

Holomorphic motions and related topics

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1 Introduction

Suppose $\overline{\mathbb{C}}$ is the extended complex plane and $E \subset \overline{\mathbb{C}}$ is a subset. For any real number $r > 0$, we let Δ_r be the disk centered at the origin in \mathbb{C} with radius r and Δ be the disk of unit radius. A map

$$h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$$

is called a holomorphic motion of E parametrized by Δ and with base point 0 if

- (i) $h(0, z) = z$ for all $z \in E$,
- (ii) for every $c \in \Delta$, $z \mapsto h(c, z)$ is injective on $\overline{\mathbb{C}}$, and
- (iii) for every $z \in E$, $c \mapsto h(c, z)$ is holomorphic for c in U .

We think of $h(c, z)$ as moving through injective mappings with the parameter c . It starts out at the identity when c is equal to the base point 0 and moves holomorphically as c varies in Δ .

We always assume E contains at least three points, p_1, p_2 and p_3 . Then since the points $h(c, p_1), h(c, p_2)$ and $h(c, p_3)$ are distinct for each $c \in \Delta$, there is a unique Möbius transformation B_c that carries these three points to 0, 1, and ∞ . Since B_c depends holomorphically on c ,

$\tilde{h}(c, z) = h(c, B_c(z))$ is also a holomorphic motion and it fixes the points $0, 1, \infty$. We shall call it a normalized holomorphic motion.

Holomorphic motions were introduced by Mãñé, Sad and Sullivan in their study of the structural stability problem for the complex dynamical systems, [MSS]. They proved the first result in the topic which is called the λ -lemma and which says that any holomorphic motion $h(c, z)$ of E parametrized by Δ and with base point 0 can be extended uniquely to a holomorphic motion of the closure \overline{E} of E parametrized by Δ and with the same base point. Moreover, $h(c, z)$ is continuous on (c, z) and for any fixed c , $z \mapsto h(c, z)$ is quasiconformal on the interior of \overline{E} . Subsequently, holomorphic motions became an important topic with applications to quasiconformal mapping, Teichmüller theory and complex dynamics. After Mãñé, Sad and Sullivan proved the λ -lemma, Sullivan and Thurston [ST] proved an important extension result. Namely, they proved that any holomorphic motion of E parametrized by Δ and with base point 0 can be extended to a holomorphic motion of $\overline{\mathbb{C}}$, but parametrized by a smaller disk, namely, by Δ_r for some universal number $0 < r < 1$. They showed that r is independent of E and independent of the motion. By a different method and published in the same journal with the Sullivan-Thurston paper, Bers and Royden [BR] proved that $r \geq 1/3$ for all motions of all closed sets E parameterized by Δ . They also showed that on $\overline{\mathbb{C}}$ the map $z \mapsto h(c, z)$ is quasiconformal with dilatation no larger than $(1 + |c|)/(1 - |c|)$. All of these authors raised the question as to whether $r = 1$ for any holomorphic motion of any subset of $\overline{\mathbb{C}}$ parametrized by Δ and with base point 0 . In [S] Slodkowski gave a positive answer by using results from the theory of polynomial hulls in several complex variables. Other authors [AM] [D] have suggested alternative proofs.

In this article we give an expository account of a recent proof of Slodkowski's theorem presented by Chirka in [C]. (See also Chirka and Rosay [CR].) The method involves an application of Schauder's fixed point theorem [CH] to an appropriate operator acting on holomorphic motions of a point and showing that this operator is compact. The compactness depends on the smoothing property of the Cauchy kernel acting on vector fields tangent to holomorphic motions. The main theorem is the following.

Theorem 12 (The Holomorphic Motion Theorem) *Suppose*

$$h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion of a closed subset E of $\overline{\mathbb{C}}$ parameterized by the unit disk. Then there is a holomorphic motion

$$H(c, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$$

which extends $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$. Moreover, for any fixed $c \in \Delta$, $H(c, \cdot) : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a quasiconformal homeomorphism whose quasiconformal dilatation

$$K(H(c, \cdot)) \leq \frac{1 + |c|}{1 - |c|}.$$

The Beltrami coefficient of $H(c, \cdot)$ given by

$$\mu(c, z) = \frac{\partial H(c, z)}{\partial \bar{z}} / \frac{\partial H(c, z)}{\partial z}$$

is a holomorphic function from Δ into the unit ball of the Banach space $\mathcal{L}^\infty(\mathbb{C})$ of all essentially bounded measurable functions on \mathbb{C} .

To prove this result we study the modulus of continuity of functions in the image of the Cauchy kernel operator. Then we follow the proof given by Chirka in [C] who introduces a non-linear operator to which the Schauder fixed point theorem [CH] applies. The existence of a fixed point of this operator implies the existence of a holomorphic extension to any disk of radius $r < 1$ and then a normal families argument allows one to take the limit as r approaches 1.

After proving this theorem, we show that tangent vectors to holomorphic motions have $|\epsilon \log \epsilon|$ moduli of continuity and then show how this type of continuity for tangent vectors can be combined with Schwarz's lemma and integration over the holomorphic variable to produce Hölder continuity on the mappings. At one point the argument requires obtaining a lower bound for the Poincaré metric on the Riemann sphere punctured at 0, 1 and ∞ . The method for obtaining this lower bound is described by Ahlfors in [A1]. A slightly improved version is given by Keen and Lakic in [KL].

We also prove that Kobayashi's and Teichmüller's metrics on the Teichmüller space $T(R)$ of a Riemann surface coincide. The proof we give is very similar to the one given in [GL]. This result was observed by Earle, Kra and Krushkal in [EKK] and had been proved earlier by Royden [R] for Riemann surfaces of finite analytic type and by Gardiner [G1] [G2] for surfaces of infinite type.

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2 The \mathcal{P} -operator and the modulus of continuity

Let $\mathcal{C} = \mathcal{C}(\mathbb{C})$ denote the Banach space of complex valued, bounded, continuous functions ϕ on \mathbb{C} with the supremum norm

$$\|\phi\| = \sup_{c \in \mathbb{C}} |\phi(c)|.$$

We use \mathcal{L}^∞ to denote the Banach space of essentially bounded measurable functions ϕ on \mathbb{C} with \mathcal{L}^∞ -norm

$$\|\phi\|_\infty = \text{ess sup}_{\mathbb{C}} |\phi(\zeta)|.$$

For the theory of quasiconformal mapping we are more concerned with the action of \mathcal{P} on \mathcal{L}^∞ . Here the \mathcal{P} -operator is defined by

$$\mathcal{P}f(c) = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - c} d\xi d\eta, \quad \zeta = \xi + i\eta, \tag{2.1}$$

where $f \in \mathcal{L}^\infty$ and has a compact support in \mathbb{C} . Then

$$\mathcal{P}f(c) \longrightarrow 0 \quad \text{as } c \longrightarrow \infty.$$

Furthermore, if f is continuous and has compact support, one can show that

$$\frac{\partial(\mathcal{P}f)}{\partial \bar{c}}(c) = f(c), \quad c \in \mathbb{C}, \tag{2.2}$$

and by using the notion of generalized derivative [AB] equation (2.2) is still true Lebesgue almost everywhere if we only know that f has compact support and is in $\mathcal{L}^p, p \geq 1$.

For the benefit of the reader we sketch the proof of (2.2) in the case that f is C^1 with compact support. In that case differentiation under the integral sign in (2.1) is permissible and so

$$\begin{aligned} \frac{\partial(\mathcal{P}f)}{\partial \bar{c}}(c) &= -\frac{1}{\pi} \frac{\partial}{\partial \bar{c}} \int \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - c} d\xi d\eta = -\frac{1}{\pi} \frac{\partial}{\partial \bar{c}} \int \int_{\mathbb{C}} \frac{f(\zeta + c)}{\zeta} d\xi d\eta \\ &= -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{f_{\bar{\zeta}}(\zeta + c)}{\zeta} d\xi d\eta = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{f_{\bar{\zeta}}(\zeta)}{\zeta - c} d\xi d\eta = \end{aligned}$$

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{f_{\bar{\zeta}}(\zeta)}{\zeta - c} d\zeta d\bar{\zeta} \\
&= -\frac{1}{2\pi i} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{df d\zeta}{\zeta - c} = -\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int \int_{|\zeta - c| > \epsilon} \frac{df d\zeta}{\zeta - c} = \\
& -\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{|\zeta - c| = \epsilon} \frac{f d\zeta}{\zeta - c} = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} f(c + \epsilon e^{i\theta}) d\theta = f(c).
\end{aligned}$$

We refer to [A2] for the verification that this relation still holds when $f \in L^p$ for $p > 2$.

We now show the classical result that \mathcal{P} transforms \mathcal{L}^∞ functions with compact support in \mathbb{C} to Hölder continuous functions with Hölder exponent $1 - 2/p$ for every $p > 2$, [A2]. We also show that \mathcal{P} carries \mathcal{L}^∞ functions with compact support to functions with an $|\epsilon \log \epsilon|$ modulus of continuity.

Lemma 15 *Suppose $p > 2$ and*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

so that $1 < q < 2$. Then for any real number $R > 0$, there is a constant $A_R > 0$ such that for any $f \in \mathcal{L}^\infty$ with a compact support contained in Δ_R

$$\|\mathcal{P}f\| \leq A_R \|f\|_\infty$$

and

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq A_R \|f\|_\infty |c - c'|^{1 - \frac{2}{p}}, \quad \forall c, c' \in \mathbb{C}.$$

Proof The norm

$$\|\mathcal{P}f\| = \sup_{c \in \mathbb{C}} \frac{1}{\pi} \left| \int \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - c} d\xi d\eta \right| \leq \sup_{c \in \mathbb{C}} \frac{1}{\pi} \int \int_{\Delta_R} \frac{|f(\zeta)|}{|\zeta - c|} d\xi d\eta.$$

So

$$\|\mathcal{P}f\| \leq \|f\|_\infty \sup_{c \in \mathbb{C}} \frac{1}{\pi} \int \int_{\Delta_R} \frac{1}{|\zeta - c|} d\xi d\eta \leq C_1 \|f\|_\infty,$$

where

$$C_1 = \frac{1}{\pi} \int \int_{\Delta_R} \frac{1}{|\zeta|} d\xi d\eta = 2R < \infty.$$

Next

$$\begin{aligned}
|\mathcal{P}f(c) - \mathcal{P}f(c')| &= \frac{1}{\pi} \left| \int_{\mathbb{C}} \int_{\mathbb{C}} f(\zeta) \left(\frac{1}{\zeta - c} - \frac{1}{\zeta - c'} \right) d\xi d\eta \right| \\
&\leq \frac{|c - c'|}{\pi} \int_{\Delta_R} \int_{\Delta_R} \frac{|f(\zeta)|}{|\zeta - c| |\zeta - c'|} d\xi d\eta \\
&\leq \frac{|c - c'|}{\pi} \left(\int_{\Delta_R} \int_{\Delta_R} |f(\zeta)|^p d\xi d\eta \right)^{\frac{1}{p}} \left(\int_{\Delta_R} \int_{\Delta_R} \left| \frac{1}{(\zeta - c)(\zeta - c')} \right|^q d\xi d\eta \right)^{\frac{1}{q}} \\
&\leq \pi^{\frac{1}{p}-1} R^{\frac{2}{p}} |c - c'| \|f\|_{\infty} \left(\int_{\Delta_R} \int_{\Delta_R} \left| \frac{1}{(\zeta - c)(\zeta - c')} \right|^q d\xi d\eta \right)^{\frac{1}{q}} \leq \\
&\quad C_2 \|f\|_{\infty} |c - c'|^{\frac{2}{q}-1},
\end{aligned}$$

where

$$C_2 = \pi^{\frac{1}{p}-1} R^{\frac{2}{p}} \left(\int_{\mathbb{C}} \int_{\mathbb{C}} \left(\frac{1}{|z||z-1|} \right)^q dx dy \right)^{\frac{1}{q}} < \infty, \quad z = x + iy.$$

Hence $A_R = \max\{C_1, C_2\}$ satisfies the requirements of the lemma. \square

Next we prove a stronger form of continuity.

Lemma 16 *Suppose the compact support of $f \in \mathcal{L}^{\infty}$ is contained in Δ . Then $\mathcal{P}f$ has an $|\epsilon \log \epsilon|$ modulus of continuity. More precisely, there is a constant B depending on R such that*

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \|f\|_{\infty} B |c - c'| \log \frac{1}{|c - c'|}, \quad \forall c, c' \in \Delta_R, \quad |c - c'| < \frac{1}{2}.$$

Proof Since

$$\begin{aligned}
|\mathcal{P}f(c) - \mathcal{P}f(c')| &= \frac{1}{\pi} \left| \int_{\mathbb{C}} \int_{\mathbb{C}} f(\zeta) \left(\frac{1}{\zeta - c} - \frac{1}{\zeta - c'} \right) d\xi d\eta \right| \\
&\leq \frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} |f(\zeta)| \left| \frac{1}{\zeta - c} - \frac{1}{\zeta - c'} \right| d\xi d\eta \\
&\leq \frac{|c - c'| \|f\|_{\infty}}{\pi} \int_{\Delta} \frac{1}{|\zeta - c| |\zeta - c'|} d\xi d\eta,
\end{aligned}$$

if we put $\zeta' = \zeta - c = \xi' + i\eta'$, then

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \frac{|c - c'| \|f\|_{\infty}}{\pi} \int \int_{\Delta_{1+R}} \frac{1}{|\zeta'| |\zeta' - (c' - c)|} d\xi' d\eta'.$$

The substitution $\zeta'' = \zeta'/(c' - c) = \xi'' + i\eta''$ yields

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \frac{|c - c'| \|f\|_\infty}{\pi} \int \int_{\Delta_{\frac{1+R}{|c'-c|}}} \frac{1}{|\zeta''||\zeta'' - 1|} d\xi'' d\eta''.$$

Since $|c - c'| < 1/2$, we have $(1 + R)/|c' - c| > 2$. This implies that

$$\begin{aligned} & |\mathcal{P}f(c) - \mathcal{P}f(c')| \\ & \leq \frac{|c - c'| \|f\|_\infty}{\pi} \left(\int \int_{\Delta_2} (\text{same}) + \int \int_{\Delta_{\frac{1+R}{|c'-c|}} - \Delta_2} (\text{same}) \right). \end{aligned}$$

Let

$$C_3 = \int \int_{\Delta_2} \frac{1}{|\zeta''||\zeta'' - 1|} d\xi'' d\eta''.$$

Then

$$\begin{aligned} & |\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \\ & \frac{|c - c'| C_3 \|f\|_\infty}{\pi} + \frac{|c - c'| \|f\|_\infty}{\pi} \int \int_{\Delta_{\frac{1+R}{|c'-c|}} - \Delta_2} \frac{1}{|\zeta''||\zeta'' - 1|} d\xi'' d\eta''. \end{aligned}$$

If $|\zeta''| > 2$ then $|\zeta'' - 1| > |\zeta''|/2$, and so

$$\begin{aligned} \frac{1}{\pi} \int \int_{\Delta_{\frac{1+R}{|c'-c|}} - \Delta_2} \frac{1}{|\zeta''||\zeta'' - 1|} d\xi'' d\eta'' & \leq \frac{1}{\pi} \int \int_{\Delta_{\frac{1+R}{|c'-c|}} - \Delta_2} \frac{2}{|\zeta''|^2} d\xi'' d\eta'' \\ & \leq \frac{1}{\pi} \int_0^{2\pi} \int_2^{\frac{1+R}{|c'-c|}} \frac{2}{r^2} r dr d\theta = 4 \int_2^{\frac{1+R}{|c'-c|}} \frac{1}{r} dr \\ & = 4 \left(\log \frac{1+R}{|c'-c|} - \log 2 \right) = 4(-\log |c - c'| + \log(1+R) - \log 2). \end{aligned}$$

Thus,

$$\begin{aligned} & |\mathcal{P}f(c) - \mathcal{P}f(c')| \leq \\ & \frac{|c - c'| C_3 \|f\|_\infty}{\pi} + 4|c - c'| \|f\|_\infty (-\log |c - c'| + \log(1+R) - \log 2) \\ & = -|c - c'| \log |c - c'| \left(\frac{4\pi \log(1+R) + C_3 \|f\|_\infty - 4\pi \log 2}{-\pi \log |c - c'|} + 4 \|f\|_\infty \right) \\ & \leq B \left(-|c - c'| \log |c - c'| \right), \end{aligned}$$

where

$$B = \frac{4\pi \log(1+R) + C_3 \|f\|_\infty - 4\pi \log 2}{\pi \log 2} + 4\|f\|_\infty.$$

□

Now we have the following theorem.

Theorem 13 *For any $f \in \mathcal{L}^\infty$ with a compact support in \mathbb{C} , $\mathcal{P}f$ has an $|\epsilon \log \epsilon|$ modulus of continuity. More precisely, for any $R > 0$, there is a constant $C > 0$ depending on R such that*

$$|\mathcal{P}f(c) - \mathcal{P}f(c')| \leq C \|f\|_\infty |c - c'| \log \frac{1}{|c - c'|}, \quad \forall c, c' \in \Delta_R, \quad |c - c'| < \frac{1}{2}.$$

Proof Suppose the compact support of f is contained in the disk Δ_{R_0} . Then $g(c) = f(R_0 c)$ has the compact support which is contained in the unit disk Δ . Then

$$\mathcal{P}g(c) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - c} d\xi d\eta = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{f(R_0 \zeta)}{\zeta - c} d\xi d\eta = \frac{1}{R_0} \mathcal{P}f(R_0 c),$$

and this implies that

$$\mathcal{P}f(c) = R_0 \mathcal{P}g\left(\frac{c}{R_0}\right).$$

Thus

$$\begin{aligned} |\mathcal{P}f(c) - \mathcal{P}f(c')| &= R_0 |\mathcal{P}g\left(\frac{c}{R_0}\right) - \mathcal{P}g\left(\frac{c'}{R_0}\right)| \\ &\leq R_0 B \|f\|_\infty \left(-\left| \frac{c}{R_0} - \frac{c'}{R_0} \right| \log \left| \frac{c}{R_0} - \frac{c'}{R_0} \right| \right) \\ &= B \|f\|_\infty \left(-|c - c'| (\log |c - c'| - \log R_0) \right) \\ &= -|c - c'| \log |c - c'| B \|f\|_\infty \left(1 - \frac{\log R_0}{\log |c - c'|} \right) \\ &\leq C \|f\|_\infty (-|c - c'| \log |c - c'|), \end{aligned}$$

where

$$C = B \left(1 + \frac{\log R_0}{\log 2} \right).$$

□

3 Extensions of holomorphic motions for $0 < r < 1$

As an application of the modulus of continuity for the \mathcal{P} -operator, we first prove, for any r with $0 < r < 1$, that for any holomorphic motion of a set E parameterized by Δ , there is an extension to $\Delta_r \times \overline{\mathbb{C}}$. We take the idea of the proof from the recent papers of Chirka [C] and Chirka and Rosay [CR].

Theorem 14 *Suppose E is a subset of $\overline{\mathbb{C}}$ consisting of a finite number of points. Suppose $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$ is a holomorphic motion. Then for every $0 < r < 1$, there is a holomorphic motion $H_r(c, z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ which extends $h(c, z) : \Delta_r \times E \rightarrow \overline{\mathbb{C}}$.*

Without loss of generality, suppose

$$E = \{z_0 = 0, z_1 = 1, z_\infty = \infty, z_2, \dots, z_n\}$$

is a subset of $n + 2 > 3$ points in the Riemann sphere $\overline{\mathbb{C}}$. Let Δ^c be the complement of the unit disk in the Riemann sphere $\overline{\mathbb{C}}$, U be a neighborhood of Δ^c in $\overline{\mathbb{C}}$ and suppose

$$h(c, z) : U \times E \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion of E parametrized by U and with base point ∞ . Define

$$f_i(c) = h(c, z_i) : U \rightarrow \overline{\mathbb{C}}$$

for $i = 0, 1, 2, \dots, n, \infty$. We assume the motion is normalized so

$$f_0(c) = 0, \quad f_1(c) = 1, \quad \text{and} \quad f_\infty(c) = \infty, \quad \forall c \in U.$$

Then we have

- a) $f_i(\infty) = z_i, i = 2, \dots, n,$
- b) for any $i = 2, \dots, n,$ $f_i(c)$ is holomorphic on U and
- c) for any fixed $c \in U,$ $f_i(c) \neq f_j(c)$ and $f_i(c) \neq 0, 1,$ and ∞ for $2 \leq i \neq j \leq n.$

Since Δ^c is compact, $f_i(c)$ is a bounded function on Δ^c for every $2 \leq i \leq n$ and so there is a constant $C_4 > 0$ such that

$$|f_i(c)| \leq C_4, \text{ for all } c \in \Delta^c \text{ and all } i \text{ with } 2 \leq i \leq n.$$

Moreover, there is a number $\delta > 0$ such that

$|f_i(c) - f_j(c)| > \delta$, for all i and j with $2 \leq i \neq j \leq n$, and for all $c \in \Delta^c$.

We extend the functions $f_i(c)$ on Δ^c to continuous functions on the Riemann sphere $\overline{\mathbb{C}}$ by defining

$$f_i(c) = f_i\left(\frac{1}{c}\right), \text{ for all } c \in \overline{\Delta}.$$

We still have

$|f_i(c) - f_j(c)| > \delta$, for all i and j with $2 \leq i \neq j \leq n$ and for all $c \in \overline{\mathbb{C}}$

and

$$|f_i(c)| \leq C_4 \text{ for all } i \text{ and } j \text{ with } 2 \leq i \neq j \leq n \text{ and for all } c \in \overline{\mathbb{C}}.$$

Since $f_i(c)$ is holomorphic in U and $f_i(\infty) = z_i$, the series expansion of $f_i(c)$ at ∞ is

$$f_i(c) = z_i + \frac{a_1}{c} + \frac{a_2}{c^2} + \cdots + \frac{a_n}{c^n} + \cdots, \quad \forall c \in \Delta^c.$$

This implies that

$$f_i(c) = f_i\left(\frac{1}{\bar{c}}\right) = z_i + a_1 \bar{c} + a_2 (\bar{c})^2 + \cdots + a_n (\bar{c})^n + \cdots, \quad \forall c \in \overline{\Delta}.$$

We have that

$$\frac{\partial f_i}{\partial \bar{c}}(c) = a_1 + 2a_2 \bar{c} + \cdots + na_n (\bar{c})^{n-1} + \cdots$$

exists at $c = 0$ and is a continuous function on $\overline{\Delta}$. Furthermore, $(\partial f_i / \partial \bar{c})(c) = 0$ for $c \in (\overline{\Delta})^c$. Since $\overline{\Delta}$ is compact, there is a constant $C_5 > 0$ such that

$$\left| \frac{\partial f_i}{\partial \bar{c}}(c) \right| \leq C_5, \quad \forall c \in \overline{\mathbb{C}}, \quad \forall 2 \leq i \leq n.$$

Pick a C^∞ function $0 \leq \lambda(x) \leq 1$ on $\mathbb{R}^+ = \{x \geq 0\}$ such that $\lambda(0) = 1$ and $\lambda(x) = 0$ for $x \geq \delta/2$. Define

$$\Phi(c, w) = \sum_{i=2}^n \lambda(|w - f_i(c)|) \frac{\partial f_i}{\partial \bar{c}}(c), \quad (c, w) \in \overline{\mathbb{C}} \times \mathbb{C}. \quad (3.1)$$

Lemma 17 *The function $\Phi(c, w)$ has the following properties:*

- i) *only one term in the sum (3.1) defining $\Phi(c, w)$ can be nonzero,*
- ii) *$\Phi(c, w)$ is uniformly bounded by C_5 on $\overline{\mathbb{C}} \times \mathbb{C}$,*

- iii) $\Phi(c, w) = 0$ for $(c, w) \in \left((\overline{\Delta})^c \times \mathbb{C} \right) \cup \left(\overline{\mathbb{C}} \times (\overline{\Delta}_R)^c \right)$ where $R = C_4 + \delta/2$,
- iv) $\Phi(c, w)$ is a Lipschitz function in w -variable with a Lipschitz constant L independent of $c \in \hat{\mathbb{C}}$.

Proof Item i) follows because if a point w is within distance $\delta/2$ of one of the values $f_i(c)$, it must be at distance greater than $\delta/2$ from any of the other values $f_j(c)$. Item ii) follows from item i) because there can be only one term in (3.1) which is nonzero and that term is bounded by the bound on $\frac{\partial f_j(c)}{\partial \bar{c}}$. Item iii) follows because if $c \in (\overline{\Delta})^c$, then $(\partial f_i / \partial \bar{c})(c) = 0$, and if $w \in (\overline{\Delta}_R)^c$, then $\Phi(c, w) = 0$. To prove item iv), we note that there is a constant $C_6 > 0$ such that $|\lambda(x) - \lambda(x')| \leq C_6|x - x'|$. Since $|\partial f_i / \partial \bar{c}(c)| \leq C_5$,

$$|\Phi(c, w) - \Phi(c, w')| \leq C_6 C_5 \sum_{i=2}^n \left| |w - f_i(c)| - |w' - f_i(c)| \right|. \quad (3.2)$$

Since only one of the terms in the sum (3.1) for $\Phi(c, w)$ is nonzero and possibly a different term is nonzero in the sum for $\Phi(c, w')$, we obtain

$$|\Phi(c, w) - \Phi(c, w')| \leq 2C_6 C_5 |w - w'|.$$

Thus $L = 2C_5 C_6$ is a Lipschitz constant independent of $c \in \hat{\mathbb{C}}$. \square

Since $\Phi(c, f(c))$ is an \mathcal{L}^∞ function with a compact support in $\overline{\Delta}$ for any $f \in \mathcal{C}$, we can define an operator \mathcal{Q} mapping functions in \mathcal{C} to functions in \mathcal{L}^∞ with compact support by

$$\mathcal{Q}f(c) = \Phi(c, f(c)), \quad f(c) \in \mathcal{C}.$$

Since $\Phi(c, w)$ is Lipschitz in the w variable with a Lipschitz constant L independent of $c \in \overline{\mathbb{C}}$, we have

$$|\mathcal{Q}f(c) - \mathcal{Q}g(c)| = |\Phi(c, f(c)) - \Phi(c, g(c))| \leq L|f(c) - g(c)|.$$

Thus

$$\|\mathcal{Q}f - \mathcal{Q}g\|_\infty \leq L\|f - g\|,$$

and $\mathcal{Q} : \mathcal{C} \rightarrow \mathcal{L}^\infty$ is a continuous operator.

From Lemma 15,

$$\|\mathcal{P}f\| \leq A_1 \|f\|_\infty$$

for any $f \in \mathcal{L}^\infty$ whose compact support is contained in Δ , and so the composition $\mathcal{K} = \mathcal{P} \circ \mathcal{Q}$, where

$$\mathcal{K}f(c) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\Phi(\zeta, f(\zeta))}{\zeta - c} d\xi d\eta, \quad \zeta = \xi + i\eta,$$

is a continuous operator from \mathcal{C} into itself.

Lemma 18 *There is a constant $D > 0$ such that*

$$\|\mathcal{K}f\| \leq D, \quad \forall f \in \mathcal{C}.$$

Proof Since $\Phi(c, w) = 0$ for $c \in \Delta^c$ and since $\Phi(c, w)$ is bounded by C_5 , we have that

$$\begin{aligned} |\mathcal{K}f(c)| &= \left| \frac{1}{\pi} \int_{\overline{\mathbb{C}}} \int_{\overline{\mathbb{C}}} \frac{\Phi(\zeta, f(\zeta))}{\zeta - c} d\xi d\eta \right| = \left| \frac{1}{\pi} \int_{\Delta} \int_{\Delta} \frac{\Phi(\zeta, f(\zeta))}{\zeta - c} d\xi d\eta \right| \\ &\leq \frac{1}{\pi} \int_{\Delta} \int_{\Delta} \frac{|\Phi(\zeta, f(\zeta))|}{|\zeta - c|} d\xi d\eta \\ &\leq \frac{C_5}{\pi} \int_{\Delta} \int_{\Delta} \frac{1}{|\zeta - c|} d\xi d\eta \leq 2C_5 = D, \end{aligned}$$

where $\zeta = \xi + i\eta$. □

Lemma 19 *Suppose $p > 2$ and q is the dual number between 1 and 2 satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then for any $f \in \mathcal{C}$, $\mathcal{K}f$ is α -Hölder continuous for

$$0 < \alpha = \frac{2}{q} - 1 < 1$$

with a Hölder constant $H = A_1 C_5$ independent of f .

Proof From Lemma 15,

$$\begin{aligned} |\mathcal{K}f(c) - \mathcal{K}f(c')| &= |\mathcal{P}(\mathcal{Q}f)(c) - \mathcal{P}(\mathcal{Q}f)(c')| \\ &\leq A_1 \|\mathcal{Q}f\|_{\infty} |c - c'|^{\alpha} \leq A_1 C_5 |c - c'|^{\alpha} = H |c - c'|^{\alpha}. \end{aligned}$$

□

The above two lemmas imply that $\mathcal{K} : \mathcal{C} \rightarrow \mathcal{C}$ is a continuous compact operator. Now for any $z \in \mathbb{C}$, let

$$\mathcal{B}_z = \{f \in \mathcal{C} \mid \|f\| \leq |z| + D\}.$$

It is a bounded convex subset in \mathcal{C} . The continuous compact operator $z + \mathcal{K}$ maps \mathcal{B}_z into itself. From the Schauder fixed point theorem [CH], $z + \mathcal{K}$ has a fixed point in \mathcal{B}_z . That is, there is a $f_z \in \mathcal{B}_z$ such that

$$f_z(c) = z + \mathcal{K}f_z(c), \quad \forall c \in \mathbb{C}.$$

Since $\mathcal{Q}f(c)$ has a compact support in $\overline{\Delta}$ for any $f \in \mathcal{C}$, $\mathcal{K}f_z(c) \rightarrow 0$ as $c \rightarrow \infty$. So f_z can be extended continuously to ∞ such that $f_z(\infty) = z$.

Lemma 20 *The solution $f_z(c)$ is the unique fixed point of the operator $z + \mathcal{K}$.*

Proof Suppose $f_z(c)$ and $g_z(c)$ are two solutions. Take

$$\phi(c) = f_z(c) - g_z(c) = \mathcal{K}(f_z)(c) - \mathcal{K}(g_z)(c).$$

Then $\phi(c) \rightarrow 0$ as $c \rightarrow \infty$. Now

$$\frac{\partial \phi}{\partial \bar{c}}(c) = \frac{\partial f_z}{\partial \bar{c}}(c) - \frac{\partial g_z}{\partial \bar{c}}(c) = \Phi(c, f_z(c)) - \Phi(c, g_z(c)).$$

So by Lemma 17

$$\frac{\partial \phi}{\partial \bar{c}}(c) = 0, \quad \forall c \in \Delta^c.$$

Since $\Phi(c, w)$ is Lipschitz in w -variable with a Lipschitz constant L ,

$$\left| \frac{\partial \phi}{\partial \bar{c}}(c) \right| = |\Phi(c, f_z(c)) - \Phi(c, g_z(c))| \leq L|f_z(c) - g_z(c)| = L|\phi(c)|.$$

Assuming that $\phi(c)$ is not equal to zero, define

$$\psi(c) = -\frac{\frac{\partial \phi}{\partial \bar{c}}(c)}{\phi(c)},$$

and otherwise, define $\psi(c)$ to be equal to zero. Then $\psi(c)$ is a function in \mathcal{L}^∞ with a compact support in $\overline{\Delta}$. So we have $\mathcal{P}\psi$ in \mathcal{C} such that

$$\frac{\partial \mathcal{P}\psi}{\partial \bar{c}}(c) = \psi(c).$$

Consider $e^{\mathcal{P}\psi} \cdot \phi$. Then

$$\frac{\partial (e^{\mathcal{P}\psi} \cdot \phi)}{\partial \bar{c}} \equiv 0.$$

This means that $e^{\mathcal{P}\psi} \cdot \phi$ is holomorphic on the complex plane \mathbb{C} .

When $c \rightarrow \infty$, $\mathcal{P}\psi \rightarrow 0$ and $\phi(c) \rightarrow 0$. This implies that $e^{\mathcal{P}\psi} \cdot \phi$ is bounded on \mathbb{C} . So $e^{\mathcal{P}\psi} \cdot \phi$ is a constant function. But $\phi(\infty) = 0$, so $e^{\mathcal{P}\psi} \cdot \phi \equiv 0$. Thus $\phi(c) \equiv 0$ and $f_z(c) = g_z(c)$ for all $c \in \mathbb{C}$. \square

For $z_i \in E$, $2 \leq i \leq n$, consider

$$\mathcal{K}f_i(c) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\Phi(\zeta, f_i(\zeta))}{\zeta - c} d\xi d\eta,$$

where $\zeta = \xi + i\eta$. From the definition of $\Phi(c, w)$, we have that

$$\Phi(\zeta, f_i(\zeta)) = \frac{\partial f_i}{\partial \bar{\zeta}}(\zeta).$$

So

$$\mathcal{K}f_i(c) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{\frac{\partial f_i}{\partial \bar{\zeta}}(\zeta)}{\zeta - c} d\xi d\eta.$$

This implies that

$$\frac{\partial \mathcal{K}f_i}{\partial \bar{c}}(c) = \frac{\partial f_i}{\partial \bar{c}}(c)$$

and that

$$\frac{\partial (f_i - \mathcal{K}f_i)}{\partial \bar{c}}(c) \equiv 0.$$

So $f_i(c) - \mathcal{K}f_i(c)$ is holomorphic on \mathbb{C} . When $c \rightarrow \infty$, $f_i(c) \rightarrow z_i$ and $\mathcal{K}f_i(c) \rightarrow 0$. So $f_i(c) - \mathcal{K}f_i(c)$ is bounded. Therefore it is a constant function. We get that

$$f_i(c) = z_i + \mathcal{K}f_i(c).$$

Thus from Lemma 20, $f_i(c) = f_{z_i}(c)$ for all $c \in \overline{\mathbb{C}}$.

By defining $H(c, z) = f_z(c)$ for $(c, z) \in \overline{\Delta}^c \times \mathbb{C} \setminus \{0, 1\}$ and $H(c, 0) = 0$ and $H(c, 1) = 1$ and $H(c, \infty) = \infty$, we get a map

$$H(c, z) = f_z(c) : \overline{\Delta}^c \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}},$$

which is an extension of

$$h(c, z) : \overline{\Delta}^c \times E \rightarrow \overline{\mathbb{C}}.$$

Lemma 21 *The map*

$$H(c, z) = f_z(c) : \overline{\Delta}^c \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}},$$

is a holomorphic motion.

Proof First $H(\infty, z) = f_z(\infty) = z$ for all $z \in \overline{\mathbb{C}}$. From the fixed point equation

$$H(c, z) = z + \mathcal{K}H(c, z),$$

$$\frac{\partial H(c, z)}{\partial \bar{c}} = \Phi(c, H(c, z)).$$

Since $\Phi(c, w) = 0$ for all $c \in \overline{\Delta}^c$,

$$\frac{\partial H(c, z)}{\partial \bar{c}} = 0, \quad \forall c \in \overline{\Delta}^c.$$

Thus, for any fixed $z \in \overline{\mathbb{C}}$, $H(c, z) : \overline{\Delta}^c \rightarrow \overline{\mathbb{C}}$ is holomorphic.

For any two points z and $z' \in \overline{\mathbb{C}}$, we claim that $H(c, z) \neq H(c, z')$ for all $c \in \mathbb{C}$. This implies that for any fixed $c \in \overline{\Delta}^c$, $H(c, z)$ is an injective map on $z \in \overline{\mathbb{C}}$ and that $H(c, z)$ is a holomorphic motion. To prove the claim take any two points $z, z' \in \overline{\mathbb{C}}$. Assume there is a point $c_0 \in \overline{\mathbb{C}}$ such that $H(c_0, z) = H(c_0, z')$. If $c_0 = \infty$, then $z = z'$, because by assumption the holomorphic motion starts out at the identity. If $c_0 \neq \infty$, then

$$f_z(c_0) - f_{z'}(c_0) = (z - z') + \mathcal{K}f_z(c_0) - \mathcal{K}f_{z'}(c_0),$$

and we can repeat the same argument given in Lemma 20.

Let $\phi(c) = f_z(c) - f_{z'}(c)$. Then $\phi(c_0) = 0$. However,

$$\frac{\partial \phi}{\partial \bar{c}}(c) = \frac{\partial f_z}{\partial \bar{c}}(c) - \frac{\partial f_{z'}}{\partial \bar{c}}(c) = \Phi(c, f_z(c)) - \Phi(c, f_{z'}(c)).$$

This implies that

$$\frac{\partial \phi}{\partial \bar{c}}(c) = 0$$

for $c \in \overline{\Delta}^c$. Since $\Phi(c, w)$ is Lipschitz in w -variable with a Lipschitz constant L ,

$$\left| \frac{\partial \phi}{\partial \bar{c}}(c) \right| = |\Phi(c, f_z(c)) - \Phi(c, f_{z'}(c))| \leq L|f_z(c) - f_{z'}(c)| = L|\phi(c)|.$$

If $\phi(c) \neq 0$, define

$$\psi(c) = -\frac{\frac{\partial \phi}{\partial \bar{c}}(c)}{\phi(c)},$$

otherwise, define $\psi(c) = 0$. Then

$$\frac{\partial e^{\mathcal{P}\psi} \cdot \phi}{\partial \bar{c}}(c) \equiv 0.$$

So $e^{\mathcal{P}\psi} \cdot \phi$ is holomorphic on \mathbb{C} . When $c \rightarrow \infty$, $\mathcal{P}\psi(c) \rightarrow 0$ and $\phi(c) \rightarrow z - z'$. So $e^{\mathcal{P}\psi(c)} \cdot \phi(c)$ is bounded on \mathbb{C} . This implies that $e^{\mathcal{P}\psi(c)} \cdot \phi(c)$ is a constant function. Since $\phi(c_0) = 0$, $e^{\mathcal{P}\psi(c)} \cdot \phi(c) \equiv 0$. So $z = z'$. \square

Proof [Proof of Theorem 14] Suppose

$$h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion. For every $0 < r < 1$, consider $\alpha_r(c) = r/c$. Let $U_r = \alpha_r(\Delta_r) \supset \overline{\Delta}^c$. Then

$$h_r(\alpha_r^{-1}(c), z) : U_r \times E \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion. From Lemmas 20 and 21, it can be extended to a holomorphic motion

$$\tilde{H}_r(c, z) : \overline{\Delta}^c \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}.$$

Then

$$H_r(c, z) = \tilde{H}_r(\alpha_r(c), z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion which is an extension of $h(c, z)$ on $\Delta_r \times E$. \square

4 Controlling quasiconformal dilatation

To control the quasiconformal dilatation of a holomorphic motion there are two methods available. One is given by the Bers-Royden paper [BR] and the other is obtained by combining methods given in the Bers-Royden paper and in the Sullivan-Thurston paper [ST]. We discuss the latter method first.

Consider a set of four points $S = \{z_1, z_2, z_3, z_4\}$ in $\overline{\mathbb{C}}$. These points are distinct if and only if the cross ratio

$$Cr(S) = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

is not equal to 0, 1, or ∞ . If one of these points is equal to ∞ , say z_4 , then this cross ratio becomes a ratio

$$Cr(S) = \frac{z_1 - z_3}{z_2 - z_3}.$$

Suppose $H : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is an orientation-preserving homeomorphism such

that $H(\infty) = \infty$. Then one of the definitions of quasiconformality [LV] of H is that

$$\limsup_{r \rightarrow 0} \sup_{a \in \mathbb{C}} \frac{\sup_{|z-a|=r} |H(z) - H(a)|}{\inf_{|z-a|=r} |H(z) - H(a)|} < \infty.$$

In [ST] Sullivan and Thurston used this definition to prove the following theorem.

Theorem 15 *Suppose $H(c, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a normalized holomorphic motion of $\overline{\mathbb{C}}$ parametrized by Δ and with base point 0. Then for each $c_0 \in \Delta$, the map $H(c_0, \cdot) : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is quasiconformal.*

Proof Let $a \in \mathbb{C}$ be any point. Let $z_3 = a$. Let z_1 and z_2 be two distinct points in \mathbb{C} not equal to a and $z_4 = \infty$. Then the cross ratio $Cr(S) = (z_1 - z_3)/(z_2 - z_3)$.

Now consider $z_1(c) = H(c, z_1)$, $z_2(c) = H(c, z_2)$, $z_3(c) = H(c, z_3)$, and $z_4(c) = H(c, z_4) = \infty$ and $S(c) = \{z_1(c), z_2(c), z_3(c), z_4(c)\}$. The cross ratio

$$Cr(S(c)) = \frac{z_1(c) - z_3(c)}{z_2(c) - z_3(c)}.$$

Since $H(c, z)$ is a holomorphic motion, $Cr(S(c)) : \Delta \rightarrow \mathbb{C} \setminus \{0, 1\}$ is a holomorphic function. Then it decreases the hyperbolic distances from ρ_Δ to $\rho_{0,1}$. So

$$\rho_{0,1}(Cr(S(c_0)), Cr(S)) \leq \rho_\Delta(0, c_0) = \log \frac{1 + |c_0|}{1 - |c_0|}.$$

This implies that there is a constant $K = K(c_0) > 0$ such that for any $|Cr(S)| = 1$,

$$|Cr(S(c_0))| \leq K.$$

So we have that

$$\limsup_{r \rightarrow 0} \sup_{a \in \mathbb{C}} \frac{\sup_{|z-a|=r} |H(c_0, z) - H(c_0, a)|}{\inf_{|z-a|=r} |H(c_0, z) - H(c_0, a)|} < \infty,$$

that is, $H(c_0, z)$ is quasiconformal. □

Suppose $\mathcal{L}^\infty(W)$ is the Banach space of all essentially bounded measurable functions on W equipped with $\|\cdot\|_\infty$ -norm. Bers and Royden [BR] proved the following theorem.

Theorem 16 Suppose $h(c, z) : \Delta \times E \rightarrow \hat{\mathbb{C}}$ is a holomorphic motion of E parametrized by Δ and with base point 0 and E has nonempty interior W , then the Beltrami coefficient of $h(c, \cdot)|_W$ given by

$$\mu(c, z) = \frac{\partial h(c, z)|_W}{\partial \bar{z}} / \frac{\partial h(c, z)|_W}{\partial z}$$

is a holomorphic function mapping $c \in \Delta$ into the unit ball of the Banach space $\mathcal{L}^\infty(W)$.

Proof Since the dual of the Banach space $\mathcal{L}^1(W)$ of integrable functions on W is $\mathcal{L}^\infty(W)$, to prove $\mu(c, \cdot)$ is a holomorphic map, it suffices to show that the function

$$c \mapsto \Psi(c) = \int \int_W \alpha(z) \mu_c(z) dx dy$$

is holomorphic in Δ for every $\alpha(z) \in \mathcal{L}^1(W)$. Furthermore, it suffices to check this for every $\alpha(z) \in \mathcal{L}^1(W)$ with a compact support in W .

Suppose $\alpha(z) \in \mathcal{L}^1(W)$ has a compact support $\text{supp}(\alpha)$ in W . There is an $\epsilon > 0$ such that the ϵ -neighborhood $U_\epsilon(\text{supp}(\alpha)) \subset W$. From Theorem 15, $h(c, \cdot)$ is quasiconformal, and it is differentiable a.e. in W . Thus

$$\begin{aligned} \Psi(c) &= \int \int_{\text{supp}(\alpha)} \alpha(z) \frac{h_x(c, z) + ih_y(c, z)}{h_x(c, z) - ih_y(c, z)} dx dy \\ &= \int \int_{\text{supp}(\alpha)} \alpha(z) \frac{1 + i \frac{h_y(c, z)}{h_x(c, z)}}{1 - i \frac{h_y(c, z)}{h_x(c, z)}} dx dy \\ &= \int \int_{\text{supp}(\alpha)} \alpha(z) \lim_{\lambda \rightarrow 0} \frac{1 + i\sigma_c(z, \lambda)}{1 - i\sigma_c(z, \lambda)} dx dy, \end{aligned}$$

where

$$\sigma_c(z, \lambda) = \frac{h(c, z + i\lambda) - h(c, z)}{h(c, z + \lambda) - h(c, z)}.$$

For any fixed $z \neq 0, 1, \infty$ and λ small,

$$\varrho(c) = \sigma_c(z, \lambda) : \Delta \mapsto \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$$

is a holomorphic function of $c \in \Delta$. So it decreases the hyperbolic distances on Δ and on $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. Since $\varrho(0) = i$, there is a number $0 < r < 1$ such that for

$$|\sigma_c(z, \lambda) - i| \leq \frac{1}{2}, \quad |c| < r.$$

Therefore

$$\left| \frac{1 + i\sigma_c(z, \lambda)}{1 - i\sigma_c(z, \lambda)} \right| = \left| \frac{-i + \sigma_c(z, \lambda)}{i + \sigma_c(z, \lambda)} \right| \leq \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{1}{3}.$$

By the Dominated Convergence Theorem, for $|c| < r$, the sequence of holomorphic functions

$$\Psi_n(c) = \int \int_{\text{supp}(\alpha)} \alpha(z) \frac{1 + i\sigma_c(z, \frac{1}{n})}{1 - i\sigma_c(z, \frac{1}{n})} dx dy$$

converges uniformly to $\Psi(c)$ as $n \rightarrow \infty$. Thus $\Psi(c)$ is holomorphic for $|c| < r$ and this implies that

$$\mu(c, \cdot) : \{c \mid |c| < r\} \rightarrow \mathcal{L}^\infty(W)$$

is holomorphic.

Now consider arbitrary $c_0 \in \Delta$. Let $s = 1 - |c_0|$ and let

$$E_0 = h(c_0, E), \quad W_0 = h(c_0, W)$$

and

$$g(\tau, \zeta) = h(c_0 + s\tau, z), \quad \zeta = h(c_0, z).$$

Then W_0 is the interior of E_0 since $h(c, z)$ is a quasiconformal homeomorphism. Also

$$g : \Delta \times E_0 \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion. So the Beltrami coefficient of g is a holomorphic function on $\{\tau \mid |\tau| < r\}$. Hence the Beltrami coefficient of h is a holomorphic function on $\{c \mid |c - c_0| < sr\}$. This concludes the proof. \square

Theorem 17 *Suppose $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$ is a holomorphic motion of E parametrized by Δ and with base point 0 and suppose E has a nonempty interior W . Then for each $c \in \Delta$, the map $h(c, z)|_W$ is a K -quasiconformal homeomorphism of W into $\overline{\mathbb{C}}$ with*

$$K \leq \frac{1 + |c|}{1 - |c|}.$$

Proof Since $c \mapsto \mu(c, \cdot)$ mapping from Δ to the unit ball of $\mathcal{L}^\infty(W)$ is a holomorphic map and since $\mu(0, \cdot) = 0$, from the Schwarz's lemma, $\|\mu\|_\infty \leq |c|$. This implies that the quasiconformal dilatation of $h(c, \cdot)$ is less than or equal to $K = \frac{1 + |c|}{1 - |c|}$. \square

5 Extension of holomorphic motions for $r = 1$

Theorem 18 (Słodkowski’s Theorem) *Suppose $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$ is a holomorphic motion. Then there is a holomorphic motion*

$$H(c, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$$

which extends $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$.

Proof Suppose E is a subset of $\overline{\mathbb{C}}$. Suppose $h(c, z) : \Delta \times E \rightarrow \overline{\mathbb{C}}$ is a holomorphic motion. Let E_1, E_2, \dots be a sequence of nested subsets consisting of finite number of points in E . Suppose

$$\{0, 1, \infty\} \subset E_1 \subset E_2 \subset \dots \subset E$$

and suppose $\cup_{i=1}^{\infty} E_i$ is dense in E . Then $h(c, z) : \Delta \times E_i \rightarrow \overline{\mathbb{C}}$ is a holomorphic motion for every $i = 1, 2, \dots$

From Theorem 14, for any $0 < r < 1$ and $i \geq 1$, there is a holomorphic motion $H_i(c, z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ such that $H_i|_{\Delta_r \times E_i} = h|_{\Delta_r \times E_i}$. From Theorem 17, $z \mapsto H_i(c, z)$ is $(1 + |c|/r)/(1 - |c|/r)$ -quasiconformal and fixes $0, 1, \infty$ for all $i > 0$. So for any $|c| \leq r$, the functions $z \mapsto H_i(c, z)$ form a normal family and there is a subsequence $H_{i_k}(c, \cdot)$ converging uniformly (in the spherical metric) to a $(1 + |c|/r)/(1 - |c|/r)$ -quasiconformal homeomorphism $H_r(c, \cdot) : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ such that $H_r(c, z) = h(c, z)$ for $z \in \cup(E_{j_k})$.

Let ζ be a point in E . Replacing E_i by $E_i \cup \{\zeta\}$ and repeating the previous construction we obtain a $(1 + |c|/r)/(1 - |c|/r)$ -quasiconformal homeomorphism \tilde{H}_r which coincides with $h(c, z)$ on $(\cup E_{i_k}) \cup \{\zeta\}$. But $z \mapsto H_r(c, z)$ and $z \mapsto \tilde{H}_r(c, z)$ are continuous everywhere and coincide on $\cup E_{i_k}$, hence on E . So $H_r(c, \zeta) = \tilde{H}_r(c, \zeta) = h(c, \zeta)$ for any $\zeta \in E$.

Now for any $z \neq 0, 1, \infty$, since $H_i(c, z) : \Delta \rightarrow \overline{\mathbb{C}}$ are holomorphic and omit three points $0, 1, \infty$. So the functions $c \mapsto H_i(c, z)$ form a normal family. Any convergent subsequence $H_{i_k}(c, z)$ still has a holomorphic limit $H_r(c, z)$, thus $H_r(c, z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a holomorphic motion which extends $h(c, z)$ on $\Delta_r \times \overline{\mathbb{C}}$.

Now we are ready to take the limit as $r \rightarrow 1$. For each $0 < r < 1$, let $H_r(c, z) : \Delta_r \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be a holomorphic motion such that $H_r = h$ on $\Delta_r \times E$. From Theorem 17, $H_r(c, \cdot)$ is $(1 + |c|/r)/(1 - |c|/r)$ -quasiconformal for every c with $|c| \leq r$.

Take a sequence $Z = \{z_i\}_{i=1}^{\infty}$ of points in $\overline{\mathbb{C}}$ such that $\overline{Z} = \overline{\mathbb{C}}$, and

assume $0, 1,$ and ∞ are not elements of Z . For each $i = 1, 2, \dots,$
 $H_r(c, z_i) : \Delta_r \rightarrow \overline{\mathbb{C}}$ is holomorphic and omits $0, 1, \infty$. Thus

$$\{H_r(c, z_i), c \in \Delta_r\}_{0 < r < 1}$$

forms a normal family. We have a subsequence $r_n \rightarrow 1$ such that $H_{r_n}(c, z_i)$ tends to a holomorphic function $\tilde{H}(c, z_i)$ defined on Δ uniformly on the spherical metric for all $i = 1, 2, \dots$. For a fixed $c \in \Delta,$ $H_{r_n}(c, \cdot)$ are $(1 + |c|/r_n)/(1 - |c|/r_n)$ -quasiconformal for all $r_n > |c|$. So $\{H_{r_n}(c, \cdot)\}_{r_n > |c|}$ is a normal family. Since $H_{r_n}(c, \cdot)$ fixes $0, 1, \infty,$ there is a subsequence of $\{H_{r_n}(c, \cdot)\},$ which we still denote by $\{H_{r_n}(c, \cdot)\},$ that converges uniformly in the spherical metric to a $(1 + |c|)/(1 - |c|)$ -quasiconformal homeomorphism $H(c, \cdot)$. Since $\tilde{H}(c, z_i) = H(c, z_i)$ for all $i = 1, 2, \dots,$ this implies that, for any fixed $z_i, H(c, z_i)$ is holomorphic. Thus $H(c, z) : \Delta \times Z \rightarrow \overline{\mathbb{C}}$ is a holomorphic motion.

For any $0 < r < 1, H(c, z)$ is $(1 + r)/(1 - r)$ -quasiconformal for all c with $|c| \leq r,$ it is α -Hölder continuous, that is,

$$d(H(c, z), H(c, z')) \leq A d(z, z')^\alpha \quad \text{for all } z, z' \in \overline{\mathbb{C}} \quad \text{and for all } |c| \leq r,$$

where $d(\cdot, \cdot)$ is the spherical distance and where A and $0 < \alpha < 1$ depend only on r .

For any $z \in Z$ such that its spherical distances to $0, 1, \infty$ are greater than $\epsilon > 0,$ the map $H(c, z)$ is a holomorphic map on $\Delta,$ which omits the values $0, 1,$ and ∞ . So $H(c, z)$ decreases the hyperbolic distance ρ_Δ on Δ and the hyperbolic distance $\rho_{0,1}$ on $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$. So we have a constant $B > 0$ depending only on r and ϵ such that

$$d(H(c, z), H(c', z)) \leq B|c - c'|$$

for all $|c|, |c'| \leq r$ and all $z \in Z$ such that spherical distances between them and $0, 1,$ and ∞ are greater than $\epsilon > 0$. Thus we get that

$$d(H(c, z), H(c', z')) \leq A\delta(z, z')^\alpha + B|c - c'|$$

for $|c|, |c'| \leq r$ and $z, z' \in Z$ such that their spherical distances from $0, 1,$ and ∞ are greater than $\epsilon > 0$. This implies that $H(c, z)$ is uniformly equicontinuous on $|c| \leq r$ and $\{z \in Z \mid d(z, \{0, 1, \infty\}) \geq \epsilon\}$. Therefore, its continuous extension $H(c, z)$ is holomorphic in c with $|c| \leq r$ for any $\{z \in \overline{\mathbb{C}} \mid d(z, \{0, 1, \infty\}) \geq \epsilon\}$. Letting $r \rightarrow 1$ and $\epsilon \rightarrow 0,$ we get that $H(c, z)$ is holomorphic in $c \in \Delta$ for any $z \in \overline{\mathbb{C}}$. Thus $H(c, z) : \Delta \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a holomorphic motion such that $H(c, z)|_{\Delta \times E} = h(c, z)$. This completes the proof. □

6 The $|\epsilon \log \epsilon|$ continuity of a holomorphic motion

In this section we show how the $|\epsilon \log \epsilon|$ modulus of continuity for the tangent vector to a holomorphic motion can be derived directly from Schwarz's lemma. Then we go on to show how the Hölder continuity of the mapping $z \mapsto w(z) = h(c, z)$ with Hölder exponent $\frac{1-|c|}{1+|c|}$ follows from the $|\epsilon \log \epsilon|$ continuity of the tangent vectors to the curve $c \mapsto h(c, z)$. In particular, since any K -quasiconformal map $z \mapsto f(z)$ coincides with $z \mapsto h(c, z)$ where $K \leq \frac{1+|c|}{1-|c|}$, we conclude that f satisfies a Hölder condition with exponent $1/K$.

Lemma 22 *Let $h(c, z)$ be a normalized holomorphic motion parametrized by Δ and with base point 0 and let $V(z)$ be the tangent vector to this motion at $c = 0$ defined by*

$$V(z) = \lim_{c \rightarrow 0} \frac{h(c, z) - z}{c}. \tag{6.1}$$

Then $V(0) = 0$, $V(1) = 0$ and $|V(z)| = o(|z|^2)$ as $z \rightarrow \infty$.

Proof Since $h(c, z)$ is normalized, $h(c, 0) = 0$ and $h(c, 1) = 1$ for every $c \in \Delta$, and therefore $V(0) = 0$ and $V(1) = 0$. Since $h(c, \infty) = \infty$ for every $c \in \Delta$ if we introduce the coordinate $w = 1/z$ and consider the motion $h_1(c, w) = 1/h(c, 1/w)$, we see that $h_1(c, 0) = 0$ for every $c \in \Delta$.

Put $p(c) = h(c, z)$ and if we think of z as a local coordinate for the Riemann sphere,

$$z \circ p(c) = z + cV^z(z) + o(c^2),$$

and in terms of the local coordinate $w = 1/z$,

$$w \circ p(c) = w + cV^w(w) + o(c^2).$$

Then $V^w(0) = 0$. Putting $g = w \circ z^{-1}$, the identity $g(z(p(c))) = w(p(c))$ yields

$$g'(z(p(0)))z'(p(0)) = w'(p(0)). \tag{6.2}$$

Since $g(z) = 1/z$, $g'(z) = -(1/z)^2$ and since

$$V^w(0) = 0, \quad \frac{d}{dc}w(p(c))|_{c=0} = V^w(w(p(0)))$$

and $V^w(w(p(c)))$ is a continuous function of c , the equation

$$V^z(z(p(c)))\frac{dw}{dz} = V^w(w(p(c)))$$

implies

$$\frac{V^z(z)}{z^2} \rightarrow 0$$

as $z \rightarrow \infty$. □

Let $\rho_{0,1}(z)$ be the infinitesimal form for the hyperbolic metric on $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ and let $\rho_{\Delta}(z) = 2/(1 - |z|^2)$ be the infinitesimal form for the hyperbolic metric on Δ . For any four distinct points a, b, c and d , the cross ratio

$$g(c) = cr(h_c(a), h_c(b), h_c(c), h_c(d))$$

is a holomorphic function of $c \in \Delta$, and omitting the values 0, 1 and ∞ . Then by Schwarz's lemma,

$$\rho_{0,1}(g(c))|g'(c)| \leq \sigma_{\Delta}(c) = \frac{2}{1 - |c|^2}$$

and

$$\rho_{0,1}(g(0))|g'(0)| \leq 2. \tag{6.3}$$

But $|g'(0)|$ is equal to

$$|g(0)| \left| \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d} \right|, \tag{6.4}$$

where $g(0) = cr(a, b, c, d) = \frac{(b-a)(d-c)}{(c-b)(a-d)}$.

Lemma 23 *If $V(b) = o(b^2)$ as $b \rightarrow \infty$, then*

$$\left(\frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} \right) \rightarrow 0 \quad \text{as } b \rightarrow \infty.$$

Proof

$$\left(\frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} \right)$$

simplifies to

$$\frac{cV(b) - bV(c) - aV(b) - cV(a) + bV(a) + aV(c)}{(b - a)(c - b)}.$$

As $b \rightarrow \infty$ the denominator grows like b^2 but the numerator is $o(b^2)$. □

Theorem 19 *For any vector field V tangent to a normalized holomorphic motion and defined by (6.1), there exists a number C depending on R such that for any two complex numbers z_1 and z_2 with $|z_1| < R$ and $|z_2| < R$ and $|z_1 - z_2| < \delta$,*

$$|V(z_2) - V(z_1)| \leq |z_2 - z_1| \left(2 + \frac{C}{\log \frac{1}{\delta}}\right) \left(\log \frac{1}{|z_2 - z_1|}\right).$$

Proof By applying Lemma 23, inequality (6.3) and equation (6.4) to $a = z_1, b = z_2, c = 0, d = \infty$, we obtain $g(0) = \frac{z_2 - z_1}{z_2}$,

$$\begin{aligned} & \left| \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d} \right| \\ &= \left| \frac{V(z_2) - V(z_1)}{z_2 - z_1} - \frac{V(z_2)}{z_2} \right|, \end{aligned}$$

and

$$\rho_{0,1} \left(\frac{z_2 - z_1}{z_2} \right) \left| \frac{z_2 - z_1}{z_2} \right| \left| \frac{V(z_2) - V(z_1)}{z_2 - z_1} - \frac{V(z_2)}{z_2} \right| \leq 2,$$

and so

$$\left| \frac{V(z_2) - V(z_1)}{z_2 - z_1} - \frac{V(z_2)}{z_2} \right| \leq \frac{2}{\rho_{0,1} \left(\frac{z_2 - z_1}{z_2} \right) \left| \frac{z_2 - z_1}{z_2} \right|}. \quad (6.5)$$

Applying (6.3) and (6.4) again with $a = 0, b = 1, c = \infty, d = z_2$, we obtain

$$\rho_{0,1}(z_2) |z_2| \left| \frac{V(z_2)}{z_2} \right| \leq 2,$$

and so

$$\frac{|V(z_2)|}{|z_2|} \leq \frac{2}{\rho_{0,1}(z_2) |z_2|}, \quad (6.6)$$

and this together with (6.5) implies

$$\left| \frac{V(z_2) - V(z_1)}{z_2 - z_1} \right| \leq \frac{2}{\rho_{0,1} \left(\frac{z_2 - z_1}{z_2} \right) \left| \frac{z_2 - z_1}{z_2} \right|} + \frac{2}{\rho_{0,1}(z_2) |z_2|}. \quad (6.7)$$

To finish the proof we need the following lemma, a form of which

appeared in [L, page 40]. We adapted similar ideas to prove the following version, which is sufficient for the proof of Theorem 19.

Lemma 24 *If $0 < |z| < 1$, then*

$$\rho_{0,1}(z) \geq \frac{1}{|z| \left(4 + \log 4 + \log \frac{1}{|z|}\right)}. \quad (6.8)$$

Proof From Agard's formula [Ag] (note that $\rho_{0,1}$ has the curvature -1),

$$\rho_{0,1}(z) = \left(\frac{1}{2\pi} \int \int_{\mathbb{C}} \left| \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)} \right| d\xi d\eta \right)^{-1}.$$

Since the smallest value of $\rho_{0,1}(z)$ on the circle $|z| = 1$ occurs at $z = -1$, we see that

$$\min_{|z|=1} \rho_{0,1}(z) = \left(\frac{1}{2\pi} \int \int_{\mathbb{C}} \left| \frac{1}{(\zeta-1)(\zeta)(\zeta+1)} \right| d\xi d\eta \right)^{-1}.$$

The infinitesimal form of the Poincaré metric on the punctured disk $\Delta_r^* = \{z \in \mathbb{C} \mid 0 < |z| < r\}$ of radius r is

$$\rho_r(z) = \frac{1}{|z| \left[\log r + \log \frac{1}{|z|} \right]}. \quad (6.9)$$

Note that $\rho_r(z)$ takes the constant value $\frac{1}{\log r}$ along $|z| = 1$. Thus, if we choose r so that $\log r$ is no less than

$$\frac{1}{2\pi} \int \int_{\mathbb{C}} \left| \frac{1}{(\zeta-1)(\zeta)(\zeta+1)} \right| d\xi d\eta \quad (6.10)$$

then

$$\rho_r(z) \leq \rho_{0,1}(z) \text{ for all } z \text{ with } |z| = 1. \quad (6.11)$$

Our next objective is to show that the same inequality

$$\rho_r(z) \leq \rho_{0,1}(z) \quad (6.12)$$

holds for all z with $|z| < \delta$ when δ is sufficiently small. To prove (6.12) we will need to assume that $\log r$ is no smaller than $4 + \log 4$. Since numerical calculation shows that $4 + \log 4$ is larger than (6.10), in the final result the number $4 + \log 4$ is the number we require for the result in (6.8). In [KL] Keen and Lakic obtain an improved lower bound by showing that inequality (6.8) is still true if $4 + \log 4$ is replaced by (6.10). This improvement is unimportant for our purposes.

In [A1, pages 17–18] Ahlfors shows that

$$\rho_{0,1}(z) \geq \frac{|\zeta'(z)|}{|\zeta(z)|} \frac{1}{4 + \log \frac{1}{|\zeta(z)|}} \quad (6.13)$$

for $|z| \leq 1$ and $|z| \leq |z - 1|$, where ζ maps the complement of $[1, +\infty]$ conformally onto the unit disk, origins corresponding to each other and symmetry with respect to the real axis being preserved. ζ satisfies

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{z\sqrt{1-z}}, \quad (6.14)$$

$$\zeta(z) = \frac{\sqrt{1-z} - 1}{\sqrt{1-z} + 1} = \frac{z}{(\sqrt{1-z} + 1)^2} \quad (6.15)$$

with $\operatorname{Re} \sqrt{1-z} > 0$, and

$$|\zeta(z)| \rightarrow \frac{|z|}{4} \quad (6.16)$$

as $z \rightarrow 0$.

We now show that there is a number $\delta > 0$ such that if $|z| < \delta$, then

$$\frac{|\zeta'|}{|\zeta|} \frac{1}{[4 + \log \frac{1}{|\zeta|}]} \geq \frac{1}{|z|[\log r + \frac{1}{|z|}]}.$$

From (6.14) this is equivalent to showing that

$$|\sqrt{1-z}|(4 + \log \frac{1}{|\zeta|}) \leq \log r + \log \frac{1}{|z|},$$

which is equivalent to

$$\begin{aligned} & |\sqrt{1-z}|(4 + \log 4) \\ & \leq \log r + \left\{ \left(\log \frac{1}{|z|} \right) \left((1 - |\sqrt{1-z}|) \left(\frac{\log \frac{1}{|\zeta|} - \log 4}{\log \frac{1}{|z|}} \right) \right) \right\}. \quad (6.17) \end{aligned}$$

From (6.16)

$$\left(\frac{\log \frac{1}{|\zeta|} - \log 4}{\log \frac{1}{|z|}} \right)$$

approaches 1 as $z \rightarrow 0$ and the expression in the curly brackets on the right-hand side of (6.17) approaches zero. Thus, in order to prove (6.12), it suffices to assume

$$\log r > 4 + \log 4.$$

We have so far established that $\rho_{0,1}(z) \geq \rho_r(z)$ on the unit circle

and on any circle $|z| = \delta$ for sufficiently small δ . To complete the proof of the lemma we observe that since both metrics $\rho_{0,1}(z)$ and $\rho_r(z)$ have constant curvatures equal to -1 , if we denote the Laplacian by

$$\Delta = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2,$$

then

$$-\rho_{0,1}^{-2}\Delta \log \rho_{0,1} = -1 \text{ and } -\rho_r^{-2}\Delta \log \rho_r = -1.$$

Therefore,

$$\Delta(\log \rho_{0,1} - \log \rho_r) = \rho_{0,1}^2 - \rho_r^2 \quad (6.18)$$

throughout the annulus $\{z : \delta \leq |z| \leq 1\}$. The minimum of $\rho_{0,1}/\rho_r$ in this annulus occurs either at a boundary point or an interior point. If it occurs at an interior point, then $\Delta \log(\rho_{0,1}/\rho_r) \geq 0$ at that point and so $\rho_{0,1}/\rho_r \geq 1$ at that point. But if it occurs on the boundary, we also have $\rho_{0,1}/\rho_r \geq 1$ at that point. So in either case

$$\rho_{0,1}/\rho_r \geq 1$$

at that point, and therefore

$$\rho_{0,1} \geq \rho_r$$

throughout the annulus provided $\log r$ is any number larger than $4 + \log 4$. But this implies the previous inequality is also true when $4 + \log 4 = \log r$, and this completes the proof of the lemma. \square

From (6.7) and this lemma we obtain

$$|V(z_2) - V(z_1)| \leq |z_2 - z_1| \left(\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| + 2 \log \frac{1}{|z_2 - z_1|} \right).$$

Therefore to prove the theorem we must show that for $\epsilon = \frac{C}{\log(1/\delta)}$

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| + 2 \log \frac{1}{|z_2 - z_1|} \leq (2 + \epsilon) \log \frac{1}{|z_2 - z_1|}.$$

This is equivalent to showing that

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| \leq \epsilon \log \frac{1}{|z_2 - z_1|}.$$

If $|z_2| < 1$, from (6.6) and Lemma 24, we have

$$\rho_{0,1}(z_2) \geq \frac{1}{|z_2|(\log r + \log \frac{1}{|z_2|})},$$

and

$$\frac{|V(z_2)|}{|z_2|} \leq 2 \log r + 2 \log \frac{1}{|z_2|}.$$

Hence

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| \leq 4 \log r.$$

If $1 \leq |z_2| \leq R$, then since $\left| \frac{V(z_2)}{z_2} \right| + 2 \log |z_2|$ is a continuous function, it is bounded by a number M_1 , so

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| \leq M_1 + 2 \log r.$$

The constant $C = M_1 + 2 \log r$ does not depend on δ and

$$\left| \frac{V(z_2)}{z_2} \right| + 2 \log r + 2 \log |z_2| \leq C$$

for any $|z_2| \leq R$. Thus, putting $\epsilon = C/\frac{1}{\log(1/\delta)}$, we obtain

$$|V(z_2) - V(z_1)| \leq |z_2 - z_1|(2 + \epsilon) \left(\log \frac{1}{|z_2 - z_1|} \right).$$

□

Applying the same argument at a variable value of c we obtain the following result.

Theorem 20 *Suppose $0 < r < 1$ and $R > 0$. If $|c| \leq r$, $|z_1(c)| \leq R$, $|z_2(c)| \leq R$ and $|z_2(c) - z_1(c)| < \delta$, then*

$$|V(z_2(c)) - V(z_1(c))| \leq \frac{2 + \epsilon}{1 - |c|^2} |z_2(c) - z_1(c)| \log \frac{1}{|z_2(c) - z_1(c)|}, \quad (6.19)$$

where $\epsilon \leq \frac{M}{\log(1/\delta)}$ and $\delta \geq |z_1(0) - z_2(0)|$. Moreover, there is a constant C such that

$$|z_2(c) - z_1(c)| \leq C \cdot |z_2 - z_1|^{\frac{1-|c|}{1+|c|}}.$$

Proof Equation (6.19) follows by the same calculations we have just completed. To prove the second inequality, put $s(c) = |z_2(c) - z_1(c)|$ and assume $0 < |c| < 1$. Then (6.19) yields

$$s'(c) \leq \frac{2 + \epsilon}{1 - |c|^2} s(c) \log \frac{1}{s(c)}.$$

So

$$-(\log \frac{1}{s(c)})' \leq \frac{2+\epsilon}{1-|c|^2} \log \frac{1}{s(c)}$$

and

$$-(\log(\log \frac{1}{s(c)}))' \leq \frac{2+\epsilon}{1-|c|^2}.$$

By integration,

$$-\log(\log \frac{1}{s(c)}) \Big|_0^c \leq -\frac{2+\epsilon}{2} \log \frac{1-|c|}{1+|c|} \Big|_0^c$$

and

$$\log \log(\frac{1}{s(c)}) - \log \log(\frac{1}{s(0)}) \geq \log \left(\frac{1-|c|}{1+|c|} \right)^{1+\frac{\epsilon}{2}}.$$

Since $\log x$ is increasing,

$$\frac{\log \frac{1}{s(c)}}{\log \frac{1}{s(0)}} \geq \left(\frac{1-|c|}{1+|c|} \right)^{1+\frac{\epsilon}{2}},$$

$$\log s(c) \leq \left(\frac{1-|c|}{1+|c|} \right)^{1+\frac{\epsilon}{2}} \log s(0)$$

and

$$s(c) \leq s(0)^{\left(\frac{1-|c|}{1+|c|}\right)^{1+\frac{\epsilon}{2}}}.$$

Putting $s = s(0)$ and $\alpha = \frac{1-|c|}{1+|c|}$, we wish to show that

$$s^{\alpha^{1+\epsilon}} \leq C s^\alpha \text{ or equivalently that } s^{(\alpha^{1+\epsilon}-\alpha)} \leq C. \quad (6.20)$$

This is equivalent to showing that

$$\alpha(\alpha^\epsilon - 1) \log s \leq \log C$$

or that

$$\alpha \left(\exp\left(\frac{M}{\log(1/s)} \log \alpha\right) - 1 \right) \log s \leq \log C.$$

Since $0 < \alpha < 1$ and since we may assume $s < e^{-1}$, by using the inequality $e^x - 1 \leq x e^{x_0}$ for $0 \leq x \leq x_0$, we see that it suffices to choose C so that

$$\alpha \frac{M}{\log(1/s)} \log(1/\alpha) e^{M \log \alpha} \log(1/s) = \alpha M \log(1/\alpha) e^{M \log \alpha} \leq \log C.$$

□

The idea for the proof of Theorem 20 is suggested but not worked out in [GK].

7 Kobayashi's metrics

Suppose \mathcal{N} is a connected complex manifold over a complex Banach space. Let $\mathcal{H} = \mathcal{H}(\Delta, \mathcal{N})$ be the space of all holomorphic maps from Δ into \mathcal{N} . For p and q in \mathcal{N} , let

$$d_1(p, q) = \log \frac{1+r}{1-r},$$

where r is the infimum of the nonnegative numbers s for which there exists $f \in \mathcal{H}$ such that $f(0) = p$ and $f(s) = q$. If no such $f \in \mathcal{H}$ exists, then $d_1(p, q) = \infty$.

Let

$$d_n(p, q) = \inf \sum_{i=1}^n d_1(p_{i-1}, p_i)$$

where the infimum is taken over all chains of points $p_0 = p, p_1, \dots, p_n = q$ in \mathcal{N} . Obviously, $d_{n+1} \leq d_n$ for all $n > 0$.

Definition 2 (Kobayashi's metric) *The Kobayashi pseudo-metric $d_K = d_{K, \mathcal{N}}$ is defined as*

$$d_K(p, q) = \lim_{n \rightarrow \infty} d_n(p, q), \quad p, q \in \mathcal{N}.$$

In general, it is possible that d_K is identically equal to 0, which is the case for example if $\mathcal{N} = \mathbb{C}$.

Another way to describe d_K is the following. Let the Poincaré metric on the unit disk Δ be given by

$$\rho_\Delta(z, w) = \log \frac{1 + \frac{|z-w|}{|1-\bar{z}w|}}{1 - \frac{|z-w|}{|1-\bar{z}w|}}, \quad z, w \in \Delta.$$

Then d_K is the largest pseudo-metric on \mathcal{N} such that

$$d_K(f(z), f(w)) \leq \rho_\Delta$$

for all z and $w \in \Delta$ and for all holomorphic maps f from Δ into \mathcal{N} . The following is a consequence of this property.

Proposition 11 *Suppose \mathcal{N} and \mathcal{N}' are two complex manifolds and $F : \mathcal{N} \rightarrow \mathcal{N}'$ is holomorphic. Then*

$$d_{K, \mathcal{N}'}(F(p), F(q)) \leq d_{K, \mathcal{N}}(p, q).$$

Lemma 25 *Suppose \mathcal{B} is a complex Banach space with norm $\|\cdot\|$. Let \mathcal{N} be the unit ball of \mathcal{B} and let d_K be the Kobayashi's metric on \mathcal{N} . Then*

$$d_K(0, \mathbf{v}) = \log \frac{1 + \|\mathbf{v}\|}{1 - \|\mathbf{v}\|} = 2 \tanh^{-1} \|\mathbf{v}\|, \quad \forall \mathbf{v} \in \mathcal{N}.$$

Proof Pick a point \mathbf{v} in \mathcal{N} . The linear function $f(c) = c\mathbf{v}/\|\mathbf{v}\|$ maps the unit disk Δ into the unit ball \mathcal{N} , and takes $\|\mathbf{v}\|$ into \mathbf{v} , and 0 into $\mathbf{0}$. Therefore

$$d_K(0, \mathbf{v}) \leq \rho_\Delta(0, \|\mathbf{v}\|),$$

where ρ_Δ is the Kobayashi's metric on Δ (it coincides with the Poincaré metric on Δ).

On the other hand, by the Hahn-Banach theorem, there exists a continuous linear function L on \mathcal{N} such that $L(\mathbf{v}) = \|\mathbf{v}\|$ and $\|L\| = 1$. Thus, L maps \mathcal{N} into the unit disk Δ , and so

$$d_K(0, \mathbf{v}) \geq \rho_\Delta(0, \|\mathbf{v}\|).$$

Therefore,

$$d_K(0, \mathbf{v}) = \rho_\Delta(0, \|\mathbf{v}\|) = \log \frac{1 + \|\mathbf{v}\|}{1 - \|\mathbf{v}\|} = 2 \tanh^{-1} \|\mathbf{v}\|.$$

□

8 Teichmüller's and Kobayashi's metrics on $T(R)$

Assume R is a Riemann surface conformal to Δ/Γ where Γ is a discontinuous, fixed point free group of hyperbolic isometries of Δ . Let $\mathcal{M} = \mathcal{M}(\Gamma)$ be the unit ball of the complex Banach space of all \mathcal{L}^∞ functions defined on Δ satisfying the Γ -invariance property:

$$\mu(\gamma(z)) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \tag{8.1}$$

for all z in Δ and all γ in Γ . An element $\mu \in \mathcal{M}$ is called a Beltrami coefficient on R . Points of the Teichmüller space $T = T(R)$ are represented by equivalence classes of Beltrami coefficients $\mu \in \mathcal{M}$. Two Beltrami coefficients $\mu, \nu \in \mathcal{M}$ are in the same Teichmüller equivalence class if the quasiconformal self maps f^μ and f^ν which preserve Δ and which are normalized to fix $0, i$ and -1 on the boundary of the unit disk coincide at all boundary points of the unit disk.

Definition 3 (Teichmüller’s metric) For two elements $[\mu]$ and $[\nu]$ of $T(R)$, Teichmüller’s metric is equal to

$$d_T([\mu], [\nu]) = \inf \log K(f^\mu \circ (f^\nu)^{-1}),$$

where the infimum is over all μ and ν in the equivalence classes $[\mu]$ and $[\nu]$, respectively. In particular,

$$d_T(0, [\mu]) = \log \frac{1 + k_0}{1 - k_0},$$

where k_0 is the minimal value of $\|\mu\|_\infty$, where μ ranges over the Teichmüller class $[\mu]$.

Lemma 26 Let d_K and d_T be Kobayashi’s and Teichmüller’s metrics of $T(R)$. Then $d_K \leq d_T$.

Proof Let a Beltrami coefficient μ satisfying (8.1) be extremal in its class and $\|\mu\|_\infty = k$. This is possible because by normal families argument every class possesses at least one extremal representative. By the definition of Teichmüller’s metric

$$d_T(0, [\mu]) = \log \frac{1 + k}{1 - k}.$$

For such a μ , let $g(c) = [c\mu/k]$. Then $g(c)$ is a holomorphic function of c for $|c| < 1$ with values in the Teichmüller space $T(R)$, $g(0) = 0$ and $g(k) = [\mu]$. Hence

$$d_K(0, [\mu]) \leq d_1(0, [\mu]) \leq d_T(0, [\mu]).$$

Now the right translation mapping $\alpha([f^\mu]) = [f^\mu \circ (f^\nu)^{-1}]$ is biholomorphic, so it is an isometry in Kobayashi’s metric. We also know that it is an isometry in Teichmüller’s metric. Therefore, the inequality

$$d_K([\nu], [\mu]) \leq d_1([\nu], [\mu]) \leq d_T([\nu], [\mu])$$

holds for an arbitrary pair of points $[\mu]$ and $[\nu]$ in the Teichmüller space $T(R)$. \square

In order to describe holomorphic maps into $T(R)$ we will use the Bers' embedding by which $T(R)$ is realized as a bounded domain in the Banach space $\mathcal{B}(R)$ of equivariant cusp forms. Here $\mathcal{B}(R)$ consists of the functions φ holomorphic in Δ^c for which

$$\sup_{z \in \Delta^c} \{(|z|^2 - 1)^2 |\varphi(z)|\} < \infty$$

and for which

$$\varphi(\gamma(z))(\gamma'(z))^2 = \varphi(z) \text{ for all } \gamma \in \Gamma.$$

We assume Γ is a Fuchsian covering group such that Δ/Γ is conformal to R . For any Beltrami differential μ supported on Δ , we let w^μ be the quasiconformal self-mapping of $\bar{\mathbb{C}}$ which fixes $1, i$ and -1 and which has Beltrami coefficient μ in Δ and Beltrami coefficient identically equal to zero in Δ^c . Let w^μ restricted to Δ^c be equal to the Riemann mapping g^μ . Then g^μ has the following properties:

- a) g^μ fixes the points $1, i$ and -1 ,
- b) $g^\mu(\partial\Delta)$ is a quasiconformal image of the circle $\partial\Delta$,
- c) g^μ is univalent and holomorphic in Δ^c ,
- d) $g^\mu \circ \gamma \circ (g^\mu)^{-1}$ is equal to a Möbius transformation $\tilde{\gamma}$, for all γ in Γ , and
- e) g^μ determines and is determined uniquely by the corresponding point in $T(R)$.

The Bers' embedding maps the Teichmüller equivalence class of μ to the Schwarzian derivative of g^μ where the Schwarzian derivative of a C^3 function g is defined by

$$S(g) = \left(\frac{g''}{g'}\right)' + \frac{1}{2} \left(\frac{g''}{g'}\right)^2.$$

In the next section we use this realization of the complex structures to prove that $d_T \leq d_K$.

9 The lifting problem

Let Φ be the natural map from the space \mathcal{M} of Beltrami differentials on R onto $T(R)$ and let f be a holomorphic map from the unit disk into $T(R)$ with $f(0)$ equal to the base point of $T(R)$. The lifting problem is

the problem of finding a holomorphic map \tilde{f} from Δ into \mathcal{M} , such that $\tilde{f}(0) = 0$ and $\Phi \circ \tilde{f} = f$.

In this section we prove the theorem of Earle, Kra and Krushkal [EKK] which says that the lifting problem always has a solution. We follow their technique which relies on proving an equivariant version of Slodkowski's extension theorem and then going on to show that the positive solution to the lifting problem implies $d_T \leq d_K$ for every Riemann surface that has a nontrivial Teichmüller space with complex structure.

Theorem 21 (An equivariant version of Slodkowski's extension theorem) *Let $h(c, z)$ be a holomorphic motion of $\Delta^c = \overline{\mathbb{C}} \setminus \Delta$ parametrized by Δ and with base point 0 and let Γ be a torsion-free group of Möbius transformations mapping Δ^c onto itself. Suppose for each $\gamma \in \Gamma$ and $c \in \Delta$ there is a Möbius transformation $\tilde{\gamma}_c$ such that*

$$h(c, \gamma(z)) = \tilde{\gamma}_c(h(c, z)), \quad \forall z \in \Delta^c.$$

Then $h(c, z)$ can be extended to a holomorphic motion $H(c, z)$ of $\overline{\mathbb{C}}$ parametrized by Δ and with base point 0 in such a way that

$$H(c, \gamma(z)) = \tilde{\gamma}_c(H(c, z))$$

holds for $\gamma \in \Gamma$, $c \in \Delta$ and $z \in \overline{\mathbb{C}}$.

Proof Observe that $\tilde{\gamma}_c$ is uniquely determined for all $c \in \Delta$ because Δ^c contains more than two points. To extend $h(c, z)$ to Δ , start with a point $w \in \Delta$. By Theorem 12, the motion $h(c, z)$ can be extended to a holomorphic motion (still denote it as $h(c, z)$) of the closed set $\Delta^c \cup \{w\}$. Furthermore, we may extend it to the orbit of w using the Γ -invariant property:

$$h(c, \gamma(w)) = \tilde{\gamma}_c(h(c, w)),$$

for all $\gamma \in \Gamma$. Since every $\gamma \in \Gamma$ is fixed point free on Δ , the motion $h(c, z)$ is well defined and satisfies the Γ -invariant property for all $c \in \Delta$ and all z in the set

$$E = \{\gamma(w) : \gamma \in \Gamma\} \cup (\overline{\mathbb{C}} \setminus \Delta).$$

So we only need to show that $h(c, z)$ is a holomorphic motion of E . Observe first that $h(0, z) = z$ since $\tilde{\gamma}_0 = \gamma$ for all $\gamma \in \Gamma$. To show $h(c, z)$ is injective for all fixed $c \in \Delta$, suppose $h(c, z_1) = h(c, z_2)$ for some $c \in \Delta$.

Since $h(c, z)$ is injective on $\Delta^c \cup \{w\}$, we may assume that $z_1 = g(w)$ for some $g \in \Gamma$. By the Γ -invariant property,

$$h(c, w) = (\tilde{g}_c)^{-1}(h(c, z_1)).$$

Thus,

$$h(c, w) = (\tilde{g}_c)^{-1}(h(c, z_2)) = h(c, g^{-1}(z_2)),$$

and we conclude that z_2 belongs to the Γ -orbit of w . Let $z_2 = \beta(w)$ for some $\beta \in \Gamma$. Then

$$h(c, w) = \tilde{\gamma}_c(h(c, w)),$$

where $\gamma = g^{-1} \circ \beta$. Therefore $h(c, w)$ is a fixed point of $\tilde{\gamma}_c$. On the other hand, since γ is a hyperbolic Möbius transformation, $\tilde{\gamma}_c$ is also hyperbolic, so unless $\tilde{\gamma}_c$ is the identity, it can only fix points on the set $h(c, \partial\Delta)$. Hence γ is the identity map and $z_1 = z_2$.

Finally, we will show that $l : c \rightarrow h(c, z)$ is holomorphic for any fixed $z \in E$. We may assume $z = g(w)$, $g \in \Gamma \setminus \{\text{identity}\}$. Then $l(c) = h(c, g(w)) = \tilde{g}_c(h(c, w))$. Since $c \rightarrow h(c, w)$ is holomorphic and \tilde{g}_c is a Möbius transformation, it is enough to prove the map $k : c \rightarrow \tilde{g}_c(\zeta)$ is holomorphic for any fixed ζ . Applying the Γ -invariant property to the three points $0, 1, \infty$, we obtain

$$\tilde{g}_c(0) = h(c, g(0)),$$

$$\tilde{g}_c(1) = h(c, g(1)),$$

$$\tilde{g}_c(\infty) = h(c, g(\infty)).$$

The right-hand sides of these three equations are holomorphic, so the maps $c \mapsto \tilde{g}_c(0)$, $c \mapsto \tilde{g}_c(1)$ and $c \mapsto \tilde{g}_c(\infty)$ are holomorphic. Since \tilde{g}_c is a Möbius transformation, $k : c \rightarrow \tilde{g}_c(\zeta)$ is holomorphic.

Therefore, we have extended $h(c, z)$ to a holomorphic motion of

$$\Delta^c \cup \{\text{the } \Gamma \text{ orbit of } z\}.$$

By repeating this extension process to a countable set of points whose Γ orbits are dense in Δ , we obtain the extension $H(c, z)$ of $h(c, z)$ with the property that

$$H(c, \gamma(z)) = \tilde{\gamma}_c(H(c, z))$$

for all $\gamma \in \Gamma$, $c \in \Delta$ and $z \in \overline{\mathbb{C}}$. □

This equivariant version of Slodkowski's extension theorem leads almost immediately to the following lifting theorem.

Theorem 22 (The lifting theorem) *If $f : \Delta \rightarrow T(R)$ is holomorphic, then there exists a holomorphic map $\tilde{f} : \Delta \rightarrow \mathcal{M}$ such that*

$$\Phi \circ \tilde{f} = f.$$

If $\mu_0 \in \mathcal{M}$ and $\Phi(\mu_0) = f(0)$, we can choose \tilde{f} such that $\tilde{f}(0) = \mu_0$.

Proof By using the translation mapping α of the Teichmüller space given by

$$\alpha([w^\mu]) = [w^\mu \circ (w^\nu)^{-1}],$$

we may assume $f(0) = 0$. For each $c \in \Delta$, let $g(c, \cdot)$ be a meromorphic function whose Schwarzian derivative is $f(c)$. Then on $\overline{\mathbb{C}} \setminus \Delta$ the map $g(c, \cdot)$ is injective, and we can specify $g(c, \cdot)$ uniquely by requiring that it fix 1, i and -1 . Thus $g(0, z) = z$. It is easy to verify that

$$g(c, z) : \Delta \times (\overline{\mathbb{C}} \setminus \Delta) \rightarrow \overline{\mathbb{C}}$$

is a holomorphic motion. For every $\gamma \in \Gamma$ and $c \in \Delta$, there exists a Möbius transformation $\tilde{\gamma}_c$ such that

$$g(c, \gamma(z)) = \tilde{\gamma}_c(g(c, z)).$$

Using the equivalent version of Slodkowski's extension theorem, we extend g to a Γ -invariant holomorphic motion (still denote it as g) of $\overline{\mathbb{C}}$. For each $c \in \Delta$, let $\tilde{f}(c)$ be the complex dilatation

$$\tilde{f}(c) = \frac{g_{\bar{z}}}{g_z}.$$

Then the Γ -invariant property of g implies that $\tilde{f}(c) \in \mathcal{M}$. From Theorem 12 in Section 1, we know that $\tilde{f}(c)$ is a holomorphic function of c . By the definition of the Bers embedding, $\Phi(\tilde{f}(c))$ is the Schwarzian derivative of g . So $\Phi(\tilde{f}(c)) = g(c)$. \square

Now we will use the lifting theorem to show that the Teichmüller metric and Kobayashi's metric of $T(R)$ coincide.

Lemma 27 *Suppose \mathcal{M} is the unit ball in the space of essentially*

bounded Beltrami differentials on a Riemann surface R . Let d_K be the Kobayashi's metric on \mathcal{M} . Then

$$d_K(\mu, \nu) = 2 \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \bar{\nu}\mu} \right\|_{\infty}$$

for all μ and ν in \mathcal{M} .

Proof From Lemma 25, for any $\nu \in \mathcal{M}$,

$$d_K(0, \nu) = 2 \tanh^{-1} \|\nu\|_{\infty}.$$

Observe the function defined by

$$\lambda \mapsto \frac{\nu - \lambda}{1 - \bar{\nu}\lambda}$$

is a biholomorphic self map of \mathcal{M} . Therefore

$$d_K(\mu, \nu) = 2 \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \bar{\nu}\mu} \right\|_{\infty}.$$

□

Theorem 23 ([R], [G1], [G2]) *The Teichmüller's and Kobayashi's metrics of $T(R)$ coincide.*

Proof In Lemma 26 we already showed that $d_K \leq d_T$, so we only need to prove $d_K \geq d_T$. Choose a holomorphic map $f : \Delta \rightarrow T(R)$ so that $f(0) = 0$ and $f(c) = [\mu]$ for some $c \in \Delta$. Then the lifting theorem implies there exists a holomorphic map $\tilde{f} : \Delta \rightarrow \mathcal{M}$ so that

$$\Phi(\tilde{f}(c)) = f(c) = [\mu].$$

So

$$d_K(0, \tilde{f}(c)) \leq \rho_{\Delta}(0, c).$$

By Lemma 27 and definition of Teichmüller metric,

$$d_T(0, [\mu]) \leq d_K(0, \tilde{f}(c)).$$

Therefore,

$$d_T(0, [\mu]) \leq \rho_{\Delta}(0, c).$$

Taking the infimum over all such f , we have

$$d_T(0, [\mu]) \leq d_K(0, [\mu]).$$

Hence $d_T \leq d_K$. \square

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