

Function Models for Teichmüller Spaces and Dual Geometric Gibbs Type Measure Theory for Circle Dynamics

Yunping Jiang

Department of Mathematics, Queens College of the City University of New York,
Flushing, NY 11367-1597

and

Department of Mathematics, Graduate School of the City University of New York,
365 Fifth Avenue, New York, NY 10016
e-mail: yunping.jiang@qc.cuny.edu

Abstract. Geometric models and Teichmüller structures have been introduced for the space of smooth expanding circle endomorphisms and for the space of uniformly symmetric circle endomorphisms. The latter one is the completion of the previous one under the Teichmüller metric. Moreover, the spaces of geometric models as well as the Teichmüller spaces can be described as the space of Hölder continuous scaling functions and the space of continuous scaling functions on the dual symbolic space. The characterizations of these scaling functions have been also investigated. The Gibbs measure theory and the dual Gibbs measure theory for smooth expanding circle dynamics have been viewed from the geometric point of view. However, for uniformly symmetric circle dynamics, an appropriate Gibbs measure theory is unavailable, but a dual Gibbs type measure theory has been developed for the uniformly symmetric case. This development extends the dual Gibbs measure theory for the smooth case from the geometric point of view. In this survey article, we give a review of these developments which combines ideas and techniques from dynamical systems, quasiconformal mapping theory, and Teichmüller theory. There is a measure-theoretical version which is called g -measure theory and which corresponds to the dual geometric Gibbs type measure theory. We briefly review it too.

Keywords. Scaling function, C^{1+} expanding circle endomorphism, uniformly symmetric circle endomorphism, Teichmüller space, symbolic dynamical system, dual symbolic dynamical system.

2000 Subject Classification: 58F23, 30C62

1. Introduction

Circle maps are basic elements in dynamical systems. The dynamics of a smooth expanding circle map presents many profound phenomena in mathematics and physics such as the structural stability theory, ergodic theory, probability theory, and, more recently, chaos theory. Teichmüller theory studies complex manifold structures of almost complex structures on Riemann surfaces. We have brought in some concepts and techniques in Teichmüller theory into the study of geometric structures of spaces

The research is partially supported by PSC-CUNY awards.

This article is prepared for the proceedings of the International Workshop on Teichmüller theory and moduli problems held at the Harish-Chandra Research Institute (HRI), Allahabad, India, January 5 to 14, 2006.

of circle expanding maps. In this survey article we review some development in this direction.

2. Circle endomorphisms

The theme here is an orientation-preserving covering map f from the unit circle $T = \{z \in \mathbb{C} \mid |z| = 1\}$ onto itself. Let d be the topological degree of f . We assume that $d \geq 2$. The universal cover of T is the real line \mathbb{R} with a covering map

$$\pi(x) = e^{2\pi xi} : \mathbb{R} \rightarrow T$$

Then f can be lifted to an orientation-preserving homeomorphism F of \mathbb{R} with the property that $F(x + 1) = F(x) + d$. Since a covering map of degree ≥ 2 has a fixed point, we assume that $z = 1$ is a fixed point of f . Then by assuming $F(0) = 0$, we set up a one-to-one correspondence between degree d circle covering maps f with $f(1) = 1$ and real line homeomorphisms F with $F(x + 1) = F(x) + d$. Thus we call in this paper f or the corresponding F a circle endomorphism. We use f^n (or F^n) to mean the composition of f (or F) by itself $n > 0$ times.

A circle endomorphism f is C^k for $k \geq 1$ if its k^{th} -derivative $F^{(k)}$ is continuous and $C^{k+\alpha}$ for some $0 < \alpha \leq 1$ if, furthermore, $F^{(k)}$ is α -Hölder continuous, that is,

$$\sup_{x \neq y \in \mathbb{R}} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x - y|^\alpha} = \sup_{x \neq y \in [0,1]} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x - y|^\alpha} < \infty.$$

A C^1 circle endomorphism f is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C\lambda^n, \quad n = 1, 2, \dots$$

3. Topological models

The topological classification of smooth expanding circle endomorphisms was first considered by Shub [24] in 1960's. He proved that two C^2 expanding circle endomorphisms f and g are topologically conjugate if and only if they have the same degree. Here f and g are topologically conjugate if there is a homeomorphism h of T such that

$$f \circ h = h \circ g.$$

By also considering the lift G of g and the lift H of h , we have an equivalent definition that f and g are topologically conjugate if there is a homeomorphism H of \mathbb{R} with $H(x + 1) = H(x) + 1$ such that

$$F \circ H = H \circ G \pmod{1}.$$

Shub's proof is an application of the contracting fixed point theorem in functional analysis. Consider the space \mathcal{C} of all continuous functions ϕ on \mathbb{R} with $\phi(x + 1) = \phi(x) + 1$ with the maximum norm

$$\|\phi\| = \sup_{x \in [0,1]} |\phi(x)|.$$

Then \mathcal{C} is a Banach space. Define an operator $\mathcal{L} = \mathcal{L}_{F,G}$ as

$$\mathcal{L}\phi(x) = F^{-1} \circ \phi \circ G : \mathcal{C} \rightarrow \mathcal{C}.$$

(Note that $F^{-1}(x + d) = F^{-1}(x) + 1$.) Without loss of generality, we assume that $C = 1$. Then one can check that

$$\|\mathcal{L}\phi - \mathcal{L}\psi\| \leq \frac{1}{\lambda} \|\phi - \psi\|.$$

So \mathcal{L} is a contracting functional from the Banach space \mathcal{C} into itself. And thus it has a unique fixed point H , that is,

$$F^{-1} \circ H \circ G = H$$

Similarly, $\mathcal{L}_{G,F}$ has a unique fixed point \tilde{H} , that is,

$$G^{-1} \circ \tilde{H} \circ F = \tilde{H}.$$

This implies that

$$H \circ \tilde{H} = id.$$

Therefore, H is a homeomorphism of \mathbb{R} such that

$$F \circ H = H \circ G \pmod{1}.$$

There is a more general theorem if we bring in the consideration of Markov partitions. Consider a partition of $[0, 1]$ by

$$I_i = I_{i,f} = F^{-1}([i, i + 1]), \quad 0 \leq i \leq d - 1.$$

It is a Markov partition in the following sense: If we consider a corresponding partition of T , which we still denote as $\{I_i\}_{i=0}^{d-1}$, then

- (1) the union of these intervals is T ;
- (2) all the intervals in the partition have pairwise disjoint interiors;
- (3) the restriction of f to the interior of every interval in the partition is injective.

We use

$$\eta_0 = \eta_{0,f} = \{I_i\}_{i=0}^{d-1}$$

to denote this initial Markov partition. We then have a sequence of Markov partitions

$$\eta_n = \eta_{n,f} = f^{-n}(\eta_0), \quad n = 0, 1, 2, \dots$$

on the unit circle T as well as the unit interval $[0, 1]$. We can label each interval in η_n as follows. Define

$$g_i(x) = F^{-1}(x + i) : [0, 1] \rightarrow I_i, \quad i = 0, 1, \dots, d - 1.$$

Each g_i is a homeomorphism. Given a word $w_n = i_0 \cdots i_k \cdots i_{n-1}$ of $\{0, \dots, d - 1\}$ of length $n \geq 1$, define

$$g_{w_n} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{n-1}}.$$

Let

$$I_{w_n} = I_{w_n, f} = g_{w_n}([0, 1]).$$

Then

$$\eta_n = \{I_{w_n} \mid w_n = i_0 \dots i_k \dots i_{n-1}, i_k \in \{0, \dots, d-1\}\}.$$

One can check that for a word $w = i_0 \dots i_{n-1} i_n \dots$ of infinite length, and with $w_n = i_0 \dots i_{n-1}$, then

$$\dots \subset I_{w_n} \subset I_{w_{n-1}} \subset \dots \subset I_{w_1} \subset [0, 1].$$

Since each I_{w_n} is compact,

$$I_w = \bigcap_{n=1}^{\infty} I_{w_n} \neq \emptyset.$$

Consider the space

$$\begin{aligned} \Sigma^+ &= \Sigma_d^+ = \prod_{n=0}^{\infty} \{0, 1, \dots, d-1\} \\ &= \{w = i_0 i_1 \dots i_k \dots i_{n-1} \dots \mid i_k \in \{0, 1, \dots, d-1\}, k = 0, 1, \dots\} \end{aligned}$$

with the product topology. Then it is a compact topological space.

If each $I_w = \{x_w\}$ contains only one point, then we define the projection $\pi_+ = \pi_{+, f}$ from Σ^+ onto T as

$$\pi_+(w) = x_w.$$

The projection π_+ is 1-1 except for a countable set B consisting of all labellings w of endpoints in the partitions $\eta_n = \{I_{w_n}\}, n = 0, 1, \dots$

Let

$$\sigma^+(w) = i_1 \dots i_{n-1} i_n \dots$$

be the left shift map. Then (Σ^+, σ^+) is called a symbolic dynamical system. From our construction, one can check that

$$\pi_+ \circ \sigma^+(w) = f \circ \pi_+(w), \quad w \in \Sigma^+.$$

Let

$$\varepsilon_n = \varepsilon_{n, f} = \max_{w_n} |I_w|$$

where w_n runs over all words of length n of $\{0, 1, \dots, d-1\}$. Then we have a more general Shub type theorem.

Theorem 1. *Let f and g be two circle endomorphisms such that both $\varepsilon_{n, f}$ and $\varepsilon_{n, g}$ tend to zero as $n \rightarrow \infty$. Then f and g are topologically conjugate if and only if their topological degrees are the same.*

Proof. Since both sets $I_{w, f} = \{x_w\}$ and $I_{w, g} = \{y_w\}$ contain only a single point for each w , we define

$$h(x_w) = y_w.$$

One can check that h is a homeomorphism with the inverse $h^{-1}(y_w) = x_w$. □

Therefore, for a fixed degree $d > 1$, there is only one topological model (Σ^+, σ^+) for the dynamics of all circle endomorphisms of degree d with $\varepsilon_n \rightarrow 0$.

4. Geometric models, part I

The next theme is the study of geometric models. A result analogous to Mostow’s rigidity theorem for closed hyperbolic 3-manifolds was proved by Shub and Sullivan [25]. The result can be stated as follows: Suppose f and g are two topologically conjugate real analytic expanding circle endomorphisms. If the conjugacy h is absolutely continuous, it must be also real analytic. Later, this result was proved for a more general case: Suppose f and g are two topologically conjugate $C^{k+\alpha}$ expanding circle endomorphisms for $1 \leq k \leq \omega$ and $0 < \alpha \leq 1$. If the conjugacy h is absolutely continuous, it must be also $C^{k+\alpha}$. Smooth invariants of a circle endomorphism have also been investigated. A quantity is called a smooth invariant if it is the same for f and g as long as f and g are smoothly conjugate (this means that the conjugacy is C^k for $k \geq 1$). A point p of f is called a periodic point of period $n \geq 1$ if $f^i(p) \neq p$ for $0 \leq i \leq n - 1$ but $f^n(p) = p$. The eigenvalue at a periodic point p of period n is defined as $e_p = (f^n)'(p)$. The eigenvalue e_p is a smooth invariant. The set of all eigenvalues of a $C^{1+\alpha}$ expanding circle endomorphism is actually a set of complete smooth invariants, where $0 < \alpha \leq 1$. This means that two $C^{1+\alpha}$ expanding circle endomorphisms f and g of degree $d > 1$ are smoothly conjugate if and only if their eigenvalues at the corresponding periodic points are the same. Therefore, one can use the set of all eigenvalues to classify geometric models of smooth expanding circle endomorphisms of the same degree. (Research in this direction has been extended to a larger class which even allows one to include maps with critical points. The reader who is interested in the smooth classification of one-dimensional dynamical systems in this direction may refer to [10–14] for more details.)

However, the structure of the set of all eigenvalues is not clear. In what follows, we define a function which is called a scaling function and will contain full information about the set of all eigenvalues in this context. (The name of the scaling function in this context was first used by Feigenbaum [8] in describing the universal geometric structure of attractors of infinitely period doubling folding maps. It was then used by Sullivan [26] for Cantor sets on the line to describe differential structures for fractal sets. The present form of the definition was formulated in [10] for any Markov map and then used to study the smooth classification of one-dimensional maps which have certain Markov properties. The reader may also refer to [11,12] for more details.)

As we have already seen, given a circle endomorphism of degree $d > 1$, there is an interval system

$$\{\eta_n\}_{n=1}^\infty = \{\{I_{w_n}\}_{w_n}\}_{n=1}^\infty$$

where w_n runs over all words of length n of $\{0, 1, \dots, d - 1\}$. When we constructed the topological model (Σ^+, σ^+) from this interval system, we read each w_n from the left to the right, i.e., $w_n = i_0 i_1 \dots i_{n-1}$. From the topological point of view, this means that we consider the set of all left cylinders

$$[w_n] = [w_n]_l = [i_0 i_1 \dots i_{n-1}]_l = \{w' = i'_0 i'_1 \dots i'_{n-1} i'_n \dots \mid i'_0 = i_0, \dots, i'_{n-1} = i_{n-1}\}$$

as a basis for the topology.

Now let us consider another topology which has a basis consisting of all right cylinders. We read w_n from the right to the left, $\kappa_n = w_n = j_{n-1}j_{n-2} \dots j_0$ and define

$$\Sigma^- = \Sigma_d^- = \{\kappa = \dots j_{n-1} \dots j_k \dots j_1 j_0 \mid j_k \in \{0, 1, \dots, d-1\}, k = 0, 1, \dots\}.$$

It is a topological space with a basis for the topology consisting of all right cylinders

$$[\kappa_n] = [\kappa_n]_r = [j_{n-1} \dots j_0]_r = \{\kappa' = \dots j'_n j'_{n-1} \dots j'_0 \mid j'_{n-1} = j_{n-1}, \dots, j'_0 = j_0\}.$$

Consider the right shift map

$$\sigma^- : \dots j_{n-1} \dots j_1 j_0 \mapsto \dots j_{n-1} \dots j_1.$$

Then we call (Σ^-, σ^-) the dual symbolic dynamical system for f .

Another way to view the symbolic dynamical system and the dual symbolic dynamical system is to consider the inverse limit of $f : T \rightarrow T$. This inverse limit can be viewed as a solenoid with the symbolic representation

$$\Sigma = \Sigma^- \times \Sigma^+.$$

Then Σ^- represents the transversal direction and Σ^+ represents the leaf direction.

On the transversal direction Σ^- , we define a function called the scaling function for f as follows. For any $\kappa = \dots j_{n-1} \dots j_1 j_0 \in \Sigma^-$, let $\kappa_n = j_{n-1} \dots j_1 j_0$, $\sigma^-(\kappa_n) = j_{n-1} \dots j_1$. Then

$$I_{\kappa_n} \subset I_{\sigma^-(\kappa_n)}.$$

Define

$$S(\kappa_n) = S_f(\kappa_n) = \frac{|I_{\kappa_n}|}{|I_{\sigma^-(\kappa_n)}|}.$$

Definition 1. If for every $\kappa \in \Sigma^-$,

$$S(\kappa) = S_f(\kappa) = \lim_{n \rightarrow \infty} S(\kappa_n)$$

exists, then we have a function

$$S = S_f : \Sigma^- \rightarrow \mathbb{R}^+.$$

We call this function the scaling function of f .

The space Σ^- is a metric space with the metric

$$d(w, w') = \sum_{k=0}^{\infty} \frac{|i_k - i'_k|}{d^k}.$$

A function S on Σ^- is called Hölder continuous if there are constants $C > 0$ and $0 < \beta \leq 1$ such that

$$|S(\kappa) - S(\kappa')| \leq C(d(\kappa, \kappa'))^\beta, \quad \kappa, \kappa' \in \Sigma^-.$$

Let $\mathcal{C}^{1+\alpha}$ denote the space of all $C^{1+\alpha}$ expanding circle endomorphisms for some $0 < \alpha \leq 1$. We have:

Theorem 2. *The scaling function S for $f \in \mathcal{C}^{1+}$ exists and is a Hölder continuous function. Furthermore, S is a completely smooth invariant. This means that $f, g \in \mathcal{C}^{1+}$ are C^1 conjugate if and only if they have the same scaling functions, i.e., $S_f = S_g$.*

This result is actually proved for a larger class of one-dimensional maps which may have critical points. The reader who is interested in this direction can refer to [11–14].

Thus geometric models of \mathcal{C}^{1+} can be represented by degrees $d > 1$ and scaling functions S as follows. We say $f \sim_s g$ if f and g are C^1 conjugate. It is an equivalence relation in \mathcal{C}^{1+} . Then we have that

$$\mathcal{C}^{1+} / \sim_s = \{d, S\}.$$

For a fixed $d > 1$, let \mathcal{C}_d^{1+} be the space of $f \in \mathcal{C}^{1+}$ with the degree d . Then

$$\mathcal{C}_d^{1+} / \sim_s = \{S\}.$$

There are two natural problems now. One is to study the geometric structure on $\mathcal{C}_d^{1+} / \sim_s$. The other is to characterize a scaling function. We will discuss these two problems.

5. Teichmüller structures, part I

A discussion of the first problem follows a similar idea to that of Teichmüller theory for Riemann surfaces with the help of the following theorem (refer to [10,11,15]).

Theorem 3. *Suppose f and g are two maps in \mathcal{C}_d^{1+} . Suppose h is the topological conjugacy between f and g . Then h is a quasisymmetric homeomorphism.*

A homeomorphism h of T is called quasisymmetric (see [1]) if there is a constant $K \geq 1$ such that

$$K^{-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq K$$

for all $x \in \mathbb{R}$ and all $t > 0$, where H is a lift of f to the real line.

Take $q_d(z) = z^d$ as a basepoint in \mathcal{C}_d^{1+} . For any $f \in \mathcal{C}_d^{1+}$, let h_f be the conjugacy from f to q_d , i.e.,

$$f \circ h_f = h_f \circ q_d.$$

Thus, we can think of \mathcal{C}_d^{1+} as pairs (f, h_f) . Two pairs satisfy $(f, h_f) \sim_t (g, h_g)$ if $h_f \circ h_g^{-1}$ is a C^1 -diffeomorphism. Then \sim_t is an equivalence relation. The Teichmüller space

$$\mathcal{TC}_d^{1+} = \{[(f, h_f)] \mid f \in \mathcal{C}_d^{1+}, \text{ with the basepoint } [(q_d, id)]\}$$

is the space of all \sim_t -equivalence classes $[(f, h_f)] = [(f, h_f)]_t$ with the basepoint $[(q_d, id)]$. This space has a Teichmüller metric $d_T(\cdot, \cdot)$ as we describe below.

We first consider the universal Teichmüller space. Let \mathcal{QS} be the set of all quasisymmetric homeomorphisms of the unit circle T factored by the space of all Möbius

transformations of the circle. (Then \mathcal{QS} may be identified with the set of all quasi-symmetric homeomorphisms of the unit circle fixing three points). For any $h \in \mathcal{QS}$, let \mathcal{E}_h be the set of all quasiconformal extensions of h into the unit disk. Let $K_{\tilde{h}}$ be the quasiconformal dilatation of $\tilde{h} \in \mathcal{E}_h$. Using quasiconformal dilatation, one defines a distance in \mathcal{QS} by

$$d_T(h_1, h_2) = \frac{1}{2} \inf\{\log K_{\tilde{h}_1\tilde{h}_2^{-1}} \mid \tilde{h}_1 \in \mathcal{E}_{h_1}, \tilde{h}_2 \in \mathcal{E}_{h_2}\}.$$

Here (\mathcal{QS}, d) is called the universal Teichmüller space. It is a complete metric space and a complex manifold with complex structure compatible with the Hilbert transform (see, for example, [1]).

A quasisymmetric homeomorphism h is called symmetric if there is a bounded positive function $\epsilon(t)$ such that $\epsilon(t) \rightarrow 0^+$ as $t \rightarrow 0^+$ and

$$1 - \epsilon(t) \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq 1 + \epsilon(t)$$

for any x in \mathbb{R} , where $H(x+1) = H(x) + 1$ is a lift of h . A C^1 -diffeomorphism of the unit circle is symmetric. However, a symmetric homeomorphism of the unit circle could be very singular. Let \mathcal{S} be the subset of \mathcal{QS} consisting of all symmetric homeomorphisms of the unit circle. The space \mathcal{S} is a closed subgroup of \mathcal{QS} . The topology coming from the metric d_T on \mathcal{QS} induces a topology on the factor space $\mathcal{QS} \text{ mod } \mathcal{S}$. Given two cosets $\mathcal{S}f$ and $\mathcal{S}g$ in this factor space, define a metric by

$$\bar{d}_T(\mathcal{S}f, \mathcal{S}g) = \inf_{A, B \in \mathcal{S}} d(Af, Bg).$$

The factor space $\mathcal{QS} \text{ mod } \mathcal{S}$ with this metric is a complete metric space and a complex manifold. The topology on $(\mathcal{QS} \text{ mod } \mathcal{S}, \bar{d}_T)$ is the finest topology which makes the projection $\pi : \mathcal{QS} \rightarrow \mathcal{QS} \text{ mod } \mathcal{S}$ continuous, and π is also holomorphic. An equivalent topology can be defined as follows. For any $h \in \mathcal{QS}$, let \tilde{h} be a quasiconformal extension of h to a small neighborhood U of T in the complex plane. Let

$$\mu_{\tilde{h}}(z) = \frac{\tilde{h}_{\bar{z}}(z)}{\tilde{h}_z(z)}, \quad k_{\tilde{h}} = \|\mu_{\tilde{h}}\|_{\infty} \quad \text{and} \quad B_{\tilde{h}} = \frac{1 + k_{\tilde{h}}}{1 - k_{\tilde{h}}}.$$

Then the boundary dilatation is defined as

$$B_h = \inf B_{\tilde{h}}$$

where the infimum is taken over all quasiconformal extensions of h near the unit circle. It is known that h is symmetric if and only if $B_h = 1$. Define

$$\tilde{d}(h_1, h_2) = \frac{1}{2} \log B_{h_2^{-1}h_1}.$$

The two metrics \bar{d} and \tilde{d} on $\mathcal{QS} \text{ mod } \mathcal{S}$ are equal. The reader may refer to [9] for this. The Teichmüller metric on \mathcal{TC}_d^{1+} is defined similarly. Let τ and τ' be two points in \mathcal{TC}_d^{1+} . Then

$$d_T(\tau, \tau') = \frac{1}{2} \log B_{h_f^{-1} \circ h_g}$$

where $(f, h_f) \in \tau$ and $(g, h_g) \in \tau'$.

Since the space of geometric models can be represented by the space of scaling functions, *the Teichmüller space can also be represented by the space of Hölder continuous scaling functions S_f for $f \in C_d^{1+}$ with the basepoint $S(\kappa) = 1/d$.*

6. Characterizations, part I

The characterization of a scaling function has been done for $d = 2$, which is the most interesting case for us. First by the definition of a scaling function, one can easily check that

$$S(\kappa 0) + S(\kappa 1) = 1, \quad \kappa \in \Sigma^- = \Sigma_2^-.$$

We call this the summation condition. In addition to this condition, a scaling function also enjoys another non-trivial condition which we call the compatibility condition,

$$\prod_{n=0}^{\infty} \frac{S(\kappa 1 \overbrace{0 \cdots 0}^n)}{S(\kappa 0 \underbrace{1 \cdots 1}_n)} = \text{const.}, \quad \kappa \in \Sigma^-.$$

Actually, this infinite product converges to the constant exponentially. (Its general term must tend to 1 as n goes to ∞ . This implies that $S(\cdots 000) = S(\cdots 111)$). We showed that the converse is also true as follows.

Theorem 4. *Let S be a positive Hölder continuous function on Σ^- . Then S is the scaling function of a map in C_2^{1+} if and only if S satisfies the summation and compatibility conditions.*

The original proof of this theorem is given in [4] and uses the Gibbs measure theory and some constructions in [22]. A proof without using the Gibbs measure theory can be founded in [16]. In the proof, we find the connection between the scaling function and the solenoid function and the linear model for a circle endomorphism and use some constructions in [5] (refer to Theorem 7). We would like to note that the solenoid function and the linear model are also interesting geometric invariants for a circle endomorphism. The solenoid function for a circle endomorphism is defined in [27] and used to describe an affine structure along leave directions Σ^+ of the solenoid $\Sigma^- \times \Sigma^+$. It has been studied in [21]. The linear model for a circle endomorphism is defined in [6]. A linear model can be thought of as a nonlinear coordinate on the unit circle. In [6], a question about what kind of nonlinear coordinate can be realized by a smooth expanding circle endomorphism arose. This question was studied in [3] by employing some results in quasiconformal theory (refer to [1]). A much simpler understanding was given in [16] by employing the naive distortion property (see [11]).

Therefore, *the Teichmüller space \mathcal{TC}_2^{1+} is represented by the space of positive Hölder continuous functions on Σ^- satisfying the summation and compatibility conditions with the basepoint $S(\kappa) = 1/2$.*

It is clear that the summation condition is true for any $d > 2$. This means that the scaling function S of $f \in \mathcal{C}_d^{1+}$ satisfies

$$S(\kappa_0) + S(\kappa_1) + \dots + S(\kappa_{d-1}) = 1, \quad \kappa \in \Sigma^- = \Sigma_d^-.$$

The compatibility condition for $d > 2$ should be similar. However, the proof of the characterization of the scaling function for a map in \mathcal{C}_d^{1+} for $d > 2$ should be slightly more complicated than the case $d = 2$, but it is a promising problem.

7. Teichmüller structures, part II

The Teichmüller space $(\mathcal{TC}_d^{1+}, d_T(\cdot, \cdot))$ is not complete. Its completion is an interesting subject to be studied. A circle endomorphism f of degree d is called uniformly symmetric if all its inverse branches for $f^n, n = 1, 2, \dots$, are symmetric uniformly. More precisely, there is a bounded positive function $\epsilon(t)$ with $\epsilon(t) \rightarrow 0^+$ as $t \rightarrow 0^+$ such that

$$1 - \epsilon(t) \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1 + \epsilon(t), \quad x \in \mathbb{R}, t > 0, n = 1, 2, \dots$$

By the naive distortion lemma (see, for example, [11]) we have

Proposition 1. *Any map $f \in \mathcal{C}^{1+}$ is uniformly symmetric.*

Let \mathcal{US} be the space of all uniformly symmetric circle endomorphisms of degree $d \geq 2$. The above proposition says that $\mathcal{C}^{1+} \subset \mathcal{US}$. However, a map in \mathcal{US} can be quite different. For example, it may not be differentiable and may not be absolutely continuous. However, we have shown that from the dual point of view, it has a lot of similarity to what we have studied for a map in \mathcal{C}^{1+} . (However, a C^1 expanding circle endomorphism is very different from what we have studied for a map in \mathcal{C}^{1+} (see [22]).)

For a fixed $d \geq 2$, let \mathcal{US}_d be the space of all uniformly symmetric circle endomorphisms of degree d . For $f \in \mathcal{US}_d$, it is certainly uniformly M -quasisymmetric for a fixed constant $M > 1$, that is,

$$M^{-1} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq M, \quad x \in \mathbb{R}, t > 0, n = 1, 2, \dots$$

We have:

Proposition 2. *If $f \in \mathcal{US}_d$, then there is a constant $C > 0$ such that*

$$S(\kappa_n) = \frac{|I_{\kappa_n}|}{|I_{\sigma^-(\kappa_n)}|} \geq C$$

for all finite words $\kappa_n = j_{n-1} \dots j_1 j_0$ of $\{0, 1, \dots, d-1\}$.

The above proposition means that the sequence of nested partitions

$$\{\eta_n\}_{n=1}^\infty = \{\{I_{w_n}\}_{w_n}\}_{n=1}^\infty$$

has bounded geometry. Using the above proposition and the summation condition, we have constants $D > 0$ and $0 < \tau < 1$ such that

$$\varepsilon_n \leq D\tau^n, \quad n = 1, 2, \dots$$

Therefore, just like in the proofs of Theorem 1 and Theorem 3, we have:

Theorem 5. *Any two maps $f, g \in \mathcal{US}_d$ are topologically conjugate and, furthermore, the conjugacy h is quasimetric.*

With this proposition, we can define the Teichmüller space for \mathcal{US}_d as we did for \mathcal{C}_d^{1+} . Take $q_d(z) = z^d$ as a basepoint in \mathcal{US}_d . For any $f \in \mathcal{US}_d$, let h_f be the conjugacy from f to q_d , i.e.,

$$f \circ h_f = h_f \circ q_d.$$

Thus we can think of \mathcal{US}_d as pairs (f, h_f) . Two pairs satisfy $(f, h_f) \sim_t (g, h_g)$ if $h_f \circ h_g^{-1}$ is symmetric. Then \sim_t is an equivalence relation. The Teichmüller space

$$\mathcal{TUS}_d = \{[(f, h_f)] \mid f \in \mathcal{US}_d, \text{ with the basepoint } [(q_d, id)]\}$$

is the space of all \sim_t -equivalence classes $[(f, h_f)] = [(f, h_f)]_t$ equipped with a Teichmüller metric

$$d_T(\tau, \tau') = \frac{1}{2} \log B_{h_f^{-1} \circ h_g}$$

where $(f, h_f) \in \tau$ and $(g, h_g) \in \tau'$.

If $f, g \in \mathcal{C}_d^{1+}$ and if the conjugacy h between f and g is symmetric, then h must be C^1 . The reason is that if the conjugacy between f and g is symmetric, then their scaling functions S_f and S_g must be the same (refer to Theorem 7). Therefore they are C^1 -conjugate and the conjugacy h must be C^1 (see Theorem 2). (A related easy but interesting fact is that the ratio of eigenvalues $e_f(p)$ and $e_g(h(p))$ of f and g at corresponding periodic point p and $h(p)$ determines the local quasimetric constant of h at p . If h is symmetric, its local quasimetric constant at p is 1, so the ratio $e_f(p)/e_g(h(p))$ is 1.) This implies that the Teichmüller space \mathcal{TC}_d^{1+} is indeed a subspace of the Teichmüller space \mathcal{TUS}_d . Furthermore, we have (refer to [4,5]):

Theorem 6. *The space $(\mathcal{TUS}_d, d_T(\cdot, \cdot))$ is a complete complex Banach manifold and is the completion of the space $(\mathcal{TC}_d^{1+}, d_T(\cdot, \cdot))$.*

The local model of the complex Banach manifold can be thought of as the set of Beltrami coefficients on the upper-half plane \mathbb{H} (complex L^∞ functions $\mu(z)$ on the upper-half plane \mathbb{H} with $\|\mu(z)\|_\infty < 1$) such that $\mu(dz) = \mu(z)$ and $|\mu(z+n) - \mu(z)| \rightarrow 0$ uniformly for n as $\Im(z) \rightarrow 0$ (refer to [3–5]).

8. Geometric models, part II

The geometric models of maps in \mathcal{US}_d can also be represented by their scaling functions. Two maps $f, g \in \mathcal{US}_d$ are called symmetrically conjugate if the conjugacy between them is symmetric. This is an equivalence relation which we denote as $f \sim_{sy} g$. The space $\mathcal{US}_d / \sim_{sy}$ of geometric models for maps in \mathcal{US}_d is the space of all equivalence classes. We have (refer to [5]):

Theorem 7. Suppose $f \in \mathcal{US}_d$. Then its scaling function

$$S = S_f : \Sigma^- \rightarrow \mathbb{R}^+.$$

exists and is a continuous function. Furthermore, it is a complete symmetric invariant for \mathcal{US}_d ; this means f and g are symmetrically conjugate if and only if their scaling functions are the same, i.e., $S_f = S_g$.

Thus \mathcal{TUS}_d can be represented by scaling functions S_f , i.e.,

$$\mathcal{US}_d / \sim_{sy} = \{S_f \mid f \in \mathcal{US}_d\}$$

and

$$\mathcal{TUS}_d = (\{S_f \mid f \in \mathcal{US}_d, \text{ with the basepoint } S = \frac{1}{7}, d_T(\cdot, \cdot)\}.$$

9. Characterizations, part II

The characterization of the scaling functions for \mathcal{US}_2 has been given as

Theorem 8. Let S be a positive continuous function on $\Sigma^- = \Sigma_2^-$. Then S is the scaling function of a map in \mathcal{US}_2 if and only if S satisfies the summation and compatibility conditions.

The proof of this theorem can be founded in [5]. In this case, the infinite product

$$\prod_{n=0}^{\infty} \frac{S(\kappa \overbrace{10 \cdots 0}^n)}{S(\kappa \underbrace{01 \cdots 1}_n)} = const., \quad \kappa \in \Sigma^-,$$

in the compatibility condition converges uniformly to a constant.

Therefore, the Tecihmüller space \mathcal{TUS}_2 is represented by the space of positive continuous functions on Σ^- satisfying the summation and compatibility conditions.

Just as in the end of §6, the characterization of a scaling function of a map in \mathcal{US}_d should be slightly more complicated than the case $d = 2$, but it is a promising problem.

10. Invariant measures and dual invariant measures

Consider the symbolic dynamical system (Σ^+, σ^+) and a positive Hölder continuous function $\psi = \psi(w)$. The standard Gibbs theory (refer to [2] or [7]) implies that there is a number $P = P(\log \psi)$ called the pressure and a σ^+ -invariant probability measure $\mu_+ = \mu_{+, \psi}$ such that

$$C^{-1} \leq \frac{\mu_+([i_0 \cdots i_{n-1}])}{\exp(-Pn + \sum_{i=0}^{n-1} \log \psi((\sigma^+)^i(w)))} \leq C$$

for any left cylinder $[i_0 \cdots i_{n-1}]$ and any $w = i_0 \dots i_{n-1} \cdots \in [i_0 \dots i_{n-1}]$, where C is a fixed constant. Here μ_+ is a σ^+ -invariant measure means that

$$\mu_+(\sigma^+)^{-1}(A) = \mu_+(A)$$

for all Borel sets of Σ^+ . A σ^+ -invariant probability measure satisfying the above inequalities is called the Gibbs measure with respect to the given potential function $\log \psi$.

Two positive Hölder continuous functions ψ_1 and ψ_2 are said to be cohomologous equivalent if there is a continuous function $u = u(w)$ on Σ^+ such that

$$\log \psi_1(w) - \log \psi_2(w) = u(\sigma^+(w)) - u(w).$$

If two functions are cohomologous to each other, they have the same Gibbs measure. Therefore, the Gibbs measure can be thought of as a representation of a cohomologous class.

The Gibbs measure is also an equilibrium state. Consider the measure-theoretical entropy $h_{\mu_+}(\sigma^+)$. Since the Borel σ -algebra of Σ^+ is generated by all left cylinders, then $h_{\mu_+}(\sigma^+)$ can be calculated as

$$\begin{aligned} h_{\mu_+}(\sigma^+) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{w_n} (-\mu_+([w_n]) \log \mu_+([w_n])) \\ &= \lim_{n \rightarrow \infty} \sum_{w_n} \left(-\mu_+([w_n]) \log \left(\frac{\mu_+([w_n])}{\mu_+(\sigma^+([w_n]))} \right) \right), \end{aligned}$$

where w_n runs over all words $w_n = i_0 \dots i_{n-1}$ of $\{0, 1, \dots, d - 1\}$ of length n . Then μ_+ is an equilibrium state in the sense that

$$P(\log \psi) = h_{\mu_+}(\sigma^+) + \int_{\Sigma^+} \log \psi(w) d\mu(w) = \sup \left\{ h_\nu(\sigma^+) + \int_{\Sigma^+} \log \psi(w) d\nu(w) \right\}$$

where ν runs over all σ^+ -invariant probability measures. The measure μ_+ is unique in this case.

There is a natural way to transfer a σ^+ -invariant probability measure μ_+ (not necessarily a Gibbs measure) to a σ^- -invariant probability measure μ_- as follows. Given any right cylinder $[j_{n-1} \dots j_0]_r$ in Σ^- , let $i_0 \dots i_{n-1} = j_{n-1} \dots j_0$ define a left cylinder

$$[i_0 \dots i_{n-1}]_l = \{w' = i'_0 \dots i'_{n-1} i'_n \dots \mid i'_0 = i_0, \dots, i'_{n-1} = i_{n-1}\}.$$

Then define

$$\mu_-([j_{n-1} \dots j_0]_r) = \mu_+([i_0 \dots i_{n-1}]_l).$$

Then

$$\begin{aligned} \mu_-([j_{n-1} \dots j_0]_r) &= \mu_+([i_0 \dots i_{n-1}]_l) = \mu_+((\sigma^+)^{-1}([i_0 \dots i_{n-1}]_l)) \\ &= \mu_+\left(\bigcup_{i=0}^{d-1} [ii_0 \dots i_{n-1}]_l\right) = \sum_{i=0}^{d-1} \mu_+([ii_0 \dots i_{n-1}]_l) \\ &= \sum_{j=0}^{d-1} \mu_-([jj_{n-1} \dots j_0]). \end{aligned}$$

This implies that μ_- satisfies the finite additive law for all cylinders, i.e., if A_1, \dots, A_k are finitely many pairwise disjoint right cylinders in Σ^- , then

$$\mu_-\left(\bigcup_{l=1}^k A_k\right) = \sum_{l=1}^k \mu_-(A_l).$$

Also μ_- satisfies the continuity law in the sense that if $\{A_n\}_{n=1}^\infty$ is a decreasing sequence of cylinders and tends to the empty set (this means $A_{n+1} \subset A_n$ and $\bigcap_{n=1}^\infty A_n = \emptyset$), then $\mu_-(A_n)$ tends to zero as n goes to ∞ . The reason is that since a cylinder of Σ^- is a compact set, a decreasing sequence of cylinders tending to the empty set must be eventually all empty. The Borel σ -algebra in Σ^- is generated by all right cylinders. So μ_- extends to measure on Σ^- . We have the following proposition.

Proposition 3. μ_- is a σ^- -invariant probability measure.

Proof. We have seen that μ^- is a measure on Σ^- . Since $\mu_-(\Sigma^-) = 1$, it is a probability measure. For any right cylinder $[j_{n-1} \dots j_0]_r$,

$$\begin{aligned} \mu_-((\sigma^-)^{-1}([j_{n-1} \dots j_0]_r)) &= \mu_-\left(\bigcup_{j=0}^{d-1} [j_{n-1} \dots j_0 j]_r\right) \\ &= \sum_{j=0}^{d-1} \mu_-([j_{n-1} \dots j_0 j]_r) = \sum_{i=0}^{d-1} \mu_+([i_0 \dots i_{n-1} i]_l) \\ &= \mu_+\left(\bigcup_{i=0}^{d-1} [i_0 \dots i_{n-1} i]_l\right) \\ &= \mu_+([i_0 \dots i_{n-1}]_l) = \mu_-([j_{n-1} \dots j_0]_r). \end{aligned}$$

So μ_- is σ^- -invariant. □

We call μ_- a dual invariant measure. A natural question now is as follows. *Is a dual invariant measure a Gibbs measure with respect to some continuous or Hölder continuous function on Σ^- ?*

A more interesting geometric question is the following. Consider a metric induced from the dual invariant measure μ_- (in the case that μ_- is supported on the whole Σ^- and is non-atomic), that is,

$$d(\kappa, \kappa') = \mu_-([j_{n-1} \cdots j_0])$$

where $[j_{n-1} \cdots j_0]$ is the smallest right cylinder containing both $\kappa = \dots j_n j_{n-1} \dots j_0$ and $\kappa' = \dots j'_n j'_{n-1} \dots j_0$, $j_n \neq j'_n$. Is σ^- differentiable with a continuous or Hölder continuous derivative under this metric? More precisely, does the limit

$$\frac{d\sigma^-}{dx}(\kappa) = \lim_{n \rightarrow \infty} \frac{\mu_-(\sigma^-([j_{n-1} \cdots j_1 j_0]))}{\mu_-([j_{n-1} \cdots j_1 j_0])} = \lim_{n \rightarrow \infty} \frac{\mu_-([j_{n-1} \cdots j_1])}{\mu_-([j_{n-1} \cdots j_1 j_0])}$$

exist for every $\kappa = \dots j_{n-1} \dots j_1 j_0 \in \Sigma^-$? If it exists, is the limiting function continuous or Hölder continuous on Σ^- ?

Actually, there is a measure-theoretical version related to these questions. I will first give a brief review of this theory.

11. g-measures

Let X be Σ^- (or Σ^+) and let f be σ^- (or σ^+). Let \mathcal{B} be the Borel σ -algebra of X . Let $\mathcal{M}(X)$ be the space of all finite Borel measures on X . Let $\mathcal{M}(X, f)$ be the space of all f -invariant probability measures in $\mathcal{M}(X)$. Let $\mathcal{C}(X)$ be the space of all continuous real functions on X . Then $\mathcal{M}(X)$ is the dual space of $\mathcal{C}(X)$. Denote

$$\langle \phi, \mu \rangle = \int_X \phi(x) d\mu, \quad \phi \in \mathcal{C}(X) \text{ and } \mu \in \mathcal{M}(X).$$

A real non-negative continuous function ψ on X is called a g -function (the historic reason to call such a function a g -function is because of Keane's paper [19]) if

$$\sum_{f(y)=x} \psi(y) = 1.$$

For a function ψ , define the transfer operator \mathcal{L}_ψ from $\mathcal{C}(X)$ into itself as

$$\mathcal{L}_\psi \phi(x) = \sum_{f(y)=x} \phi(y) \psi(y), \quad \phi \in \mathcal{C}(X).$$

One can check that $\mathcal{L}_\psi \phi = \mathcal{L}_1(\psi \phi)$ and if ψ is a g -function, then $\mathcal{L}_\psi 1 = 1$. Let \mathcal{L}_ψ^* be the dual operator of \mathcal{L}_ψ , that is, \mathcal{L}_ψ^* is the operator from $\mathcal{M}(X)$ into itself satisfying

$$\langle \phi, \mathcal{L}_\psi^* \mu \rangle = \langle \mathcal{L}_\psi \phi, \mu \rangle, \quad \forall \phi \in \mathcal{C}(X) \text{ and } \forall \mu \in \mathcal{M}(X).$$

Suppose ψ is a g -function. Then a probability measure $\mu \in \mathcal{M}(X)$ is called a g -measure if it is a fixed point of \mathcal{L}_ψ , that is,

$$\mathcal{L}_\psi^* \mu = \mu.$$

A g -measure is a f -invariant measure because

$$\begin{aligned} \mu(f^{-1}(B)) &= \langle 1_{f^{-1}(B)}, \mu \rangle = \langle 1_B \circ f, \mathcal{L}_\psi^* \mu \rangle \\ &= \langle \mathcal{L}_\psi 1_B, \mu \rangle = \langle 1_B, \mu \rangle = \mu(B), \quad \forall B \in \mathcal{B}. \end{aligned}$$

For any $\mu \in \mathcal{M}(X)$, let $\tilde{\mu} = \mathcal{L}_1^* \mu$.

Proposition 4.

$$\tilde{\mu}(B) = \sum_{j=0}^{d-1} \mu(f(B \cap [j]))$$

where B is any Borel subset in \mathcal{B} and $[j]$ is the right cylinder of j . Moreover, if $\mu \in \mathcal{M}(X, f)$, μ is absolutely continuous with respect to $\tilde{\mu}$.

Proof. For any Borel subset $B \in \mathcal{B}$,

$$\tilde{\mu}(B) = \langle 1_B, \mathcal{L}_1^* \mu \rangle = \langle \mathcal{L}_1 1_B, \mu \rangle.$$

But

$$\mathcal{L}_1 1_B(x) = \sum_{j=0}^{d-1} 1_B(xj) = \sum_{j=0}^{d-1} 1_{f(B \cap [j])}(x).$$

So we have that

$$\tilde{\mu}(B) = \sum_{j=0}^{d-1} \mu(f(B \cap [j])).$$

If μ is f -invariant, then we have

$$\tilde{\mu}(B) = \sum_{j=0}^{d-1} \mu(f(B \cap [j])) = \sum_{j=0}^{d-1} \mu(f^{-1}(f(B \cap [j]))) \geq \sum_{j=0}^{d-1} \mu(B \cap [j]) = \mu(B).$$

Therefore, $\mu(B) = 0$ whenever $\tilde{\mu}(B) = 0$. So μ is absolutely continuous with respect to $\tilde{\mu}$. □

Suppose $\mu \in \mathcal{M}(X, f)$. Then μ is absolutely continuous with respect to $\tilde{\mu}$. So the Radon-Nikodým derivative

$$D_\mu(x) = \frac{d\mu}{d\tilde{\mu}}(x), \quad \tilde{\mu} - a.e. x$$

of μ with respect to $\tilde{\mu}$ exists $\tilde{\mu}$ -a.e. and is a $\tilde{\mu}$ -measurable function. We would like to note that $\tilde{\mu}$ may not be absolutely continuous with respect to μ .

The following theorem was proved by Ledrappier in [20] and was used by Walters in [28] in the study of a generalized version of Ruelle’s theorem.

Theorem 9. *Suppose ψ is a g -function and $\mu \in \mathcal{M}(X)$ is a probability measure. The followings are equivalent:*

- i) μ is a g -measure, i.e., $\mathcal{L}_\psi^* \mu = \mu$.
- ii) $\mu \in \mathcal{M}(X, f)$ and $D_\mu(x) = \psi(x)$ for $\tilde{\mu}$ -a.e. x .
- iii) $\mu \in \mathcal{M}(X, f)$ and

$$E[\phi|f^{-1}(\mathcal{B})](x) = \mathcal{L}_\psi \phi(fx) = \sum_{fy=fx} \psi(y)\phi(y), \text{ for } \mu\text{-a.e. } x$$

where $E[\phi|f^{-1}(\mathcal{B})]$ is the conditional expectation of ϕ with respect to $f^{-1}(\mathcal{B})$.

iv) $\mu \in \mathcal{M}(X, f)$ and is an equilibrium state in the meaning that

$$0 = h_\mu(f) + \int_X \log \psi \, d\mu = \sup \left\{ h_\nu(f) + \int_X \log \psi \, d\nu \mid \nu \in \mathcal{M}(X, f) \right\}.$$

(Note that the pressure $P(\log \psi) = 0$ for a g -function ψ .)

For any σ^+ -invariant probability measure μ_+ , let μ_- be the dual σ^- -invariant probability measure which we have constructed in the previous section. Then we have a $\tilde{\mu}_+$ -measurable function

$$D_{\mu_+}(w) = \lim_{n \rightarrow \infty} \frac{\mu_+([i_0 i_1 \dots i_{n-1}])}{\mu_+([i_1 \dots i_{n-1}])}, \quad \text{for } \tilde{\mu}_+ \text{-a.e. } w = i_0 i_1 \dots i_{n-1} \dots$$

and a $\tilde{\mu}_-$ -measurable function

$$D_{\mu_-}(\kappa) = \lim_{n \rightarrow \infty} \frac{\mu_-([j_{n-1} \dots j_1 j_0])}{\mu_-([j_{n-1} \dots j_1])}, \quad \text{for } \tilde{\mu}_- \text{-a.e. } \kappa = \dots j_{n-1} \dots j_0.$$

Now the question related to those at the end of the previous section is as follows. Can we extend D_{μ_-} as well as D_{μ_+} to a continuous g -function or a Hölder continuous g -function?

The Borel σ -algebra of Σ^+ (or of Σ^-) is generated by all left cylinders (or all right cylinders). The measure-theoretical entropy $h_{\mu_+}(\sigma^+)$ can be calculated as

$$\begin{aligned} h_{\mu_+}(\sigma^+) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{w_n} (-\mu_+([w_n]) \log \mu_+([w_n])) \\ &= \lim_{n \rightarrow \infty} \sum_{w_n} \left(-\mu_+([w_n]) \log \left(\frac{\mu_+([w_n])}{\mu_+(\sigma^+([w_n]))} \right) \right), \end{aligned}$$

where w_n runs over all words $w_n = i_0 \dots i_{n-1}$ of $\{0, 1, \dots, d - 1\}$ of length n . The measure-theoretical entropy $h_{\mu_-}(\sigma^-)$ can be calculated as

$$\begin{aligned} h_{\mu_-}(\sigma^-) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\kappa_n} (-\mu_-([\kappa_n]) \log \mu_-([\kappa_n])) \\ &= \lim_{n \rightarrow \infty} \sum_{\kappa_n} \left(-\mu_-([\kappa_n]) \log \left(\frac{\mu_-([\kappa_n])}{\mu_-(\sigma^-([\kappa_n]))} \right) \right), \end{aligned}$$

where κ_n runs over all words $\kappa_n = j_{n-1} \dots j_0$ of $\{0, 1, \dots, d - 1\}$ of length n . We would like to know when is μ_+ (or μ_-) an equilibrium state? We have studied these questions for \mathcal{C}^{1+} and for \mathcal{US} .

12. Geometric Gibbs measures and dual geometric Gibbs measures

Consider $f \in \mathcal{C}^{1+}$. Then $1/f'(x)$ can be lifted to a positive Hölder continuous function $\psi(w) = \psi_f(w) = 1/f'(\pi_+(w))$ on the symbolic space Σ^+ . By thinking of

$\log \psi$ as a potential function for the dynamical system (Σ^+, σ^+) , there is a unique σ^+ -invariant measure $\mu_+ = \mu_{+, \psi}$ as we have mentioned in the previous section such that

$$C^{-1} \leq \frac{\mu_+([i_0 \dots i_{n-1}])}{\prod_{i=0}^{n-1} \psi((\sigma^+)^i(w))} \leq C$$

for any left cylinder $[i_0 \dots i_{n-1}]$ and any $w = i_0 \dots i_{n-1} \dots \in [i_0 \dots i_{n-1}]$, where C is a fixed constant. (Note that $P = P(\log \psi) = 0$ in this case.)

The geometric model $[f]_s$ in \mathcal{C}^{1+} can also be represented by the Gibbs measure μ_+ with respect to $\psi(w) = 1/f'(\pi_+(w))$. The reason is that any $g \in [f]_s$ is smoothly conjugate to f , so there is a C^1 diffeomorphism h of T such that $f(h(x)) = h(g(x))$. Then $f'(h(x))h'(x) = h'(g(x))g'(x)$. Therefore,

$$\log \psi_f(w) - \log \psi_g(w) = \log h'(w) - \log h'(\sigma^+(w)).$$

So ψ_g and ψ_f are cohomologous to each other. We call this μ_+ a geometric Gibbs measure because it enjoys the following geometric property too: The push-forward measure $\mu = (\pi_+)_* \mu_+$ is a smooth f -invariant measure. This means there is a continuous function ρ on T such that

$$\mu(A) = \int_A \rho(x) dx, \quad \text{for all Borel subsets } A \text{ on } T.$$

There is another way to find the density ρ . First it is a standard method to find an invariant measure for a dynamical system f . Let ν_0 be the Lebesgue measure. Consider the push-forward measure $\nu_n = (f^n)_* \nu_0$ by the n^{th} iterates of f . Sum up these measures to get

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_k$$

Any limit μ of a subsequence of $\{\mu_n\}$ will be an f -invariant measure. Since we start with an $f \in \mathcal{C}^{1+}$, we can prove that the sequence $\{\mu_n\}$ is actually convergent in \mathcal{C}^1 topology. This means that each $\nu_n = (f^n)_* \nu_0$ has a Hölder continuous density

$$\rho_n(x) = \sum_{f^n(y)=x} \frac{1}{(f^n)'(y)}.$$

Following the theory of transfer operators (refer to [17,7]), $\rho_n(x)$ converges uniformly to a continuous function $\rho(x)$. The density of μ_n is just

$$\frac{1}{n} \sum_{k=0}^{n-1} \rho_k.$$

So it also converges to ρ uniformly. Thus $\mu(A) = \int_A \rho(x) dx$ is the limit of μ_n and is a smooth f -invariant measure.

Let $y = h(x) = \mu([0, x])$. Then $y = h(x)$ is a C^1 -diffeomorphism of T . Let

$$g(y) = h \circ f \circ h^{-1}(y), \quad x = h^{-1}(y)$$

(Note that g here means a circle endomorphism not a g -function!) Then g preserves the Lebesgue measure dy (which means that $g_*(dy) = dy$, or equivalently, the Lebesgue measure is g -invariant). Since the Lebesgue measure is an ergodic g -invariant measure, g is unique in the geometric model $[f]_S$.

By considering $\psi(w) = 1/g'(\pi(w))$, it is a g -function on Σ^+ and μ_+ is a g -measure. Thus μ_+ is an equilibrium state. It follows that μ is also an equilibrium state, that is,

$$\begin{aligned} 0 &= P(-\log f'(x)) = h_\mu(f) - \int_T \log f'(x) d\mu = h_\mu(f) - \int_T \log f'(x) \rho(x) dx \\ &= \sup \left\{ h_\nu(f) - \int_T \log f'(x) d\nu \mid \nu \text{ is an } f\text{-invariant probability measure} \right\} \\ &= h_{Leb}(g) - \int_T \log g'(y) dy, \end{aligned}$$

where $h_{Leb}(g)$ denotes the measure-theoretical entropy with respect to the Lebesgue measure. The equilibrium state μ is unique in this case.

Now by considering the dual invariant measure μ_- for this geometric Gibbs measure μ_+ , we have:

Theorem 10. *Suppose $f \in C^{1+}$. Consider Σ^- with the metric $d(\cdot, \cdot)$ induced from μ_- . Then the right shift σ^- is a C^{1+} differentiable with respect to d . The derivative is one over the scaling function S_f , i.e.,*

$$\frac{d\sigma^-}{dx}(\kappa) = \frac{1}{S(\kappa)}, \quad \kappa \in \Sigma^-.$$

The proof follows the proof of Theorem 2 and the definition of μ_- . Note that by the definition of the derivative for $\kappa = \dots j_{n-1} \dots j_1 j_0 \in \Sigma^-$,

$$\frac{d\sigma^-}{dx}(\kappa) = \lim_{n \rightarrow \infty} \frac{\mu_-(\sigma^-([j_{n-1} \dots j_1 j_0]))}{\mu([j_{n-1} \dots j_1 j_0])} = \lim_{n \rightarrow \infty} \frac{\mu_-([j_{n-1} \dots j_1])}{\mu_-([j_{n-1} \dots j_1 j_0])}.$$

The theorem says that it equals to $1/S(\kappa)$ pointwise. Moreover, this convergence is exponentially fast. Then, following the fact that Σ^- is a compact space, we have automatically the Gibbs inequality that

$$C^{-1} \leq \frac{\mu_-([j_{n-1} \dots j_0])}{\prod_{l=0}^{n-1} S((\sigma^-)^l(\kappa))} \leq C$$

for any right cylinder $[j_{n-1} \dots j_0]$ and any κ in this cylinder, where $C > 0$ is a fixed constant. Thus μ_- is a Gibbs measure with respect to the potential function $\log S_f$. We call μ_- a dual geometric Gibbs measure.

Corollary 1. *The dual geometric Gibbs measure μ_- is a g -measure with respect to the g -function S_f whose pressure $P(\log S_f) = 0$. Moreover, $D_{\mu_-} = S_f$ for $\tilde{\mu}$ -a.e. κ and μ_- is a unique equilibrium state in the sense that*

$$\begin{aligned}
 0 &= P(\log S) = h_{\mu_-}(\sigma^-) + \int_{\Sigma^-} \log S(\kappa) d\mu_-(\kappa) \\
 &= \sup \left\{ h_\nu(\sigma^-) + \int_{\Sigma^-} \log S(\kappa) d\nu(\kappa) \mid \nu \text{ is a } \sigma^- \text{-invariant measure} \right\}.
 \end{aligned}$$

So following Theorem 4 and Theorem 10, we have:

Theorem 11. *Suppose $S(\kappa)$ is a Hölder continuous function on Σ^- satisfying the summation and compatibility conditions. Then there is a unique measure μ_- and the metric $d(\cdot, \cdot)$ induced from μ_- on Σ^- such that $1/S$ is the derivative of the right shift σ^- with respect to this metric. Moreover, μ_- is an equilibrium state for the dynamical system $\sigma^- : (\Sigma^-, d(\cdot, \cdot)) \rightarrow (\Sigma^-, d(\cdot, \cdot))$ from a metric space into itself with the potential $\log S$.*

13. Dual geometric Gibbs type measures

A map $f \in \mathcal{US}$ may not be differentiable everywhere (it may not be even be absolutely continuous). There is no suitable Gibbs theory to be used in the study of geometric properties of a σ^+ -invariant measure. We have turned to the dual symbolic dynamical system (Σ^-, σ^-) and produced a similar dual geometric Gibbs type measure theory.

An f -invariant measure μ can be found as we did in the previous section. Let ν_0 be the Lebesgue measure. Consider the push-forward measure $\nu_n = (f^n)_* \nu_0$ and sum them up to get

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \nu_i.$$

Take a weak limit μ of a subsequence of $\{\mu_n\}$. Then μ is an f -invariant measure.

Each $h_n(x) = \mu_n([0, x])$ defines a homeomorphism on T . Since f is uniformly symmetric, the sequence $\{h_n\}$ is also uniformly symmetric. The space of all quasisymmetric homeomorphisms with a fixed quasisymmetric constant is a normal family (refer to [1]). So there is a subsequence of $\{h_n\}$ that converges uniformly to a function which is a symmetric homeomorphism $h(x)$ in this case. Furthermore, we have

$$h(x) = \mu([0, x]).$$

Moreover, by considering

$$g = h \circ f \circ h^{-1},$$

we see that g is a uniformly symmetric circle endomorphism in the geometric model $[f]_{\text{sy}}$ preserving the Lebesgue measure.

We can lift μ to Σ^+ to get a σ^+ -invariant measure μ_+ as follows. For any finite word $w_n = i_0 \cdots i_{n-1}$, consider the cylinder $[w_n]$. Define

$$\mu_+([w_n]) = \mu(I_{w_n}),$$

where I_{w_n} is the interval in the interval system labeled by w_n . One can check that it satisfies the finite additive law and the continuity law. So it can be extended to a σ^+ -invariant probability measure μ_+ on Σ^+ such that $(\pi_+)_*\mu_+ = \mu$. For μ_+ , we can construct its dual invariant measure μ_- on Σ^- as we did in the previous two sections. Then we have the following geometric Gibbs type property as we had before in the smooth case:

Theorem 12. *Suppose $f \in \mathcal{US}$. Consider Σ^- with the metric $d(\cdot, \cdot)$ induced from μ_- . Then the right shift σ^- is C^1 differentiable. The derivative is one over the scaling function S_f , i.e.,*

$$\frac{d\sigma^-}{dx}(\kappa) = \frac{1}{S_f(\kappa)}, \quad \kappa \in \Sigma^-.$$

The proof of the theorem follows the proof of Theorem 7 and the definition of μ_- .

Definition 2. *Suppose $\psi(\kappa)$ is a positive continuous function on Σ^- . A σ^- -invariant measure ν is called a geometric Gibbs type measure with the potential $-\log \psi(\kappa)$ if*

$$\lim_{n \rightarrow \infty} \frac{\nu([j_{n-1} \cdots j_1 j_0])}{\nu([j_{n-1} \cdots j_1])} = \psi(\kappa), \quad \forall \kappa = \dots j_{n-1} \dots j_0 \in \Sigma^-.$$

Corollary 2. *The measure μ_- in Theorem 12 is a geometric Gibbs type measure with the potential $\log S_f$. Furthermore, μ_- is a g -measure with respect to the g -function S_f and $D_{\mu_-}(\kappa) = S_f(\kappa)$ for $\tilde{\mu}_-$ -a.e. κ . Moreover, μ_- is an equilibrium state in the sense that*

$$\begin{aligned} 0 &= P(-\log S) = h_{\mu_-}(\sigma^-) + \int_{\Sigma^-} \log S(\kappa) d\mu_-(\kappa) \\ &= \sup \left\{ h_\nu(\sigma^-) + \int_{\Sigma^-} \log S(\kappa) d\nu(\kappa) \right\} \end{aligned}$$

where ν runs over all σ^- -invariant probability measures.

So following Theorem 8 and Theorem 12, we have:

Theorem 13. *Suppose $S(\kappa)$ is a continuous function on Σ^- satisfying the summation and compatibility conditions. Then there is a geometric Gibbs type measure μ_- with the potential $\log S$. Moreover, μ_- is an equilibrium state for the dynamical system σ^- with the potential $\log S$.*

The Teichmüller space \mathcal{TUS} is a complex Banach manifold and consists of certain positive functions on the dual symbolic space Σ^- . It is an interesting problem now to study the change of μ_- when the potential $\log S$ is changed in the manifold. The reader who is interested in this direction may refer to [18,23] for some results about differentiating the absolutely continuous invariant measure of a map with respect to this map.

Acknowledgement

During my research in this direction and related topics, I have had many interesting conversations with Professors Guizhen Cui, Aihua Fan, Fred Gardiner, Jihua Ma, Sudeb Mitra, Huyi Hu, Anthony Quas, David Ruelle, and Sheldon Newhouse. Professor Dennis Sullivan introduced me to this interesting topic. I would like to express my thanks to them all.

References

- [1] L. V. Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand Mathematical studies 10, D. Van Nostrand Co. Inc., Toronto-New York-London (1966).
- [2] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Springer-Verlag, Berlin (1975).
- [3] G. Cui, Circle expanding maps and symmetric structures, *Ergod. Th. & Dynamical Sys.*, **18** (1998) 831–842.
- [4] G. Cui, Y. Jiang and A. Quas, Scaling functions, g -measures, and Teichmüller spaces of circle endomorphisms, *Discrete and Continuous Dynamical Sys.*, **3** (1999) 534–552.
- [5] G. Cui, F. Gardiner and Y. Jiang, Scaling functions for circle endomorphisms, *Contemporary Mathematics, AMS Series*, **355** (2004) 147–163.
- [6] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. Éc. Norm. Sup.*, **18** (1985) 287–343.
- [7] A. Fan and Y. Jiang, On Ruelle-Perron-Frobenius operators. I. Ruelle's theorem and II. Convergence speed, *Commun. Math. Phys.*, **223** (2001) no. 1, 125–141 and 143–159.
- [8] M. Feigenbaum, The universal metric properties of non-linear transformations, *J. Stat. Phys.*, **21** (1979), 669–706.
- [9] F. Gardiner and D. Sullivan, Symmetric and quasisymmetric structures on a closed curve, *Amer. J. of Math.*, **114** (1992) no. 4, 683–736.
- [10] Y. Jiang, Generalized Ulam-von Neuman Transformations, Ph.D Thesis (1990) CUNY Graduate Center.
- [11] Y. Jiang, Renormalization and Geometry in One-Dimensional and Complex Dynamics, World Scientific, Singapore-New Jersey-London-Hong Kong 10 (1996).
- [12] Y. Jiang, Smooth classification of geometrically finite one-dimensional maps, *Trans. of the Amer. Math. Soc.*, **348**(6) (1996) 2391–2412.
- [13] Y. Jiang, On rigidity of one-dimensional maps, *Contemporary Mathematics, AMS series*, **211** (1997) 319–341.
- [14] Y. Jiang, Differentiable rigidity and smooth conjugacy, *Annales Academiæ Scientiarum Fennicæ Mathematica*, **30** (2005) 361–383.
- [15] Y. Jiang, Geometry of geometrically finite one-dimensional mappings, *Commun. in Math. Phys.*, **156** (1993) 639–647.
- [16] Y. Jiang, Metric invariants in dynamical systems. *Journal of Dynamics and Differentiable Equations*, **17** (2005) no. 1, 51–71.
- [17] Y. Jiang, A Proof of existence and simplicity of a maximal eigenvalue for Ruelle-Perron-Frobenius operators, *Letters in Mathematical Physics*, **48** (1999) 211–219.
- [18] Y. Jiang and D. Ruelle, Analyticity of the susceptibility function for unimodal Markovian maps of the interval, *Nonlinearity*, **18** (2005) 2447–2453.
- [19] M. Keane, Strongly mixing g -measures, *Invent. Math.*, **16** (1972) 309–324.
- [20] F. Ledrappier, Principe variationnel et systèmes symboliqué, *Comm. Math. Phys.*, **33** (1973) 119–128.
- [21] A. Pinto and D. Sullivan, Dynamical systems applied to asymptotic geometry, preprint.

- [22] A. Quas, Non-ergodicity for C^1 expanding maps and g -measures, *Erg. Th. & Dynam. Sys.*, **16** (1996) no. 3, 531–543.
- [23] D. Ruelle, Differentiating the absolutely continuous invariant measure of an interval map f with respect to f , *Commun. Math. Phys.*, **258** (2005) 445–453.
- [24] M. Shub, Endomorphisms of compact differentiable manifolds, *Amer. J. Math.*, **91** (1969) 129–155.
- [25] M. Shub and D. Sullivan, Expanding endomorphisms of the circle revisited, *Ergod. Th & Dynam. Sys.*, **5** (1985) 285–289.
- [26] D. Sullivan, Differentiable structure on fractal like sets determined by intrinsic scaling functions on dual Cantor sets, *The Proceedings of Symposia in Pure Mathematics*, **48** (1988).
- [27] D. Sullivan, Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers, *Topological Methods in Modern Mathematics*, (1993) 543–563.
- [28] P. Walters, Ruelle's operator theorem and g -measures, *Trans. Amer. Math. Soc.*, **214** (1975) 375–387.