

Proof. If σ_f has a fixed point $[\mu]$ in T_f, then $\tilde{\mu} = f^*\mu \sim \mu$. Let $\mathrm{Ext}(\mu)$ be the extension of μ to $\widehat{\mathbb{C}}$. Let $\phi_{\mathrm{Ext}(\mu)}$ and $\phi_{f^*(\mathrm{Ext}(\mu))}$ be the corresponding quasiconformal homeomorphisms which fix 0, 1, and the infinity. Let ϕ_μ and $\phi_{\tilde{\mu}}$ be their restrictions to $\widehat{\mathbb{C}} \setminus Q$, respectively. It follows that there is a conformal isomorphism

$$h : \widehat{\mathbb{C}} \setminus \phi_\mu(Q) \to \widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}}(Q)$$

such that $\phi_{\tilde{\mu}}$ and $h \circ \phi_\mu$ are isotopic to each other rel X. As in the proof of Lemma 4.1, one can show that such h is actually equal to the identity map. In fact, we can again define a homeomorphism $\mathrm{Ext}(h) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by setting

$$\mathrm{Ext}(h)(z) = \begin{cases} h(z) & \text{for } z \in \widehat{\mathbb{C}} \setminus \phi_{\mathrm{Ext}(\mu)}(Q), \\ \phi_{f^*(\mathrm{Ext}(\mu))} \circ \phi_{\mathrm{Ext}(\mu)}^{-1}(z) & \text{for otherwise.} \end{cases} \tag{7}$$

It is clear that $\mathrm{Ext}(h)$ is holomorphic everywhere except those points in $\phi_{\mathrm{Ext}(\mu)}(X)$. Since $\phi_{\mathrm{Ext}(\mu)}(X)$ is the union of finitely many points and finitely many quasicircles by Lemma 2.1, it follows that $\mathrm{Ext}(h)$ is a holomorphic homeomorphism of $\widehat{\mathbb{C}}$ to itself, and therefore a Möbius map. By the normalization condition, $\mathrm{Ext}(h)$ fixes 0, 1, and ∞ also. So $\mathrm{Ext}(h) = \mathrm{id}$. This implies that ϕ_μ and $\phi_{\tilde{\mu}}$ are isotopic to each other rel X. It follows that $\phi_{\mathrm{Ext}(\mu)}$ and $\phi_{f^*(\mathrm{Ext}(\mu))}$ are isotopic to each other rel Q. Note that when restricted to D, $\phi_{\mathrm{Ext}(\mu)}$ and $\phi_{f^*(\mathrm{Ext}(\mu))}$ are analytic and equal to each other. This implies that f is CLH-equivalent to the rational map $g = \phi_{\mathrm{Ext}(\mu)} \circ f \circ \phi_{f^*(\mathrm{Ext}(\mu))}^{-1}$.

If f is CLH-equivalent to g, then we have a Beltrami coefficient μ defined on $\widehat{\mathbb{C}} \setminus Q$ such that $g = \phi_{\mathrm{Ext}(\mu)} \circ f \circ \phi_{f^*(\mathrm{Ext}(\mu))}^{-1}$ and moreover, $\phi_{\mathrm{Ext}(\mu)}$ and $\phi_{f^*(\mathrm{Ext}(\mu))}$ are isotopic to each other rel Q. This implies that ϕ_μ and $\phi_{\tilde{\mu}}$ are isotopic to each other rel X. It follows that $[f^*(\mu)] = [\mu]$ and thus $\sigma_f([\mu]) = [\mu]$.

It is clear that the fixed point $[\mu]$ is unique is equivalent to say that g is unique up to Möbius conjugations. □

5. Bounded geometry

Let $d(X, Y)$ denote the spherical distance between two subsets of $\widehat{\mathbb{C}}$. Recall that

$$D = \bigcup_i D_i, \quad P_1 = P_f \setminus D, \quad \text{and} \quad P_f' = \{a_i\}.$$

Definition 5.1. Let $b > 0$ be a constant. Let $T_{f,b} \subset T_f$ be the subspace such that for every $[\mu] \in T_{f,b}$, if $\phi_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the quasiconformal homeomorphism which fixes 0, 1, and the infinity, and which solves the Beltrami equation given by $\mathrm{Ext}(\mu)$, then the following conditions hold,

(1) for all $z_i \neq z_{i'} \in P_1$,

$$d\big(\phi_\mu(z_i), \phi_\mu(z_{i'})\big) \geqslant b;$$

(2) for all $z_j \in P_1$ and all D_i,

$$d\big(\phi_\mu(z_j), \phi_\mu(D_i)\big) \geqslant b;$$

(3) for all $D_i \neq D_{i'}$,

$$d\big(\phi_\mu(D_i), \phi_\mu(D_{i'})\big) \geqslant b;$$

(4) for every D_i, $\phi_\mu(D_i)$ contains a round disk of radius b centered at $\phi_\mu(a_i)$.

Let $K > 1$. Then the family of all the K-quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$ to itself, which fix 0, 1, and the infinity, is compact. We thus have

Lemma 5.1. *Let $K > 1$. Then for every $\delta > 0$, there is an $\epsilon > 0$ depending only on K and δ such that for every two points $x, y \in \widehat{\mathbb{C}}$ with $d(x, y) > \delta$, and every K-quasiconformal homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes 0, 1, and the infinity, we have $d(\phi(x), \phi(y)) > \epsilon$.*

By Definitions 3.3 and 5.1, and Lemma 5.1, we have

Lemma 5.2. *Let $b, D > 0$. Then there is a $b' > 0$ depending only on b and D such that for any two Beltrami coefficients μ and ν defined on $\widehat{\mathbb{C}} \setminus Q$, if $d_T([\mu], [\nu]) < D$ and $\mu \in T_{f,b}$, then $\nu \in T_{f,b'}$.*

Definition 5.2. Let Z be a subset of Q with $\#(Z) \geqslant 4$. Let $[\mu] \in T_f$ and $\gamma \subset \widehat{\mathbb{C}} \setminus Z$ be a simple closed and non-peripheral curve. We use $\|\gamma\|_{\mu,Z}$ to denote the hyperbolic length of the unique simple closed geodesic ξ which is homotopic to $\phi_\mu(\gamma)$ in the hyperbolic Riemann surface $\widehat{\mathbb{C}} \setminus \phi_\mu(Z)$. We say γ is a (μ, Z)-simple closed geodesic if $\phi_\mu(\gamma)$ is a simple closed geodesic in $\widehat{\mathbb{C}} \setminus \phi_\mu(Z)$.

For each holomorphic disk D_i, fix a point b_i on the boundary ∂D_i. Set

$$E = P_1 \cup \bigcup_i \{a_i, b_i\}.$$

Note that P_1 contains 0, 1, and the infinity by our assumption. Since $P_1 \subset E$ and ϕ_μ fixes 0, 1, and the infinity, it follows that E and $\phi_\mu(E)$ contain 0, 1, and the infinity also. By the proof of Lemma 4.1, for every non-peripheral curve $\gamma \subset \widehat{\mathbb{C}} \setminus Q$ and every $[\mu] \in T_f$, the quantity $\|\gamma\|_{\mu,E}$ is well defined.

Lemma 5.3. *Let $a > 0$. Then there is a $b > 0$ depending only on a such that for every Beltrami coefficient μ defined on $\widehat{\mathbb{C}} \setminus Q$ with $\mu(z) = 0$ on $\bigcup_i A_i$, if every (μ, E)-simple closed geodesic $\gamma \subset \widehat{\mathbb{C}} \setminus Q$ has hyperbolic length not less than a, then $\mu \in T_{f,b}$.*

Proof. Note that $\#(\phi_\mu(E)) = \#(E)$ is finite. Since $\phi_\mu(E)$ contains 0, 1, and the infinity, it follows that the spherical distance between any two points in $\phi_\mu(E)$ has a positive lower bound which depends only on a and $\#(E)$. Since ϕ_μ is holomorphic in every topological disk $\overline{D_i} \cup A_i$ and since $\phi_\mu(\overline{D}_i)$ contains $\phi_\mu(a_i)$ and $\phi_\mu(b_i)$, it follows from Koebe's distortion theorem that every $\phi_\mu(D_i)$ contains a round disk centered at $\phi_\mu(a_i)$, the radius of which has a positive lower bound depending only on a. Since $\{0, 1, \infty\} \notin \phi_\mu(\overline{D_i} \cup A_i)$, it follows that the diameter of each component of $\widehat{\mathbb{C}} \setminus \phi_\mu(A_i)$ has a positive lower bound depending only on a. Since ϕ_μ is analytic on every A_i, we have

$$\operatorname{mod}\bigl(\phi_\mu(A_i)\bigr) = \operatorname{mod}(A_i).$$

It follows that every $\phi_\mu(A_i)$ has definite thickness which depends only on a. All of these implies that there is a constant $b > 0$ depending only on a such that the four conditions in Definition 5.1 hold. The proof of the lemma is completed. $\square$

The next lemma is a direct consequence of Proposition 6.1 and Theorem 6.3 of [6].

Lemma 5.4. *Let X be a hyperbolic Riemann surface and $\gamma \subset X$ be a simple closed geodesic with hyperbolic length l. Then there exists a topological annulus $A \subset X$ such that*

1. *γ is the core curve of A,*
2. $\frac{\pi}{2l} - 1 < \operatorname{mod}(A) < \frac{\pi}{2l}$.

From the modulus inequality of Teichmüller extremal problem (for instance, see Chapter III of [1]), we have

Lemma 5.5. *Let $T \in \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Let $H \subset \widehat{\mathbb{C}}$ be an annulus which separates $\{0, 1\}$ and $\{T, \infty\}$. Then*

$$\operatorname{mod}(H) \leqslant \frac{1}{2\pi} \log 16\big(|T| + 1\big).$$

Lemma 5.6. *There exists an $\eta > 0$ such that for any Beltrami coefficient μ defined on $\widehat{\mathbb{C}} \setminus Q$ with $\mu(z) = 0$ on $\bigcup_i A_i$ and any (μ, E)-simple closed geodesic $\gamma \subset \widehat{\mathbb{C}} \setminus E$ with $\|\gamma\|_{\mu,E} < \eta$, we have $\gamma \subset \widehat{\mathbb{C}} \setminus Q$. Moreover, for any $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\|\gamma\|_{\mu,E} > (1 - \epsilon)\|\gamma\|_{\mu,Q} \tag{8}$$

provided that $\|\gamma\|_{\mu,E} < \delta$.

Proof. Let $\gamma \subset \widehat{\mathbb{C}} \setminus E$ be a (μ, E)-simple closed geodesic. By Lemma 5.4, there is an annulus $A \subset \widehat{\mathbb{C}} \setminus \phi_\mu(E)$ such that $\phi_\mu(\gamma)$ is the core curve of A and

$$\frac{\pi}{2\|\gamma\|_{\mu,E}} - 1 < \operatorname{mod}(A) < \frac{\pi}{2\|\gamma\|_{\mu,E}}. \tag{9}$$

We may assume that A separates 0 and the infinity. Let K_1 and K_2 be the two components of $\widehat{\mathbb{C}} \setminus A$ such that $0 \in K_1$ and $\infty \in K_2$. Let

$$r = \max\big\{|z| \,\big|\, z \in K_1\big\} \quad \text{and} \quad R = \min\big\{|z| \,\big|\, z \in K_2\big\}.$$

By Lemma 5.5, when $\|\gamma\|_{\mu,E}$ is small, R/r is large. Consider the round annulus

$$H = \big\{z \,\big|\, r < |z| < R\big\}.$$

It follows that $H \subset A$ and that the core curve of H is in the same homotopic class as γ. By Lemma 5.5 and (9), it follows that there is a uniform constant $0 < C < \infty$ such that

$$\operatorname{mod}(H) \geqslant \operatorname{mod}(A) - C \tag{10}$$

holds provided that $\|\gamma\|_{\mu,E}$ is small. Note that every pair $\{\phi_\mu(a_i), \phi_\mu(b_i)\}$ is contained either in $\{z \mid |z| < r\}$ or in $\{z \mid |z| > R\}$. Since ϕ_μ is holomorphic in $\overline{D_i} \cup A_i$ and $\{\phi_\mu(a_i), \phi_\mu(b_i)\} \subset \phi_\mu(\overline{D_i})$, it follows from Koebe's distortion theorem that there is an $1 < M < \infty$, which depends only on $\{D_i\}$ and $\{A_i\}$, such that every $\phi_\mu(\overline{D_i})$ is contained either in $\{z \mid |z| < Mr\}$ or in $\{z \mid |z| > R/M\}$. By (9) and (10), we have

$$R/M > Mr$$

provided that $\|\gamma\|_{\mu,E}$ is small enough. All of these implies that the annulus

$$H_M = \{z \mid Mr < |z| < R/M\}$$

is contained in $\widehat{\mathbb{C}} \setminus \phi_\mu(Q)$ provided that $\|\gamma\|_{\mu,E}$ is small enough.

Now the first assertion of the lemma follows if we can show that

$$\phi_\mu(\gamma) \subset H_M$$

provided that $\|\gamma\|_{\mu,E}$ is small enough. Suppose this were not true. Then there are two cases. In the first case, there exist two points z and z' such that

1. $z \in K_2$ with $|z| = R$,
2. $|z'| = R/M$,
3. $\phi_\mu(\gamma)$ separates $\{0, z'\}$ and $\{z, \infty\}$.

In the second case, there exist two points z and z' such that

1. $|z| = Mr$,
2. $z' \in K_1$ and $|z'| = r$,
3. $\phi_\mu(\gamma)$ separates $\{0, z'\}$ and $\{z, \infty\}$.

Suppose we are in the first case. Note that the curve $\phi_\mu(\gamma)$ separates A into two sub-annuli such that the modulus of each of them is equal to $\mathrm{mod}(A)/2$. But on the other hand, the outer one separates $\{0, z'\}$ and $\{z, \infty]\}$, and thus by Lemma 5.5, its modulus has an upper bound depending only on M. By (9) this is impossible when $\|\gamma\|_{\mu,E}$ is small enough. The same argument can be used to get a contradiction in the second case. This proves the first assertion of the lemma.

Now let us prove the second assertion. Let l denote the hyperbolic length of the core curve of H_M with respect to the hyperbolic metric of H_M. Since $H_M \subset \widehat{\mathbb{C}} \setminus \phi_\mu(Q)$ when $\|\gamma\|_{\mu,E}$ is small enough, it follows that $l > \|\gamma\|_{\mu,Q}$. Thus we have

$$\mathrm{mod}(H_M) = \frac{\pi}{2l} < \frac{\pi}{2\|\gamma\|_{\mu,Q}}.$$

From (9) and (10), there is a constant $0 < C' < \infty$ such that

$$\mathrm{mod}(H_M) \geqslant \frac{\pi}{2\|\gamma\|_{\mu,E}} - C'$$

holds provided that $\|\gamma\|_{\mu,E}$ is small enough. Thus we have

$$\frac{\pi}{2\|\gamma\|_{\mu,Q}} \leqslant \frac{\pi}{2\|\gamma\|_{\mu,E}} \leqslant \frac{\pi}{2\|\gamma\|_{\mu,Q}} + C'.$$

The second assertion follows. □

6. From bounded geometry to strictly contracting

The main purpose of this section is to prove that bounded geometry implies the strict contracting property of the operator $\sigma_f : T_f \to T_f$. Let us first prove a technical lemma.

Lemma 6.1. *Let $H = \{z \mid 1 < |z| < R\}$ be an annulus. Let $F_n(w)$ be a sequence of integrable and holomorphic functions defined on H such that*

$$\int_H \left|F_n(w)\right| |dw|^2 \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Then for any $1 < r < R$,

$$\int_{|w|=r} \left|F_n(w)\right| |dw| \to 0 \quad \text{as } n \to \infty.$$

Proof. Let $1 < r < R$ be fixed. Take $\delta > 0$ such that $1 + \delta < r < R - \delta$. Let

$$C(r, \delta) = \min\{r - 1 - \delta, R - \delta - r\}.$$

It follows that $C(r, \delta) > 0$. For any $\epsilon > 0$, by (11), there is an N such that for every $n > N$, there exist $1 < R_1 < 1 + \delta$ and $R - \delta < R_2 < R$, such that

$$\int_{|z|=R_1} \left|F_n(z)\right| |dz| < \epsilon$$

and

$$\int_{|z|=R_2} \left|F_n(z)\right| |dz| < \epsilon.$$

Let $A = \{z \mid R_1 < |z| < R_2\}$. For $|w| = r$, by Cauchy formula, we have

$$\left|F_n(w)\right| \leqslant \left|\frac{1}{2\pi i} \int_{\partial A} \frac{F_n(z)}{z - w}\, dz\right|.$$

Note that $|z - w| \geqslant C(r, \delta)$ for $|w| = r$ and $z \in \partial A$. This implies that

$$\left|F_n(w)\right| \leqslant \frac{\epsilon}{\pi C(r, \delta)}$$

holds for all $|w| = r$ and $n > N$. It follows that for all $n > N$,

$$\int_{|w|=r} \left|F_n(w)\right| |dw| \leqslant \frac{2r\epsilon}{C(r, \delta)}.$$

The lemma follows. □

For a Beltrami coefficient μ defined on $\widehat{\mathbb{C}} \setminus Q$, we use $\tilde{\mu}$ to denote $f^*(\mu)$.

Lemma 6.2. *Let $b > 0$. Then there is a constant $0 < a < 1$ depending only on b such that if both $[\mu]$ and $[\tilde{\mu}]$ belong to $T_{f,b}$, then*

$$\int_{\cup \phi_{\tilde{\mu}}(A_i)} \left|\tilde{q}(w)\right| |dw|^2 \geqslant a$$

where $\tilde{q}(w)\, dw^2$ is any integrable holomorphic quadratic differential defined on $\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}}(Q)$ with

$$\int_{\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}}(Q)} \left|\tilde{q}(w)\right| |dw|^2 = 1.$$

Proof. Let us prove it by contradiction. We will use a geometric limit-type argument. By using a Möbius transformation which fixes 0 and 1, and maps $\phi_{\tilde{\mu}}(a_1)$ to the infinity, we may assume that $\infty \in D_1$. Since $\tilde{\mu} \in T_{f,b}$, such Möbius transformation lies in a compact family and therefore the assumption does not affect the validity of the proof.

Now let us suppose that there exist a sequence of pairs $(\tilde{\mu}_n, \mu_n)$ in $T_{f,b}$ and a sequence of holomorphic quadratic differentials $\tilde{q}_n$ over $\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}_n}(Q)$ such that

$$\int_{\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}_n}(Q)} \left|\tilde{q}_n(w)\right| |dw|^2 = 1, \tag{12}$$

and

$$\int_{\cup \phi_{\tilde{\mu}_n}(A_i)} \left|\tilde{q}_n(w)\right| |dw|^2 \to 0 \quad \text{as } n \to \infty. \tag{13}$$

By Lemma 2.1 $f(\bigcup_i \overline{A}_i) \subset \bigcup_i D_i$ and f is holomorphic in $\overline{D_i} \cup A_i$. This, together with the fact that ϕ_{μ_n} is holomorphic on $\bigcup_i D_i$, implies that $\phi_{\tilde{\mu}_n}$ is holomorphic and thus univalent on $\bigcup_i (\overline{D}_i \cup A_i)$.

Note that every ring A_i is holomorphically isomorphic to some annulus

$$H_i = \left\{z \;\middle|\; 1 < |z| < R_i\right\}.$$

Let $\Phi_i : H_i \to A_i$ be a holomorphic isomorphism and let $\mathbb{T}_r$ denote the circle $\{z \mid |z| = r\}$. We claim that for every $1 < r < R_i$,

$$\int_{\phi_{\tilde{\mu}_n}(\Phi_i(\mathbb{T}_r))} \left|\tilde{q}_n(w)\right| |dw| \to 0 \quad \text{as } n \to \infty. \tag{14}$$

In fact, from (13), we have

$$\int_{H_i} \left|\tilde{q}_n\big((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z)\big)\right|\left|(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)\right|^2 |dz|^2 \to 0 \quad \text{as } n \to \infty. \tag{15}$$

By Lemma 6.1, we have

$$\int_{\mathbb{T}_r} \left|\tilde{q}_n\big((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z)\big)\right|\left|(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)\right|^2 |dz| \to 0 \quad \text{as } n \to \infty.$$

Since $\phi_{\tilde{\mu}_n} \circ \Phi_i$ is univalent on H_i, it follows from Koebe's distortion theorem that for every $1 < r < R_i$, there is a $C > 1$ depending only on r, R_i, and b such that

$$1/C < \left|(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)\right| < C \tag{16}$$

holds for all $z \in \mathbb{T}_r$. We thus have

$$\int_{\mathbb{T}_r} \left|\tilde{q}_n\big((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z)\big)\right|\left|(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)\right||dz| \to 0 \quad \text{as } n \to \infty.$$

This implies (14) and the claim has been proved.

Now for every A_i, take an arbitrary $1 < r_i < R_i$ and let

$$\gamma_{i,n} = (\phi_{\tilde{\mu}_n} \circ \Phi_i)(\mathbb{T}_{r_i}). \tag{17}$$

For every n, let R_n denote the component of $\widehat{\mathbb{C}} \setminus \bigcup_i \gamma_{i,n}$ such that

$$\partial R_n = \bigcup_i \gamma_{i,n}.$$

Recall that $P_1 = \{z_j\}$ and $P_f' = \{a_i\}$ are both finite sets and each $\tilde{q}_n = \tilde{q}_n(w)\, dw^2$ has at most simple poles at the points in $\{\phi_{\tilde{\mu}_n}(z_j)\}$. This implies that one can write

$$\tilde{q}_n(w) = \sum_j \frac{b_{j,n}}{w - \phi_{\tilde{\mu}_n}(z_j)} + g_n(w) \tag{18}$$

where $g_n(w)$ is a holomorphic function on $\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}_n}(\overline{D})$.

Since $\tilde{\mu}_n \in T_{f,b}$, it follows by taking a subsequence if necessary, that we can assume that for every a_i, the sequence

$$a_{i,n} = \phi_{\tilde{\mu}_n}(a_i)$$

converges to a point e_i with respect to the spherical distance as n goes to ∞. Since $\phi_{\tilde{\mu}_n}$ is holomorphic in $\overline{D_i} \cup A_i$, similarly, we can assume that for every D_i, the sequence

$$D_{i,n} = \phi_{\tilde{\mu}_n}(D_i)$$

converges to a topological disk E_i with respect to the Hausdorff metric. It follows that each E_i contains a round disk of radius b centered at e_i. Note that by taking each A_i thinner, we may assume that $\phi_{\tilde{\mu}_n}$ is univalent in a larger disk containing $\overline{D_i \cup A_i}$ in its interior. So by taking a subsequence if necessary, we can also assume that

$$A_{i,n} = \phi_{\tilde{\mu}_n}(A_i)$$

converges to a topological annulus B_i with respect to the Hausdorff metric. It is clear that

$$\mathrm{mod}(B_i) = \mathrm{mod}(A_i).$$

Recall that $\gamma_{i,n} = (\phi_{\tilde{\mu}_n} \circ \Phi_i)(\mathbb{T}_{r_i})$. Since $(\phi_{\tilde{\mu}_n} \circ \Phi_i)$ maps H_i univalently into $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ and since $\tilde{\mu}_n \in T_{f,b}$, it follows again by taking a subsequence if necessary, that we may assume that $\phi_{\tilde{\mu}_n} \circ \Phi_i$ converges to some univalent function Λ_i defined on H_i, and moreover,

$$(\phi_{\tilde{\mu}_n} \circ \Phi_i)(z) \to \Lambda_i(z) \quad \text{uniformly in any compact set of } H_i. \tag{19}$$

Let

$$\gamma_i = \Lambda_i(\mathbb{T}_{r_i}).$$

It is not difficult to see that every γ_i is a real analytic and simple closed curve which is homotopic to the core curve of B_i.

Again by taking a subsequence if necessary, we may assume that as $n \to \infty$, for every $z_j \in P_1$,

$$w_{j,n} = \phi_{\tilde{\mu}_n}(z_j)$$

converges to some w_j in the spherical distance. It is important to note that the objects in $\{E_i\}$ and $\{w_j\}$ still satisfy the bounded geometry properties in Definition 5.1 since the defining conditions are closed. Let

$$\mathcal{R} = \widehat{\mathbb{C}} \setminus \left(\bigcup_i \overline{E}_i \cup \{w_j\} \right).$$

Since $g_n(w)$ is a holomorphic function on $\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}_n}(Q)$, it follows that for any compact set $W \subset \mathcal{R}$, the function $g_n(w)$ is defined on W provided n is large enough. Moreover, from (17), for any such compact set W, we can always take r_i close to 1 or R_i such that

$$W \subset R_n.$$

For any $w \in W$, from (18) and Cauchy formula, we have

$$\begin{aligned} g_n(w) &= \frac{1}{2\pi i} \int\limits_{\bigcup_i \gamma_{i,n}} \frac{g_n(\xi)}{\xi - w}\, d\xi \\ &= \frac{1}{2\pi i} \int\limits_{\bigcup_i \gamma_{i,n}} \frac{\tilde{q}_n(\xi)}{\xi - w}\, d\xi - \frac{1}{2\pi i} \sum_j \int\limits_{\bigcup_i \gamma_{i,n}} \frac{b_{j,n}}{(\xi - w_{j,n})(\xi - w)}\, d\xi. \end{aligned}$$

Note that by assumption $\infty \in D_1$ and hence $\infty \notin R_n$. It follows that as a function of ξ,

$$\frac{b_{j,n}}{(\xi - w_{j,n})(\xi - w)}$$

is holomorphic in R_n and the residues at the two simple poles are negative of each other. It follows that its integral along $\bigcup_i \gamma_{i,n}$ is zero. We thus have

$$g_n(w) = \frac{1}{2\pi i} \int\limits_{\bigcup_i \gamma_{i,n}} \frac{\tilde{q}_n(\xi)}{\xi - w}\, d\xi.$$

By (14) and the fact that $d(W, \bigcup_i \gamma_{i,n}) > 0$, it follows that $g_n(w) \to 0$ uniformly in W as $n \to \infty$. In particular, since $\bigcup_i \gamma_{i,n}$ is a compact subset of $\mathcal{R}$, it follows that $g_n(w) \to 0$ uniformly for $w \in \bigcup_i \gamma_{i,n}$. This, together with (14) and (18), implies

$$\int\limits_{\bigcup_i \gamma_{i,n}} \left| \sum_j \frac{b_{j,n}}{w - w_{j,n}} \right| |dw| \to 0 \quad \text{as } n \to \infty. \tag{20}$$

We claim that $b_{j,n} \to 0$ as $n \to \infty$ for each j. Let us prove the claim by contradiction. Let $\beta_n = \max_j\{|b_{j,n}|\}$. By taking a subsequence we may assume that there is an $\epsilon > 0$ such that $\beta_n \geqslant \epsilon$ for all $n \geqslant 0$. Let

$$h_{j,n} = b_{j,n}/\beta_n.$$

Then $\max_j\{|h_{j,n}|\} = 1$. By (20), we have

$$\int\limits_{\bigcup_i \gamma_{i,n}} \left| \sum_j \frac{h_{j,n}}{w - w_{j,n}} \right| |dw| \to 0 \quad \text{as } n \to \infty. \tag{21}$$

By taking a convergent subsequence again, we may assume that every $h_{j,n}$ converges to a number h_j as n goes to infinity. We thus have

$$\max_j \bigl\{|h_j|\bigr\} = 1. \tag{22}$$

From (19) and (21), we have

$$\int\limits_{\bigcup_i \gamma_i} \left| \sum_j \frac{h_j}{w - w_j} \right| |dw| = 0.$$

This implies that

$$\sum_j \frac{h_j}{w - w_j} = 0 \quad \text{for all } w \in \textstyle\bigcup_i \gamma_i \text{ and thus equal to zero everywhere.}$$

Since all w_j are distinct with each other, it follows by computing the residue at each w_j that all h_j are equal to zero. This contradicts with (22) and the claim has been proved.

Since $g_n(z) \to 0$ uniformly on any compact set of $\mathcal{R}$ and $b_{j,n} \to 0$ as $n \to \infty$ for every j, it follows from (18) that

$$\int_{R_n} \left|\tilde{q}_n(w)\right| |dw|^2 \to 0 \quad \text{as } n \to \infty.$$

This, together with (13), implies

$$\int_{\widehat{\mathbb{C}} \setminus \phi_{\tilde{\mu}_n}(Q)} \left|\tilde{q}_n(w)\right| |dw|^2 \to 0 \quad \text{as } n \to \infty.$$

This contradicts with the assumption (12) and completes the proof of the lemma. □

Given any $[\mu_0] \in T_f$. Let $[\mu_n] = \sigma_f^n([\mu_0]) = [(f^*)^n \mu_0]$ for $n \geqslant 0$.

Lemma 6.3. *Suppose that there exist a $b > 0$ and a point $[\mu_0] \in T_f$ such that $\{[\mu_n]\}_{n=0}^{\infty} \subset T_{f,b}$. Then σ_f has a unique fixed point in T_f.*

Proof. Let $\gamma_0 \subset T_f$ be a piecewise smooth curve segment joining $[\mu_0]$ and $[\mu_1]$. Then $\gamma_n = \sigma_f^n(\gamma_0)$ is a piecewise smooth curve segment joining $[\mu_n]$ and $[\mu_{n+1}]$. By Corollary 4.1 it follows that for all $n \geqslant 1$, one has

$$l_T(\gamma_n) \leqslant l_T(\gamma_0)$$

where $l_T(\cdot)$ denotes the length with respect to the Teichmüller metric in T_f. Since $\{[\mu_n]\}_{n=0}^{\infty} \subset T_{f,b}$ for all $n \geqslant 0$, it follows that there is a constant c depending only on b and $l_T(\gamma_0)$ such that

$$\bigcup_{n \geqslant 0}^{\infty} \gamma_n \subset T_{f,c}.$$

By Lemma 6.2, there is a constant $0 < \delta < 1$ depending only on c such that for every $n \geqslant 0$ and every $\tau \in \gamma_n$, one has

$$\|d\sigma_f|_\tau\| \leqslant \delta.$$

This implies that $l_T(\gamma_n) \leqslant \delta^n l_T(\gamma_0)$ and $\{[\mu_n]\}_{n=0}^{\infty}$ is a Cauchy sequence. Since T_f is complete, $[\mu_n]$ converges to a limit point $[\mu]$ in T_f, that is,

$$\lim_{n \to \infty} [\mu_n] = [\mu].$$

It follows that $\sigma_f([\mu]) = [\mu]$.

Suppose that there are two fixed points $[\mu]$ and $[\nu]$ of σ_f. Let $\gamma_0 \subset T_f$ be a piecewise smooth curve segment joining $[\mu]$ and $[\nu]$. Let $\gamma_n = \sigma_f^n(\gamma_0)$. It follows that γ_n is a piecewise smooth segment joining $[\mu]$ and $[\nu]$. Then by the same argument as above, we have $l_T(\gamma_n) \leqslant \delta^n l_T(\gamma_0)$. Let $n \to \infty$, we get $[\mu] = [\nu]$. This implies the uniqueness of the fixed point. The proof of Lemma 6.3 is completed. □

7. No Thurston obstruction implies bounded geometry

The argument used this section is almost an entire adaptation of Section 8 of [6]. The subtle point here is that the set E is not forward invariant. This will make some difference when we prove Lemma 7.5. We overcome this by introducing a bigger set P_2 and showing that any short simple closed geodesic does not intersect the holomorphic disks (see Lemma 7.4).

Lemma 7.1. *Suppose that f has no Thurston obstructions. Then there is an integer $k > 0$ such that for every f-stable multi-curve $\Gamma = \{\gamma_i\}$ with $\gamma_i \subset \widehat{\mathbb{C}} \setminus Q$ and the associated linear transformation matrix A_Γ, we have*

$$\max_j \sum_i b_{ij} < 1/2 \tag{23}$$

where $A_\Gamma^k = (b_{ij})$.

Proof. Let $\Gamma = \{\gamma_i\}$ be a f-stable multi-curve with $\gamma_i \subset \widehat{\mathbb{C}} \setminus Q$. It is clear that the number of the elements in Γ has an upper bound which depends only on $\#(E)$. This implies that there can be only finitely many distinct A_Γ. The lemma follows. □

Let $Z \subset Q$ be a subset with $\#(Z) \geqslant 4$ and $\gamma \subset \widehat{\mathbb{C}} \setminus Z$ be a non-peripheral simple closed curve. For $[\mu] \in T_f$, define

$$w_Z\big(\gamma, [\mu]\big) = -\log \|\gamma\|_{\mu,Z}.$$

By using the same argument as in the proof of Proposition 7.2 of [6], we have

Lemma 7.2. *Let $Z \subset Q$ be a subset with $\#(Z) \geqslant 4$ and $\gamma \subset \widehat{\mathbb{C}} \setminus Z$ be a non-peripheral simple closed curve. Then the function*

$$[\mu] \to w_Z\big(\gamma, [\mu]\big) : T_f \to \mathbb{R}$$

is Lipschitz with Lipschitz constant 2.

Recall that $E = P_1 \cup \bigcup_i \{a_i, b_i\}$. Let $[\mu] \in T_f$ such that $\mu(z) = 0$ for all $z \in \bigcup_i A_i$. Let b be a real number. Define

$$\Gamma_\mu^b = \big\{\gamma \mid \gamma \text{ is a } (\mu, E)\text{-simple closed geodesic with } w_E\big(\gamma, [\mu]\big) \geqslant b\big\},$$

and

$$L_\mu = \big\{w_E\big(\gamma, [\mu]\big) \mid \gamma \text{ is a } (\mu, E)\text{-simple closed geodesic}\big\}.$$

Lemma 7.3. *There exists an* $A > -\log\log\sqrt{2}$ *such that for any* $[\mu] \in T_f$ *with* $\mu(z) = 0$ *for all* $z \in \bigcup_i A_i$ *and any real numbers* $a < b$, *if*

1. $a > A$,
2. $b - a \geqslant \log d + 2d_T([\mu], [f^*\mu]) + 1$,
3. $[a, b] \cap L_\mu = \emptyset$,
4. $\Gamma_\mu^b \neq \emptyset$,

then Γ_μ^b *is a* f*-stable multi-curve in* $\widehat{\mathbb{C}} \setminus Q$.

Proof. Let $\gamma \in \Gamma_\mu^b$. By the first assertion of Lemma 5.6, γ is a non-peripheral and simple closed curve in $\widehat{\mathbb{C}} \setminus Q$ provided that A is big and thus $\|\gamma\|_{\mu,E}$ is small. By the second assertion of Lemma 5.6, we have

$$w_Q\big(\gamma, [\mu]\big) > w_E\big(\gamma, [\mu]\big) - 1$$

provided that A is big and thus $\|\gamma\|_{\mu,E}$ is small. Now suppose that γ' is a non-peripheral component of $f^{-1}(\gamma)$. Since f is a degree d branched covering of the two sphere, it follows that

$$w_{f^{-1}(Q)}\big(\gamma', [f^*\mu]\big) \geqslant w_Q\big(\gamma, [\mu]\big) - \log d.$$

Since $E \subset f^{-1}(Q)$, it follows that

$$w_E\big(\gamma', [f^*\mu]\big) > w_{f^{-1}(Q)}\big(\gamma', [f^*\mu]\big).$$

By Lemma 7.2, we have

$$w_E\big(\gamma', [\mu]\big) \geqslant w_E\big(\gamma', [f^*\mu]\big) - 2d_T\big([\mu], [f^*\mu]\big).$$

This implies that

$$w_E\big(\gamma, [\mu]\big) - w_E\big(\gamma', [\mu]\big) < \log d + 2d_T\big([\mu], [f^*\mu]\big) + 1 \leqslant b - a.$$

Since $w_E(\gamma, [\mu]) > b$ and $[a, b] \cap L_\mu = \emptyset$, it follows that $w_E(\gamma', [\mu]) > b$. This implies that γ' is homotopic to some element in Γ_μ^b. The lemma follows. □

Let $k \geqslant 0$ be the integer in Lemma 7.1. Let

$$P_2 = E \cup f^k(E) \cup \bigcup_{1 \leqslant j \leqslant k} f^j(\Omega_f). \tag{24}$$

Lemma 7.4. *There exists an* $\epsilon_0 > 0$ *such that for any* $[\mu] \in T_f$ *with* $\mu(z) = 0$ *for all* $z \in \bigcup_i A_i$ *and any* (μ, P_2)*-simple closed geodesic* η, *if* $\|\eta\|_{\mu,P_2} \leqslant \epsilon_0$, *then there is a* (μ, E)*-simple closed geodesic* γ *such that* η *is homotopic to* γ *in* $\widehat{\mathbb{C}} \setminus P_2$.

Proof. Suppose that η is not homotopic to any (μ, E)-simple closed geodesic in $\widehat{\mathbb{C}} \setminus P_2$. Then there is at least one holomorphic disk D_i such that η separates the points in $D_i \cap P_2$. Let $x, y \in D_i \cap P_2$ which are separated by η. Let $z \in P_2 \setminus \overline{D_i}$. Let $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the quasiconformal homeomorphism which solves the Beltrami equation given by μ and which maps x, y and z respectively to 0, 1 and the infinity. It follows that $\phi(\eta)$ is a simple closed curve in $\mathbb{C}$ and separates 0 and 1. Now we have two cases.

In the first case, $\phi(\eta)$ is contained in $\phi(\overline{D_i} \cup A_i)$. Note that A_i is the shielding ring attached to the outside of D_i. So $\phi(\eta)$ must enclose the ϕ-images of at least two points in $D_i \cap P_2$. Since ϕ is univalent in $\overline{D_i} \cup A_i$ and $\phi(\eta)$ separates 0 and 1, it follows from Koebe's distortion theorem that there exists a $D > 0$ independent of μ such that $\phi(\eta)$ intersect the unit disk and the Euclidean diameter of $\phi(\eta)$ is greater than D. This implies there is an ϵ_0 independent of μ such that the hyperbolic length of $\phi(\eta)$ in $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is greater than ϵ_0. It follows that $\|\eta\|_{\mu, P_2} > \epsilon_0$.

In the second case, $\phi(\eta)$ is not contained in $\phi(\overline{D_i} \cup A_i)$. Since $\phi(\eta)$ separates 0 and 1, it follows that $\phi(\eta)$ must cross through the annulus $\phi(A_i)$. Since ϕ is univalent in $\overline{D_i} \cup A_i$ and since ϕ maps x and y to 0 and 1 respectively, by Koebe's distortion theorem, the annulus $\phi(A_i)$ has definite thickness. This again implies that there exists a $D > 0$ independent of μ such that $\phi(\eta)$ intersect the unit disk and the Euclidean diameter of $\phi(\eta)$ is greater than D. By the same reasoning as above, we have an ϵ_0 independent of μ such that $\|\eta\|_{\mu, P_2} > \epsilon_0$. This completes the proof of Lemma 7.4. □

By the definition of P_2, it follows that

$$f^k : \widehat{\mathbb{C}} \setminus f^{-k}(P_2) \to \widehat{\mathbb{C}} \setminus P_2 \tag{25}$$

is a covering of degree d^k. Let $A > -\log\log\sqrt{2}$ and $\epsilon_0 > 0$ be the constants in Lemma 7.3 and Lemma 7.4 respectively.

Lemma 7.5. *Let $B > A$. Then there exists a constant $M > 0$ depending only on the numbers k, B, $\#(E)$, ϵ_0 and the degree d of f, such that for any $[\mu] \in T_f$ with $\mu(z) = 0$ for all $z \in \bigcup_i A_i$ and any real numbers $a < b$, if*

1. $A < a < B$,
2. $b - a \geqslant \log d + 2d_T([\mu], [f^*\mu]) + 1$,
3. $[a, b] \cap L_\mu = \emptyset$,
4. $\Gamma_\mu^b \neq \emptyset$,

then

$$\sum_{\gamma \in \Gamma_\mu^b} \frac{1}{\|\gamma\|_{\nu, E}} \leqslant \frac{1}{2} \sum_{\gamma \in \Gamma_\mu^b} \frac{1}{\|\gamma\|_{\mu, E}} + M,$$

where $\nu = (f^k)^*(\mu)$ and $k \geqslant 0$ is the integer in Lemma 7.1.

Proof. By Lemma 7.3, Γ_μ^b is a f-stable multi-curve in $\widehat{\mathbb{C}} \setminus Q$. For each $\gamma_j \in \Gamma_\mu^b$, let $\gamma_{i,j,\alpha}$ be any component of $f^{-k}(\gamma_j)$ homotopic to γ_i in $\widehat{\mathbb{C}} \setminus Q$. Then $\gamma_{i,j,\alpha}$ is also homotopic to γ_i in $\widehat{\mathbb{C}} \setminus E$.

Let $g = \phi_\mu \circ f^k \circ \phi_\nu^{-1}$. Then g is a rational map. It follows from (25) that

$$g : \widehat{\mathbb{C}} \setminus \phi_\nu\big(f^{-k}(P_2)\big) \to \widehat{\mathbb{C}} \setminus \phi_\mu(P_2)$$

is a holomorphic covering map, and therefore,

$$\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)} = d_{i,j,\alpha} \|\gamma_j\|_{\mu,P_2}$$

where $d_{i,j,\alpha} \leqslant d^k$ is the degree of

$$f^k : \gamma_{i,j,\alpha} \to \gamma_j.$$

Thus

$$\sum_\alpha \frac{1}{\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)}} = \left(\sum_\alpha \frac{1}{d_{i,j,\alpha}}\right) \frac{1}{\|\gamma_j\|_{\mu,P_2}} = b_{ij} \frac{1}{\|\gamma_j\|_{\mu,P_2}}.$$

Since $E \subset P_2$ by (24), it follows that $\|\gamma_j\|_{\mu,P_2} > \|\gamma_j\|_{\mu,E}$, and therefore

$$\frac{1}{\|\gamma_j\|_{\mu,P_2}} < \frac{1}{\|\gamma_j\|_{\mu,E}}.$$

This implies

$$\sum_\alpha \frac{1}{\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)}} < b_{ij} \frac{1}{\|\gamma_j\|_{\mu,E}}. \tag{26}$$

Note that $E \subset f^{-k}(P_2)$ by (24). Let p denote the number of the points in $f^{-k}(P_2) \setminus E$. It follows from (24) that there is a constant

$$0 < C\big(k, d, \#(E)\big) < \infty$$

depending only on d, k, and $\#(E)$ such that $p \leqslant C(k, d, \#(E))$.

Now we claim that for any $(\nu, f^{-k}(P_2))$-simple closed geodesic γ which is homotopic to γ_i in $\widehat{\mathbb{C}} \setminus E$, either γ is homotopic to some $\gamma_{i,j,\alpha}$ in $\widehat{\mathbb{C}} \setminus f^{-k}(P_2)$, or

$$\|\gamma\|_{\nu, f^{-k}(P_2)} > \min\big\{e^{-B}, \epsilon_0\big\}.$$

Let us prove the claim. In fact, if γ is not homotopic in $\widehat{\mathbb{C}} \setminus f^{-k}(P_2)$ to some $\gamma_{i,j,\alpha}$, then $f^k(\gamma)$ is a (μ, P_2)-simple closed geodesic which is not homotopic to any γ_j in $\widehat{\mathbb{C}} \setminus P_2$. Then there are two cases.

In the first case, $f^k(\gamma)$ is homotopic in $\widehat{\mathbb{C}} \setminus P_2$ to some (μ, E)-simple closed geodesic ξ which does not belong to Γ_μ^b. By the assumption that $L_\mu \cap [a, b] = \emptyset$, we have

$$\big\| f^k(\gamma)\big\|_{\mu,P_2} > \big\| f^k(\gamma)\big\|_{\mu,E} = \|\xi\|_{\mu,E} > e^{-a} > e^{-B}.$$

In the second case, $f^k(\gamma)$ is not homotopic in $\widehat{\mathbb{C}} \setminus P_2$ to any (μ, E)-simple closed geodesic. By Lemma 7.4, we have

$$\left\| f^k(\gamma) \right\|_{\mu, P_2} > \epsilon_0.$$

We thus have

$$\|\gamma\|_{\nu, f^{-k}(P_2)} \geqslant \left\| f^k(\gamma) \right\|_{\mu, P_2} > \min\left\{e^{-B}, \epsilon_0\right\}.$$

Now from the left hand of the inequality given by (c) in Theorem 7.1 of [6], we have

$$\frac{1}{\|\gamma_i\|_{\nu,E}} \leqslant \sum_j \sum_\alpha \frac{1}{\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)}} + \frac{2}{\pi} + \frac{C(k,d,\#(E)) + 1}{\min\{e^{-B}, \epsilon_0\}}.$$

Let

$$M' = \frac{2}{\pi} + \frac{C(k,d,\#(E)) + 1}{\min\{e^{-B}, \epsilon_0\}}.$$

Thus

$$\sum_{\gamma \in \Gamma_\mu^b} \frac{1}{\|\gamma\|_{\nu,E}} \leqslant \sum_i \sum_j \sum_\alpha \frac{1}{\|\gamma_{i,j,\alpha}\|_{\nu, f^{-k}(P_2)}} + KM'$$

where K is the number of the curves in Γ which is bounded above by $\#(E) - 3$. Let

$$M = \left(\#(E) - 3\right)M'.$$

By (26), we have

$$\sum_{\gamma \in \Gamma_\mu^b} \frac{1}{\|\gamma\|_{\nu,E}} \leqslant \sum_j \left(\sum_i b_{ij}\right) \frac{1}{\|\gamma_j\|_{\mu,E}} + M \leqslant \frac{1}{2} \sum_{\gamma \in \Gamma_\mu^b} \frac{1}{\|\gamma\|_{\mu,E}} + M.$$

This completes the proof of the lemma. □

The following is a technical lemma from calculus.

Lemma 7.6. *Let $b_0 > 1$, c_0, $M_0 > 0$, and integer $m_0 > 1$ be given. Then for any sequence $\{x_n\}_{n=0}^\infty$ of positive numbers satisfying*

(1) $x_0 \leqslant c_0$,
(2) $x_{n+1}/x_n \leqslant b_0$,
(3) *if* $x_n \geqslant M_0$, *then* $x_{n+m_0} \leqslant x_n$,

one has

$$x_n \leqslant \max\left\{b_0^{m_0-1} c_0, b_0^{m_0} M_0\right\}, \quad \forall n \geqslant 0.$$

Proof. Let $C = \max\{b_0^{m_0-1}c_0, b_0^{m_0}M_0\}$. It is sufficient to prove that

$$x_{i+lm_0} \leqslant C$$

for all $0 \leqslant i \leqslant m_0 - 1$ and $l \geqslant 0$. Take an arbitrary integer $0 \leqslant i \leqslant m_0 - 1$. Let us prove that

$$x_{i+lm_0} \leqslant C$$

for all $l \geqslant 0$ by induction. For $l = 0$, we have

$$x_i \leqslant b_0^i x_0 \leqslant b_0^{m_0-1}c_0 \leqslant C.$$

Now assume that

$$x_{i+km_0} \leqslant C \tag{27}$$

for some integer $l = k \geqslant 0$. Let us prove that

$$x_{i+(k+1)m_0} \leqslant C.$$

In fact, there are two cases by assumption (27). In the first case, $x_{i+km_0} < M_0$. In this case, we have

$$x_{i+(k+1)m_0} \leqslant b_0^{m_0}x_{i+km_0} < b_0^{m_0}M_0 \leqslant C.$$

In the second case, $x_{i+km_0} \geqslant M_0$. Then we have

$$x_{i+(k+1)m_0} \leqslant x_{i+km_0} \leqslant C.$$

This proves that $x_{i+lm_0} \leqslant C$ for all $l \geqslant 0$. Since this holds for any $0 \leqslant i \leqslant m_0 - 1$, the lemma follows. □

Lemma 7.7. *If f has no Thurston obstructions, then for any $[\mu_0] \in T_f$, there exists a constant $b > 0$ such that for all $n \geqslant 1$,*

$$[\mu_n] \in T_{f,b},$$

where $\mu_n = (f^*)^n(\mu_0)$.

Proof. Since f is holomorphic on $\bigcup A_i$ and $f(\bigcup A_i) \subset \bigcup D_i$, it follows that for all $n \geqslant 1$, $\mu_n(z) = 0$ on $\bigcup A_i$. By Lemma 5.3, it suffices to prove that there is a uniform positive lower bound of the length of all the (μ_n, E)-simple closed geodesics.

Let $D = d_T([\mu_0], [\mu_1])$. Then by Lemma 3.2 and Corollary 4.1, we have

$$d_T\big([\mu_n], [\mu_{n+1}]\big) \leqslant D \quad \text{for all } n \geqslant 0.$$

Let $K = \#(E) - 3 \geqslant 1$ and $k \geqslant 1$ be the integer in Lemma 7.1. Let $l_0 \geqslant 1$ be the least integer such that

$$K \leqslant 2^{l_0 - 1}. \tag{28}$$

Let

$$x_n = \max_{\gamma}\left\{\|\gamma\|_{\mu_n, E}^{-1}\right\},$$

where max is taken over all the (μ_n, E)-simple closed geodesics. Now it suffices to prove that there exist positive constants c_0, $M_0 > 0$, $b_0 > 1$, and an integer $m_0 > 0$, such that the sequence $\{x_n\}$ satisfies the three conditions in Lemma 7.6.

By Corollary 6.6 of [6], there are at most K (μ_n, E)-simple closed geodesics which has hyperbolic length less than $\log(\sqrt{2} + 1)$. This implies that we can have $c_0 > 0$ such that

$$x_0 \leqslant c_0.$$

It is the first condition in Lemma 7.6. From Lemma 7.2 we can take $b_0 = e^{2D}$.

Recall that we use d to denote the degree of f. Let $k_0 = \log d + 2D$ and $m_0 = k_0 l_0$. Let

$$k_1 = k_0 + 4m_0 D + 1. \tag{29}$$

In particular, $k_1 > \log d + 2D + 1$. Let $A > -\log\log(\sqrt{2} + 1)$ be the constant in Lemma 7.3. In Lemma 7.5, take

$$B = A + (K + 1)k_1$$

and let M denote the corresponding constant there. Let

$$M_0 = \max\left\{e^B, 2^{l_0+1} M\right\}.$$

It remains to prove that if $x_n \geqslant M_0$, then $x_{n+m_0} \leqslant x_n$. To see this, suppose that $x_n \geqslant M_0$. It follows that there is a (μ_n, E)-simple close geodesic such that $w_E(\gamma, [\mu_n]) \geqslant B$. Then by the choice of the numbers k_1 and B, and the fact that there are at most K (μ_n, E)-simple closed geodesics which have hyperbolic length less than $\log(\sqrt{2} + 1)$, one can take an interval $[a, b]$ such that

1. $A < a < b < B$,
2. $b - a = k_1$,
3. $[a, b] \cap L_{\mu_n} = \emptyset$.

It follows that $\Gamma_{\mu_n}^b \neq \emptyset$ and therefore is a f-stable multi-curve by Lemma 7.3. Now for each $i = 0, 1, \ldots, l_0$, let

$$[a_i, b_i] = [a + 2kiD, b - 2kiD].$$

By Lemma 7.2, the gap condition $b - a = k_1$, and (29), it follows that each family $\Gamma_{\mu_{n+ki}}^{b_i}$, $0 \leqslant i \leqslant l_0$, contains the same set of homotopy classes of simple closed curves as $\Gamma_{\mu_n}^b$. Let us simply denote each of them by Γ. Now for each $0 \leqslant i \leqslant l_0 - 1$, let $\mu = \mu_{n+ki}$ and $\nu = \mu_{n+k(i+1)}$, and let $[a_i, b_i]$ be the corresponding gap interval. Then the conditions in Lemma 7.5 are satisfied with the constants A and B given as above. By Lemma 7.5, we have

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+k(i+1)},E}} \leqslant \frac{1}{2} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+ki},E}} + M$$

for $0 \leqslant i \leqslant l_0 - 1$. It follows from $m_0 = kl_0$ that

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+m_0},E}} \leqslant \frac{1}{2^{l_0}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n,E}} + 2M. \tag{30}$$

Since

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n,E}} \geqslant x_n \geqslant M_0 \geqslant 2^{l_0+1} M,$$

it follows that

$$M \leqslant \frac{1}{2^{l_0+1}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n,E}}. \tag{31}$$

From (30) and (31), we have

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+m_0},E}} \leqslant \frac{1}{2^{l_0-1}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n,E}}. \tag{32}$$

Since the number of the elements in Γ is at most K, it follows that

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n,E}} \leqslant K x_n.$$

From (28) and (32), we have

$$x_{n+m_0} \leqslant \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_{n+m_0},E}} \leqslant \frac{1}{2^{l_0-1}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n,E}} \leqslant \frac{K}{2^{l_0-1}} x_n \leqslant x_n. \qquad \square$$

The Main Theorem now follows from Lemmas 4.3, 6.3, and 7.7.

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