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Quasisymmetric property for conjugacies between Anosov diffeomorphisms of the two-torus

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract We prove that the restrictions of the conjugacy between two Anosov diffeomorphisms of the two-torus to the stable and unstable manifolds are quasisymmetric homeomorphisms.

Keywords quasisymmetry, Anosov diffeomorphisms, conjugacy, Hölder condition, Markov partition

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1 Introduction

The study of the quasisymmetric property for a conjugacy between two one-dimensional maps has led to solutions of many important problems in one-dimensional dynamical systems and in complex dynamical systems. We give a partial list of references in this direction [5, 8–11, 15, 18, 19, 21, 23].

A quasisymmetric homeomorphism could be very singular, i.e., it could map a set of positive Lebesgue measure to a set of zero Lebesgue measure or vice versa. Generally speaking, as a conjugacy between certain two one-dimensional dynamical systems, a homeomorphism must be either totally singular or smooth (see, for examples, [9, 12–16]). However, a quasisymmetric homeomorphism has many important geometric properties. For example, a quasisymmetric homeomorphism of the real line can be extended to the whole complex plan as a quasiconformal homeomorphism [2].

We would like to push the study of the quasisymmetric property into higher dimensional dynamical systems but with either one-dimensional stable manifolds or one-dimensional unstable manifolds. In [3], Cawley did a similar study and more emphasized on the geometric structure of the space of Anosov diffeomorphisms of the two-torus parametrized by potentials on stable and unstable manifolds. In this paper, we study the quasisymmetric property of a conjugacy between two Anosov diffeomorphisms of the two-torus when the conjugacy is restricted to stable and unstable manifolds. The main technique we use in this paper is the Markov partition [22] which has been used in the study of the quasisymmetric property of one-dimensional dynamical systems [9].

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2 Notations and the main theorem

Let $\mathbb{T}^2$ be the two-torus. Let f be an Anosov diffeomorphism of the two-torus. Suppose that f is at least $C^{1+\alpha}$ for some $0 < \alpha \leqslant 1$. By the definition, f is an Anosov diffeomorphism if there is an invariant splitting of the tangent bundle $T\mathbb{T}^2 = E^s \oplus E^u$, where the subbundle E^s is contracted by f, and the subbundle E^u is expanded by f. That is, by considering the Lebesgue metric $\|\cdot\|$ on $\mathbb{T}^2$, there are two constants $0 < \mu < 1$ and $C_0 > 0$ such that for all $n \geqslant 0$,

$$\|Df^n v\| \leqslant C_0 \mu^n \|v\|, \quad \forall\, v \in E^s,$$

and

$$\|Df^{-n} v\| \leqslant C_0 \mu^n \|v\|, \quad \forall\, v \in E^u.$$

The only 2-dimensional smooth manifold that supports an Anosov diffeomorphism is the two-torus [4] (see also [20]).

The stable manifold theorem [6] says that for an Anosov diffeomorphism f, $\mathbb{T}^2$ can be foliated by two transversal $C^{1+\alpha}$ submanifolds W^s and W^u such that $TW^s = E^s$ and $TW^u = E^u$. Here W^s and W^u are called the stable and unstable manifolds for f. For each x in $\mathbb{T}^2$, the stable manifold $W^s(x)$ and unstable manifold $W^u(x)$ passing x are

$$W^s(x) = \{y \in \mathbb{T}^2 \mid d(f^n(x), f^n(y)) \to 0,\ n \to \infty\}$$

and

$$W^u(x) = \{y \in \mathbb{T}^2 \mid d(f^{-n}(x), f^{-n}(y)) \to 0,\ n \to \infty\}.$$

Either $W^s(x)$ or $W^u(x)$ is a connecting $C^{1+\alpha}$ immersed submanifold.

Suppose that f and g are two Anosov diffeomorphisms of the two-torus. We say that f and g are topologically conjugate if there is a homeomorphism h of the two-torus such that

$$h \circ f = g \circ h.$$

Franks [4] and Manning [20] showed that every Anosov diffeomorphism f of the two-torus is topologically conjugate to a linear example; i.e., to an automorphism defined by a hyperbolic element A of $\mathrm{GL}(2, \mathbb{Z})$ whose determinant has absolute value one. Thus every Anosov diffeomorphism f of the two-torus has a fixed point, which we always take it as 0. It is known that the conjugacy h between any two Anosov diffeomorphisms is Hölder continuous (This will also be a corollary of our main theorem in this paper).

There is another important geometric concept for homeomorphisms of the real line, called *quasisymmetry* in complex analysis. A homeomorphism H of the real line $\mathbb{R}$ is called quasisymmetric if there is a constant $M \geqslant 1$ such that

$$\frac{1}{M} \leqslant \left| \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \right| \leqslant M, \quad \forall\, x \in \mathbb{R},\ \forall\, 0 < t \leqslant 1.$$

Suppose that $W_f^s(0)$ and $W_f^u(0)$, $W_g^s(0)$ and $W_g^u(0)$ are the stable and unstable manifolds for f and g. Since they are all connecting $C^{1+\alpha}$ submanifolds of the two-torus, we have $C^{1+\alpha}$ embeddings $\rho_{s,f}$, $\rho_{u,f}$, $\rho_{s,g}$, and $\rho_{u,g}$ from $\mathbb{R}$ onto $W_f^s(0)$, $W_f^u(0)$, $W_g^s(0)$, and $W_g^u(0)$, respectively. We assume that $\rho_{s,f}$, $\rho_{u,f}$, $\rho_{s,g}$, and $\rho_{u,g}$ preserve the arc-length.

For the conjugacy h from f to g, define

$$H_s = \rho_{s,g}^{-1} \circ h \circ \rho_{s,f} \quad \text{and} \quad H_u = \rho_{u,g}^{-1} \circ h \circ \rho_{u,f}.$$

Then they are two homeomorphisms of the real lines. We say that $h|W_f^s(0)$ and $h|W_f^u(0)$ are quasisymmetric if H_s and H_u are quasisymmetric. We will prove the following

Theorem 1. *Suppose that f and g are two conjugated Anosov diffeomorphisms of the two-torus and h is a conjugacy between f and g, i.e., $h \circ f = g \circ h$. Then*

$$h|W_f^s(0) : W_f^s(0) \to W_g^s(0) \quad \textit{and} \quad h|W_f^u(0) : W_f^u(0) \to W_g^u(0)$$

are both quasisymmetric homeomorphisms.

It is known that a quasisymmetric homeomorphism of the real line is Hölder continuous [2]. So the Hölder continuity property of h, which is a known result for a long time among experts, is a corollary of the above theorem.

Corollary 1. *A conjugacy h between Anosov diffeomorphisms f and g of the two-torus is Hölder continuous.*

The proof of Theorem 1 is given in Section 8 and the proof of Corollary 1 is given in Section 9.

3 Markov partitions

Suppose that f is an Anosov diffeomorphism of $\mathbb{T}^2$. Then f has a local product structure, i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) \leqslant \delta$, $W^s_\epsilon(x) \cap W^y_\epsilon(y)$ contains exactly one point, denoted by $[x, y]$, where $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ are the local stable and unstable manifold at x given by

$$W^s_\epsilon(x) = \{y \in \mathbb{T}^2 \mid d(f^n(x), f^n(y)) \leqslant \epsilon,\ \forall\, n \geqslant 0\},$$

and

$$W^u_\epsilon(x) = \{y \in \mathbb{T}^2 \mid d(f^{-n}(x), f^{-n}(y)) \leqslant \epsilon,\ \forall\, n \geqslant 0\}.$$

A set R whose diameter is less than δ is called a *rectangle* if $x, y \in R$ implies $[x, y] \in R$. A rectangle R is proper if it is the closure of its interior. It is easy to check that if R is a rectangle, so is $f(R)$, and if R and S are rectangles, so is $R \cap S$, provided the diameters of the rectangles involved are all small.

For a rectangle R and a point $x \in R$, we denote $W^s(x, R) = W^s_\epsilon(x) \cap R$ and $W^u(x, R) = W^u_\epsilon(x) \cap R$. Note that if R is connected, then both $W^s(x, R)$ and $W^u(x, R)$ are connected curves.

A Markov partition for f is a set $\mathcal{R} = \{R_1, \ldots, R_n\}$ of proper connected rectangles satisfying:

(1) $\mathbb{T}^2 = \bigcup_{i=1}^n R_i$;

(2) $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $1 \leqslant i \neq j \leqslant n$;

(3) $fW^s(x, R_i) \subset W^s(f(x), R_j)$ if $x \in R_i$ and $f(x) \in R_j$;

(4) $fW^u(x, R_i) \supset W^u(f(x), R_j)$ if $x \in R_i$ and $f(x) \in R_j$.

Sinai proved that any Anosov diffeomorphism has a Markov partition of arbitrarily small diameter [22]. Since we only consider Anosov diffeomorphisms of the two-torus and since every such an Anosov diffeomorphism is topologically conjugate to a linear one, we can construct a canonical Markov partition for every f as follows. Note that diameters of rectangles in this canonical Markov partition may not be small.

Suppose that A is a hyperbolic automorphism of $\mathbb{T}^2$ conjugating to f. We first construct a canonical Markov partition for A [1]. Note that A can be defined by a hyperbolic matrix whose absolute value of the determinant is 1. So the matrix has an eigenvalue whose absolute value is greater than 1 (called the unstable eigenvalue) and an eigenvalue whose absolute value is less than 1 (called the stable eigenvalue).

Suppose that E^s and E^u are the stable and unstable eigenspaces of the matrix respectively. Then they are two transversal lines passing through the origin of $\mathbb{R}^2$. Suppose that the unit square $[0, 1) \times [0, 1)$ is a copy of $\mathbb{T}^2$ on the plane. Project into the $\mathbb{T}^2$ a segment in E^s through the origin, and a segment in E^u through the origin. Extend these segments until they cut the $\mathbb{T}^2$ into parallelograms. The set of these parallelograms is our canonical Markov partition $\mathcal{R}_A$ for A. The reader may refer to [17, pp. 84–86] for more details and some pictures of a canonical Markov partition. Let h_A be the conjugacy from A to f, i.e., $h_A \circ A = f \circ h_A$. Then $\mathcal{R}_f = h_A(\mathcal{R}_A)$ is our canonical Markov partition for f.

4 Nested sequence of partitions on $W^s(0)$ and $W^u(0)$

For a canonical Markov partition $\mathcal{R} = \mathcal{R}_f = \{R_1, \ldots, R_n\}$ for f, we define

$$\kappa^s_0 = \{W^s(x, R_i) \mid x \in W^s(0), 1 \leqslant i \leqslant n\}$$

and

$$\kappa_0^u = \{W^u(x, R_i) \mid x \in W^u(0), 1 \leqslant i \leqslant n\}.$$

So κ_0^s is a partition of $W^s(0)$ into countably many segments $W^s(x, R_i)$, $x \in W^s(0)$, $R_i \in \mathcal{R}$ and κ_0^u is a partition of $W^u(0)$ into countably many segments $W^u(x, R_i)$, $x \in W^u(0)$, $R_i \in \mathcal{R}$. Then we define $\kappa_n^s = f^n \kappa_0^s$ and $\kappa_n^u = f^{-n}\kappa_0^u$ for any $n \geqslant 1$. That is, κ_n^s consists of all segments l^s in $W^s(0)$ such that $f^{-n}(l^s) \in \kappa_0^s$ and κ_n^u consists of all segments l^u in $W^u(0)$ such that $f^n(l^u) \in \kappa_0^u$. By the conditions (3) and (4) we know that each element of κ_n^s or κ_n^u is a union of some elements of κ_{n+1}^s or κ_{n+1}^u respectively.

5 Holonomy map

For any two segments l^s and $\tilde{l}^s$ of κ_0^s in a same rectangle $R \in \mathcal{R}$, a holonomy map $\theta^s(x) : l^s \to \tilde{l}^s$ is defined by sliding along the unstable curves, i.e., for any $x \in l^s$, $\theta^s(x) = [z, x]$, the only point contained in the intersection $W^s(z) \cap W^u(x)$, where z is any point in $\tilde{l}^s$. Similarly, for any two segments l^u and $\tilde{l}^u$ of κ_0^u in a same rectangle $R \in \mathcal{R}$, a holonomy map $\theta^u(y) : l^u \to \tilde{l}^u$ is defined by sliding along the stable curves, i.e., for any $y \in l^s$, $\theta^u(y) = [y, z]$, the only point contained in the intersection $W^s(y) \cap W^u(z)$, where z is any point in $\tilde{l}^u$. The proof of the following lemma can be found in [6, 17] (also, refer to [7, Proposition 3.2]), using the facts that both stable and unstable foliations are of codimension one.

Lemma 1. *All holonomies are Lipschitz continuous with a uniform Lipschitz constant. More precisely, there is a constant $C_1 > 0$ such that for any two segments l^s and $\tilde{l}^s$ of κ_0^s in a same rectangle $R \in \mathcal{R}$,*

$$d(\theta^s(x), \theta^s(x')) \leqslant C_1 d(x, x'), \quad \forall\, x,\ x' \in l^s,$$

and for any two segments l^u and $\tilde{l}^u$ of κ_0^s in a same rectangle $R \in \mathcal{R}$,

$$d(\theta^u(y), \theta^u(y')) \leqslant C_1 d(y, y'), \quad \forall\, y,\ y' \in l^u.$$

This lemma implies the following

Lemma 2. *There is a constant $C_1 > 1$ such that*

$$\frac{1}{C_1} \leqslant \frac{|l^s|}{|m^s|}, \frac{|l^u|}{|m^u|} \leqslant C_1$$

for all $l^s, m^s \in \kappa_0^s$ and $l^u, m^u \in \kappa_0^u$, where $|\cdot|$ means the length of the segment.

Remark 1. Following the method used in one-dimensional dynamical systems [9], Cawley [3] studied the quasisymmetric property of holonomies.

6 Distortions

For an Anosov diffeomorphism, we have that

Lemma 3 (Distortion)**.** *For any $\epsilon > 0$, there is a constant $C_2 = C_2(\epsilon) > 0$ such that for any $x, y \in W^s(0)$ with $d^s(x, y) \leqslant \epsilon$ and $n > 0$,*

$$\frac{1}{C_2} \leqslant \frac{\|Df^n(y)|_{E_y^s}\|}{\|Df^n(x)|_{E_x^s}\|} \leqslant C_2,$$

and for any $x, y \in W^u(0)$ with $d^u(x, y) \leqslant \epsilon$ and $n > 0$,

$$\frac{1}{C_2} \leqslant \frac{\|Df^{-n}(y)|_{E_y^u}\|}{\|Df^{-n}(x)|_{E_x^u}\|} \leqslant C_2,$$

where d^s and d^u are the distances along $W^s(0)$ and $W^u(0)$ respectively.

The proof of this lemma is the same as the proof of the naive distortion lemma in one-dimensional dynamical systems [9, Chapter 1] and can be found in many books for hyperbolic dynamical systems, see, for example, [17].

7 Bounded nearby geometry

Definition 1. *The nested sequences of partitions* $\kappa^s = \{\kappa_n^s\}$ *or* $\kappa^u = \{\kappa_n^u\}$ *are said to have bounded nearby geometry if there is a constant* $C > 0$ *such that for any two adjacent segments* $l^s, m^s \in \kappa_n^s$ *or* $l^u, m^u \in \kappa_n^u$, $n \geqslant 0$,

$$\frac{1}{C} \leqslant \frac{|l^s|}{|m^s|} \leqslant C \quad \textit{or} \quad \frac{1}{C} \leqslant \frac{|l^u|}{|m^u|} \leqslant C,$$

respectively.

Theorem 2. *Suppose that* f *is a* $C^{1+\alpha}$ *Anosov diffeomorphism for some* $\alpha \in (0,1]$. *Then the nested sequences of partitions* κ^s *and* κ^u *have the bounded nearby geometry.*

Proof. By Lemma 2, there is a constant $C_1 > 1$ such that

$$\frac{1}{C_1} \leqslant \frac{|l^s|}{|m^s|}, \frac{|l^u|}{|m^u|} \leqslant C_1$$

for any two adjacent segments $l^s, m^s \in \kappa_0^s$ or $l^u, m^u \in \kappa_0^u$.

For any $n \geqslant 1$ and for any two adjacent segments l^s and m^s in κ_n^s or l^u and m^u in κ_n^u, $f^{-n}(l^s)$ and $f^{-n}(m^s)$ or $f^n(l^u)$ and $f^n(m^u)$ are two adjacent segments in κ_0^s or κ_0^u respectively. Then we apply the distortion lemma, Lemma 3, to get the result. □

Lemma 4. *For any* $c > 0$, *there exists* $k = k(c) > 0$ *such that for any* $n > 0$, $l^s \in \kappa_n^s$ *and* $m^s \in \kappa_{n+k}^s$ *with* $m^s \subset l^s$,

$$|m^s| \leqslant c|l^s|,$$

and for any $l^u \in \kappa_n^u$ *and* $m^u \in \kappa_{n+k}^u$ *with* $m^u \subset l^u$,

$$|m^u| \leqslant c|l^u|.$$

Proof. By the above theorem we know that $\{|\tilde{l}^s| \mid \tilde{l}^s \in \kappa_0^s\}$ and $\{|\tilde{l}^u| \mid \tilde{l}^u \in \kappa_0^u\}$ are bounded above and below. Since f is uniformly contracting along the stable direction and f^{-1} is uniformly contracting along the unstable direction, we can take $k > 0$ such that for any $\tilde{l}^s \in \kappa_0^s$ and $\tilde{m}^s \in \kappa_k^s$ with $\tilde{m}^s \subset \tilde{l}^s$,

$$|\tilde{m}^s| \leqslant cC_2^{-1}|\tilde{l}^s|,$$

and for any $\tilde{l}^u \in \kappa_0^u$ and $\tilde{m}^u \in \kappa_k^u$ with $\tilde{m}^u \subset \tilde{l}^u$,

$$|\tilde{m}^u| \leqslant cC_2^{-1}|\tilde{l}^u|.$$

Note that if $l^s \in \kappa_n^s$ and $m^s \in \kappa_{n+k}^s$ with $m^s \subset l^s$, then $f^{-n}(l^s) \in \kappa_0^s$ and $f^{-n}(m^s) \in \kappa_k^s$ with $f^{-n}(m^s) \subset f^{-n}(l^s)$. Hence, we have

$$|f^{-n}(m^s)| \leqslant cC_2^{-1}|f^{-n}(l^s)|,$$

and similarly if $l^u \in \kappa_n^u$ and $m^u \in \kappa_{n+k}^u$ with $m^u \subset l^u$, then

$$|f^n(m^u)| \leqslant cC_2^{-1}|f^n(l^u)|,$$

Now we apply Lemma 3 to get $|m^s| \leqslant c|l^s|$ and $|m^u| \leqslant c|l^u|$. □

8 Quasisymmetric property

Proof of Theorem 1. We adapt a technique used in [9, 10] to prove that the bounded nearby geometry implies the quasisymmetric property.

Suppose that $W^s_f(0)$ and $W^u_f(0)$, $W^s_g(0)$ and $W^u_g(0)$ are the stable and unstable manifolds for f and g at 0. Suppose that $\rho_{s,f}:\mathbb{R}\to W^s_f(0)$, $\rho_{u,f}:\mathbb{R}\to W^u_f(0)$, $\rho_{s,g}:\mathbb{R}\to W^s_g(0)$, $\rho_{u,g}:\mathbb{R}\to W^u_g(0)$ are embedding maps preserving arc length.

We prove that $H_u=\rho^{-1}_{u,g}\circ h\circ\rho_{u,f}:\mathbb{R}\to\mathbb{R}$ is a quasisymmetric homeomorphism. The proof that $H_s=\rho^{-1}_{s,g}\circ h\circ\rho_{s,f}:\mathbb{R}\to\mathbb{R}$ is a quasisymmetric homeomorphism is the exactly the same just by replacing u by s.

Let $\xi_{n,f}=\rho^{-1}_{u,f}\kappa^u_{n,f}$ and $\xi_{n,g}=\rho^{-1}_{u,g}\kappa^u_{n,g}$ for $n\geqslant 0$. Then they are two sequences of nested partitions on the real line and $H_u\xi_{n,f}=\xi_{n,g}$.

Let Ω be the set of all endpoints of intervals $I\in\xi_{n,f}$, $n=0,1,\dots$ It is a dense subset in $\mathbb{R}$.

For $x\in\Omega$, consider the interval $[x-t,x]$. There is a largest integer $n\geqslant 0$ such that there is an interval $I=[a,x]\in\xi_{n,f}$ satisfying $[x-t,x]\subseteq I$. Suppose $J=[b,x]\in\xi_{n+1,f}$. Then $J\subseteq[x-t,x]$. Let $J'=[x,c]\in\xi_{n+1,f}$. From Theorem 2 for f, there is a constant $C_1>0$ such that

$$C_1^{-1}\leqslant\frac{|J'|}{|J|}\leqslant C_1.$$

If $|J'|>t$, we have $|J'|\leqslant C_1|J|\leqslant C_1t$. Take $k=k(C_1^{-1})$ as in Lemma 2, and let $J'_k=[x,c_k]\in\xi_{n+k,f}$. Then $J'_k\subset J'$ and by the lemma we have $|J'_k|\leqslant C_1^{-1}|J'|\leqslant t$. This implies that $J'_k\subseteq[x,x+t]$. So we have

$$\frac{|H(J'_k)|}{|H(I)|}\leqslant\frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|}\leqslant\frac{|H(J')|}{|H(J)|},$$

where $H(I)\in\xi_{n,g}$, $H(J),H(J')\in\xi_{n+1,g}$, and $H(J'_k)\in\xi_{n+k+1,g}$. Now from Theorem 2 for g, we have a constant $C>0$ such that

$$C^{-1}\leqslant\frac{|H(J'_k)|}{|H(I)|}\leqslant\frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|}\leqslant\frac{|H(J')|}{|H(J)|}\leqslant C.$$

If $|J'|\leqslant t$, we have $|J'|\geqslant C_1^{-1}t$. We take the same $k=k(C_1^{-1})$ as above, and let $J'_{-k}=[x,c_{-k}]\in\xi_{n-k,f}$. Then $J'_{-k}\supset J'$ and by Lemma 4 we have $|J'|\leqslant C_1^{-1}|J'_{-k}|$ and therefore $|J'_{-k}|\geqslant C_1|J'|\geqslant t$. This implies that $J'_{-k}\supseteq[x,x+t]$. So we have

$$\frac{|H(J')|}{|H(I)|}\leqslant\frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|}\leqslant\frac{|H(J'_{-k})|}{|H(J)|},$$

where $H(I)\in\xi_{n,g}$, $H(J),H(J')\in\xi_{n+1,g}$, and $H(J'_{-k})\in\xi_{n-k+1,g}$. Now from Theorem 2 for g, we have a constant $C>0$, such that

$$C^{-1}\leqslant\frac{|H(J')|}{|H(I)|}\leqslant\frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|}\leqslant\frac{|H(J'_{-k})|}{|H(J)|}\leqslant C.$$

For any $x\in\mathbb{R}$, since Ω is dense in $[0,1]$, we have a sequence $x_n\in\Omega$ such that $x_n\to x$ as $n\to\infty$. For any $t>0$, we have that

$$C^{-1}\leqslant\frac{|H(x_n+t)-H(x_n)|}{|H(x_n)-H(x_n-t)|}\leqslant C.$$

Since H is continuous on $\mathbb{R}$, we get that

$$C^{-1}\leqslant\frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|}\leqslant C.$$

We now have proved the theorem. □

9 Hölder condition

Proof of Corollary 1. By Theorem 1, $h|W^s_f(0)$ and $h|W^u_f(0)$ are quasisymmetric, and therefore is Hölder continuous [2]. That is, there are constants $K>0$ and $\gamma\in(0,1]$ such that

$$d^s_g(h(x),h(y))\leqslant Kd^s_f(x,y)^\gamma,\qquad\forall\,x,y\in W^s_f(0)$$

and

$$d_g^u(h(x), h(y)) \leqslant K d_f^u(x, y)^\gamma, \qquad \forall\, x, y \in W_f^u(0),$$

where d_f^s and d_f^u are the distances along $W_f^s(0)$ and $W_f^u(0)$ respectively, and d_g^s and d_g^u are understood in a similar way.

Since the splitting $E_f^s(x) \oplus E_f^u(x)$ is continuous, there exist $\epsilon > 0$, $\beta_1 > \beta_2 > 0$ such that for any $x, y \in \mathbb{T}^2$ with $d(x, y) \leqslant \epsilon$, the angle between $E_f^s(x)$ and $E_f^u(y)$ is greater than β_1, while the angle between $E_f^s(x)$ and $E_f^s(y)$ and that between $E_f^u(x)$ and $E_f^u(y)$ are less than β_2. The latter implies that there is $L > 1$ such that

$$d_f^s(x, y) \leqslant L d(x, y), \quad \forall\, x, y \in W_f^s(0)$$

provided $d_f^s(x, y) \leqslant \epsilon$ and

$$d_f^u(x, y) \leqslant L d(x, y), \quad \forall\, x, y \in W_g^u(0)$$

provided $d_f^u(x, y) \leqslant \epsilon$.

For the ϵ chosen as above, take $\delta > 0$ as in the beginning in Section 3 for diffeomorphism f. Hence if $d(x, y) < \delta$, then $[x, y]$ is well defined and $d_f^s(x, [x, y])$, $d_f^u(y, [x, y]) \leqslant \epsilon$. By the choice of ϵ and β_1, we know that the angle between the line segment passing through x and $[x, y]$ and that passing through y and $[x, y]$ is greater than β_1. Hence, by the law of sine, we have

$$d(x, y) \geqslant \sin\beta_1 d(x, [x, y]), \qquad d(x, y) \geqslant \sin\beta_1 d(y, [x, y]).$$

Now applying the inequalities we get that for any $x \in W_f^s(0)$, $y \in W_f^u(0)$ with $d(x, y) < \delta$,

$$\begin{aligned} d(h(x), h(y)) &\leqslant d_g^s(h(x), [h(x), h(y)]) + d_g^u(h(y), [h(x), h(y)]) \\ &= d_g^s(h(x), h[x, y]) + d_g^u(h(y), h[x, y]) \\ &\leqslant K d_f^s(x, [x, y])^\gamma + K d_f^u(x, [x, y])^\gamma \\ &\leqslant K L^\gamma d(x, [x, y])^\gamma + K L^\gamma d(x, [x, y])^\gamma \\ &\leqslant 2 K L^\gamma (\sin\beta_1)^{-\gamma} d(x, y)^\gamma. \end{aligned}$$

In general, for $x, y \in \mathbb{T}^2$ with $d(x, y) < \delta$, we can choose $x_n \in W_f^s(0)$ with $x_n \to x$ and $y_n \in W_f^u(0)$ with $y_n \to y$. This is possible because $W_f^s(0)$ and $W_f^u(0)$ are dense in $\mathbb{T}^2$. Then we apply the above inequality for each pair x_n and y_n, and take limit. □

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