Douady-Earle section, holomorphic motions, and some applications

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For Professor Clifford Earle on his 75th birthday

Abstract. We review several applications of Douady-Earle section to holomorphic motions over infinite dimensional parameter spaces. Using Douady-Earle section we study group-equivariant extensions of holomorphic motions. We also discuss the relationship between extending holomorphic motions and lifting holomorphic maps. Finally, we discuss several applications of holomorphic motions in complex analysis.

Introduction

This is a survey article on holomorphic motions and Teichmüller spaces, and some applications of holomorphic motions in complex analysis. Our paper is divided into two parts. In Part 1, we study the applications of Douady-Earle section to holomorphic motions over infinite dimensional parameter spaces. It is well-known that holomorphic motions were first introduced in the study of the dynamics of rational maps in the paper [30]. Since its inception, a fundamental topic in this subject has been about extending holomorphic motions. In their famous paper [39], Sullivan and Thurston asked two important questions on extending holomorphic motions over the open unit disk. We use Douady-Earle section to study these two questions over infinite dimensional parameter spaces. There is an intimate relationship between extending holomorphic motions and lifting holomorphic maps into appropriate Teichmüller spaces, first observed by Bers and Royden in [5]. We study that in the fullest generality, which is another application of Douady-Earle section. In particular, we discuss some new results on group-equivariant extensions of holomorphic motions. In Part 2, we focus on holomorphic motions over the open unit disk to study some problems in complex analysis. We first review a proof of a theorem on gluing holomorphic germs on the Riemann sphere. Using the same idea, we give outlines of new proofs of König’s theorem, Böttcher’s theorem, and...
their generalizations. Finally, we use holomorphic motions to discuss a proof of quasiconformal rigidity for parabolic germs.

Throughout this paper we will use \( \mathbb{C} \) for the complex plane, \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) for the Riemann sphere, and \( \Delta \) for the open unit disk \( \{z \in \mathbb{C} : |z| < 1\} \).

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Part 1. Applications of Douady-Earle section in holomorphic motions

One of the fundamental contributions of Clifford Earle is the paper he wrote with Adrien Douady (see [8]). That paper has found many applications in Teichmüller theory; some examples are [9], [13], [14], and [17]. In Part 1 of our paper, we highlight several applications in the study of holomorphic motions. The key point is to study the Douady-Earle section for the Teichmüller space of a closed set in \( \hat{\mathbb{C}} \).

1. Basic definitions

In this section we review some important definitions.

Definition 1.1. Let \( V \) be a connected complex manifold, and let \( E \) be a subset of \( \hat{\mathbb{C}} \). A holomorphic family of injections of \( E \) over \( V \) is a family of maps \( \{\phi_x\}_{x \in V} \) that has the following properties:

(i) for each \( x \) in \( V \), the map \( \phi_x : E \to \hat{\mathbb{C}} \) is an injection, and,
(ii) for each \( z \) in \( E \), the map \( x \mapsto \phi_x(z) \) is holomorphic.

It is convenient to define \( \phi : V \times E \to \hat{\mathbb{C}} \) as the map \( \phi(x, z) := \phi_x(z) \) for all \( (x, z) \in V \times E \).

If \( V \) is a connected complex manifold with a basepoint \( x_0 \), then a holomorphic motion of \( E \) over \( V \) is a holomorphic family of injections such that \( \phi(x_0, z) = z \) for all \( z \) in \( E \).

We say that \( V \) is the parameter space of the holomorphic motion \( \phi \).

We will always assume that 0, 1, and \( \infty \) belong to \( E \) and that \( \phi \) is normalized, i.e. 0, 1, and \( \infty \) are fixed points of the map \( \phi(x, \cdot) \) for every \( x \) in \( V \).

We next review the definition of quasiconformal motions introduced in [39]. Let \( V \) be a connected Hausdorff space with a basepoint \( x_0 \), and let \( E \) be any subset in \( \hat{\mathbb{C}} \).

For any map \( \phi : V \times E \to \hat{\mathbb{C}} \), \( x \) in \( V \), and any quadruplet \( a, b, c, d \) of points in \( E \), let \( \phi_x(a, b, c, d) \) denote the cross-ratio of the values \( \phi(x, a), \phi(x, b), \phi(x, c), \) and \( \phi(x, d) \). As in Definition 1.1, we often write \( \phi(x, z) \) as \( \phi_x(z) \) for \( x \) in \( V \) and \( z \) in \( E \). So we have:

\[
\phi_x(a, b, c, d) = \frac{(\phi_x(a) - \phi_x(c))(\phi_x(b) - \phi_x(d))}{(\phi_x(a) - \phi_x(d))(\phi_x(b) - \phi_x(c))}
\]

for each \( x \) in \( V \).

It is clear that condition (i) in Definition 1.1 holds if and only if \( \phi_x(a, b, c, d) \) is a well-defined point in the thrice-punctured sphere \( \hat{\mathbb{C}} \setminus \{0, 1, \infty\} \) for all \( x \) in \( V \) and all quadruplets \( a, b, c, d \) of distinct points in \( E \).
Let $\rho$ be the Poincaré metric on $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$. In their paper \cite{39}, Sullivan and Thurston introduced the following definition.

**Definition 1.2.** A quasiconformal motion of $E$ over $V$ is a map $\phi : V \times E \to \hat{\mathbb{C}}$ such that:

(i) $\phi(x_0, z) = z$ for all $z$ in $E$,

(ii) for each $x$ in $V$, the map $\phi_x : E \to \hat{\mathbb{C}}$ is injective, and

(iii) given any $x$ in $V$ and any $\epsilon > 0$, there exists a neighborhood $U_x$ of $x$ such that for any quadruplet of distinct points $a, b, c, d$ in $E$, we have

$$\rho(\phi_y(a, b, c, d), \phi_{y'}(a, b, c, d)) < \epsilon \quad \text{for all } y \text{ and } y' \text{ in } U_x.$$ 

We will always assume that $\phi$ is a normalized quasiconformal motion; i.e. $0, 1, \infty$ belong to $E$ and are fixed points of the map $\phi_x(\cdot)$ for every $x$ in $V$.

**Definition 1.3.** Let $V$ be a path-connected Hausdorff space with a basepoint $x_0$. A normalized continuous motion of $\hat{\mathbb{C}}$ over $V$ is a continuous map $\phi : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that:

(i) $\phi(x_0, z) = z$ for all $z$ in $\hat{\mathbb{C}}$, and

(ii) for each $x$ in $V$, the map $\phi_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a homeomorphism that fixes $0, 1, \infty$.

**Remark 1.4.** Suppose $\phi : V \times E \to \hat{\mathbb{C}}$ is a holomorphic motion (we assume that $V$ is Kobayashi-hyperbolic). For any quadruplet of points $a, b, c, d$ in $E$, the map $x \mapsto \phi_x(a, b, c, d)$ from $V$ into $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is holomorphic. Therefore, it is distance-decreasing with respect to the Kobayashi metrics on $V$ and $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$. It easily follows that $\phi$ is also a quasiconformal motion.

**Remark 1.5.** Let $V$ and $W$ be connected Hausdorff spaces with basepoints, and $f$ be a basepoint preserving continuous map of $W$ into $V$. If $\phi$ is a quasiconformal motion of $E$ over $V$ its pullback by $f$ is the quasiconformal motion

$$f^*(\phi)(x, z) = \phi(f(x), z) \quad \forall (x, z) \in W \times E$$

of $E$ over $W$.

**Remark 1.6.** If $V$ and $W$ are path-connected Hausdorff spaces with basepoints, $f$ is a basepoint preserving continuous map of $W$ into $V$, and $\phi : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a continuous motion, then its pullback $f^*(\phi) : W \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (defined as in Equation (1.2) with $E = \hat{\mathbb{C}}$) is a continuous motion.

**Remark 1.7.** If $V$ and $W$ are connected complex manifolds with basepoints, $f$ is a basepoint preserving holomorphic map of $W$ into $V$, and $\phi : V \times E \to \hat{\mathbb{C}}$ is a holomorphic motion, then its pullback $f^*(\phi) : W \times E \to \hat{\mathbb{C}}$ (defined as in Equation (1.2)) is a holomorphic motion.

**Definition 1.8.** Let $\phi : V \times E \to \hat{\mathbb{C}}$ be a holomorphic motion, where $E$ is a closed set in $\hat{\mathbb{C}}$ ($0, 1, \infty$ belong to $E$). Let $G$ be a group of Möbius transformations, such that $E$ is invariant under $G$ (which means $g(E) = E$ for all $g$ in $G$). We say that $\phi$ is $G$-equivariant if and only if for each $g$ in $G$, and $x$ in $V$, there exists a Möbius transformation $\theta_x(g)$ such that:

$$\phi(x, g(z)) = (\theta_x(g))(\phi(x, z)) \quad \text{for all } z \text{ in } E.$$
2. Teichmüller space of a closed set in \( \hat{\mathbb{C}} \)

Henceforth we will always assume that \( E \) is a closed subset of \( \hat{\mathbb{C}} \), and \( 0, 1, \infty \) belong to \( E \). The Teichmüller space of \( E \), denoted by \( T(E) \), is intimately related with holomorphic motions of \( E \). In this section we review some basic properties of \( T(E) \). This section is mainly expository. For further details the reader is referred to \[15\].

Recall that a homeomorphism of \( \hat{\mathbb{C}} \) is called normalized if it fixes the points 0, 1, and \( \infty \).

2.1. Definition. Two normalized quasiconformal self-mappings \( f \) and \( g \) of \( \hat{\mathbb{C}} \) are said to be \( E \)-equivalent if and only if \( f^{-1} \circ g \) is isotopic to the identity rel \( E \). The Teichmüller space \( T(E) \) is the set of all \( E \)-equivalence classes of normalized quasiconformal self-mappings of \( \hat{\mathbb{C}} \).

The basepoint of \( T(E) \) is the \( E \)-equivalence class of the identity map.

2.2. \( T(E) \) as a complex manifold. Let \( M(\mathbb{C}) \) be the open unit ball of the complex Banach space \( L^\infty(\mathbb{C}) \). Each \( \mu \) in \( M(\mathbb{C}) \) is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism \( w^\mu \) of \( \hat{\mathbb{C}} \) onto itself. The basepoint of \( M(\mathbb{C}) \) is the zero function.

We define the quotient map

\[
P_E : M(\mathbb{C}) \to T(E)
\]

by setting \( P_E(\mu) \) equal to the \( E \)-equivalence class of \( w^\mu \), written as \([w^\mu]_E\). Clearly, \( P_E \) maps the basepoint of \( M(\mathbb{C}) \) to the basepoint of \( T(E) \).

In his doctoral dissertation \([29]\), G. Lieb proved that \( T(E) \) is a complex Banach manifold such that the projection map \( P_E : M(\mathbb{C}) \to T(E) \) is a holomorphic split submersion. See \S 2.3 for more details.

2.3. Lieb’s isomorphism theorem. In what follows, we shall assume that \( E \) is infinite, and has a nonempty complement \( E^c = \hat{\mathbb{C}} \setminus E \). Let \( \{X_n\} \) be the connected components of \( E^c \). Each \( X_n \) is a hyperbolic Riemann surface; let \( \text{Teich}(X_n) \) denote its Teichmüller space. (For standard facts of Teichmüller theory, see any of the following texts: \[18, 20, 22, 37\].) If the number of components is finite, \( \text{Teich}(E^c) \) is, by definition, the cartesian product of the spaces \( \text{Teich}(X_n) \). If there are infinitely many components, we define the product Teichmüller space \( \text{Teich}(E^c) \) as follows.

For each \( n \geq 1 \), let \( 0_n \) be the basepoint of the Teichmüller space \( \text{Teich}(X_n) \), and let \( d_n \) be the Teichmüller metric on \( \text{Teich}(X_n) \). As usual, let \( M(X_n) \) denote the open unit ball of the complex Banach space \( L^\infty(X_n) \), for each \( n \geq 1 \). By definition, the product Teichmüller space \( \text{Teich}(E^c) \) is the set of sequences \( t = \{t_n\}_{n=1}^\infty \) such that \( t_n \) belongs to \( \text{Teich}(X_n) \) for each \( n \geq 1 \), and

\[
\sup\{d_n(0_n, t_n) : n \geq 1\} < \infty.
\]

The basepoint of \( \text{Teich}(E^c) \) is the sequence \( 0 = \{0_n\} \) whose \( n \)th term is the basepoint of \( \text{Teich}(X_n) \).

Let \( L^\infty(E^c) \) be the complex Banach space of sequences \( \mu = \{\mu_n\} \) such that \( \mu_n \) belongs to \( L^\infty(X_n) \) for each \( n \geq 1 \) and the norm \( \|\mu\|_\infty = \sup\{\|\mu_n\|_\infty : n \geq 1\} \)
is finite. Let \( M(E^c) \) be the open unit ball of \( L^\infty(E^c) \). Note that if \( \mu \) belongs to \( M(E^c) \), then \( \mu_n \) belongs to \( M(X_n) \) for all \( n \geq 1 \) (but the converse is false).

For each \( n \geq 1 \), let \( \Phi_n \) be the standard projection from \( M(X_n) \) to \( \text{Teich}(X_n) \) (see, for example, [22] or [37] for the basic definitions). For \( \mu \) in \( M(E^c) \), let \( \Phi(\mu) \) be the sequence \( \{\Phi_n(\mu_n)\} \). It is easy to see that \( \Phi(\mu) \) belongs to \( \text{Teich}(E^c) \), and the map \( \Phi \) is surjective. We call \( \Phi \) the standard projection of \( M(E^c) \) onto \( \text{Teich}(E^c) \).

In [29] it was shown that \( \text{Teich}(E^c) \) is a complex Banach manifold such that the map \( \Phi \) is a holomorphic split submersion (see also [15] or [32]).

Let \( M(E) \) be the open unit ball in \( L^\infty(E) \). The product \( \text{Teich}(E^c) \times M(E) \) is a complex Banach manifold. (If \( E \) has zero area, then \( M(E) \) contains only one point, and \( \text{Teich}(E^c) \times M(E) \) is then isomorphic to \( \text{Teich}(E^c) \).

For \( \mu \) in \( L^\infty(C) \), let \( \mu|E^c \) and \( \mu|E \) be the restrictions of \( \mu \) to \( E^c \) and \( E \) respectively. We define the projection map \( P_E \) from \( M(C) \) to \( \text{Teich}(E^c) \times M(E) \) by the formula:

\[
\tilde{P}_E(\mu) = (\Phi(\mu|E^c), \mu|E) \quad \text{for all } \mu \in M(C).
\]

**Proposition 2.1** (Lieb’s isomorphism theorem). For all \( \mu \) and \( \nu \) in \( M(C) \) we have \( P_E(\mu) = P_E(\nu) \) if and only if \( \tilde{P}_E(\mu) = \tilde{P}_E(\nu) \).


**Remark 2.2.** It follows from Proposition 2.1, that there is a well-defined bijection \( \Theta : T(E) \rightarrow \text{Teich}(E^c) \times M(E) \) such that \( \Theta \circ P_E = \tilde{P}_E \), and \( T(E) \) has a unique complex manifold structure such that \( P_E \) is a holomorphic split submersion.

**Remark 2.3.** Let \( E \) be a finite set. Its complement \( \Omega = \widehat{C} \setminus E \) is the Riemann sphere with punctures at the points of \( E \). It follows from Remark 2.2, that \( T(E) \) is biholomorphic to the classical Teichmüller space \( \text{Teich}(\Omega) \). This canonical identification will be useful in our paper. The reader is referred to [18], [20], [22], or [37] for standard facts on classical Teichmüller theory.

**Remark 2.4.** When \( E = \widehat{C}, T(\widehat{C}) \) is biholomorphically identified with \( M(C) \).

**2.4. Forgetful maps.** Let \( E \) and \( \widehat{E} \) be two closed sets such that \( E \subset \widehat{E} \); as usual, \( 0, 1, \) and \( \infty \) belong to both \( E \) and \( \widehat{E} \). If \( \mu \) is in \( M(C) \), then the \( \widehat{E} \)-equivalence class of \( w^\mu \) is contained in the \( E \)-equivalence class of \( w^\mu \). Therefore, there is a well-defined ‘forgetful map’ \( P_{\widehat{E}, E} \) from \( T(\widehat{E}) \) to \( T(E) \) such that \( P_E = P_{\widehat{E}, E} \circ P_{\widehat{E}} \). It is easy to see that this forgetful map is a basepoint preserving holomorphic split submersion.

**2.5. Teichmüller metric on \( T(E) \).** The Teichmüller distance \( d_M(\mu, \nu) \) between \( \mu \) and \( \nu \) on \( M(C) \) is defined by

\[
d_M(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \mu \nu} \right\|_\infty.
\]

The Teichmüller metric on \( T(E) \) is the quotient metric

\[
d_{T(E)}(s, t) = \inf \{d_M(\mu, \nu) : \mu \text{ and } \nu \in M(C), P_E(\mu) = s \text{ and } P_E(\nu) = t \}.
\]

It is proved in [15] that the Teichmüller metric on \( T(E) \) is the same as its Kobayashi metric.
2.6. Changing the basepoint. Let $w$ be a normalized quasiconformal self-mapping of $\hat{\mathbb{C}}$, and let $\hat{E} = w(E)$. By definition, the allowable map $g : T(\hat{E}) \to T(E)$ maps the $\hat{E}$-equivalence class of $f$ to the $E$-equivalence class of $f \circ w$, for every normalized quasiconformal self-mapping $f$ of $\hat{\mathbb{C}}$.

**Proposition 2.5.** The allowable map $g : T(\hat{E}) \to T(E)$ is biholomorphic. If $\mu$ is the Beltrami coefficient of $w$, then $g$ maps the basepoint of $T(\hat{E})$ to the point $P_E(\mu)$ in $T(E)$.

3. Douady-Earle section

3.1. Background. We give a brief background of the *Douady-Earle section* (sometimes called the *barycentric section*) for classical Teichmüller spaces; see [8] for details. Let $M(\Delta)$ denote the open unit ball of the complex Banach space $L^\infty(\Delta, \mathbb{C})$. For each $\mu \in M(\Delta)$, there exists a unique quasiconformal map $f^\mu$ of $\Delta$ onto itself fixing the points 1, $i$, and $-1$. Let $\varphi^\mu$ be the restriction of $f^\mu$ to the unit circle $S^1$. Let $ex(\varphi^\mu) : \Delta \to \Delta$ denote the Douady-Earle extension of $\varphi^\mu$. By Theorem 2 in [8], $ex(\varphi^\mu)$ is quasiconformal and so its complex dilatation belongs to $M(\Delta)$. That determines a map $\sigma : M(\Delta) \to M(\Delta)$ that sends $\mu$ to the Beltrami coefficient of $ex(\varphi^\mu)$; see §6 of [8] for details. In §6 of [8] it is shown that the map $\sigma$ is conformally natural.

Let $\Gamma$ be a Fuchsian group, $M(\Gamma)$ be the $\Gamma$-invariant elements of $M(\Delta)$, $Teich(\Gamma)$ be the Teichmüller space of $\Gamma$ and $\pi : M(\Gamma) \to Teich(\Gamma)$ be the usual projection; see §7 of [8] for the details. In Lemma 5 of [8] it is shown that:

(i) $\sigma$ maps $M(\Gamma)$ into itself,
(ii) there is a continuous map $S : Teich(\Gamma) \to M(\Gamma)$ such that $S \circ \pi = \sigma : M(\Gamma) \to M(\Gamma)$, and
(iii) $\pi \circ \sigma = \pi : M(\Gamma) \to Teich(\Gamma)$.

We call the continuous map $S : Teich(\Gamma) \to M(\Gamma)$ the *Douady-Earle section* of $\pi$ for the Teichmüller space $Teich(\Gamma)$.

3.2. Douady-Earle section for product Teichmüller spaces. We now study Douady-Earle section for the product Teichmüller space $Teich(E^c)$ defined in §2.3.

**Proposition 3.1.** There is a continuous basepoint preserving map $\hat{s}$ from $Teich(E^c)$ to $M(E^c)$ such that $\Phi \circ \hat{s}$ is the identity map on $Teich(E^c)$.

**Proof.** Let $\tau \in Teich(E^c)$ where $\tau = \Phi(\mu)$ for $\mu \in M(E^c)$. Following our discussion in §2.3, $\tau = \{\tau_n\}$ where $\tau_n = \Phi_n(\mu_n) \in Teich(X_n)$ where $\mu_n \in M(X_n)$ for each $n \geq 1$. By Lemma 5 in [8] (also the above discussion), for each $n \geq 1$, there is a continuous basepoint preserving map $\hat{s}_n$ from $Teich(X_n)$ into $M(X_n)$ such that $\Phi_n \circ \hat{s}_n$ is the identity map on $Teich(X_n)$. Let $\sigma_n$ denote the continuous map $\hat{s}_n \circ \Phi_n$ from $M(X_n)$ to itself.

Since $\tau \in Teich(E^c)$, we have by (2.1), $\sup\{d_n(0_n, \tau_n) : n \geq 1\} < \infty$. Let $\|\mu_n\|_\infty \leq k$ for all $n \geq 1$. Then, by Proposition 7 in [8], there exists $0 \leq c(k) < 1$, where $c(k)$ depends only on $k$ and is independent of $n$, such that $\|\sigma_n(\mu_n)\|_\infty \leq c(k)$ for all $n \geq 1$.

Define $\sigma(\mu) := \{\sigma_n(\mu_n)\}$. It is easy to check that $\sigma$ is a continuous map of $M(E^c)$ into itself (see, for example, Proposition 7.11 in [15]). Furthermore, there exists a
unique well defined map \( \hat{s} \) from \( \text{Teich}(E^c) \) to \( M(E^c) \) such that \( \sigma = \hat{s} \circ \Phi \). Since \( \sigma \) is continuous and \( \Phi \) is a holomorphic split submersion, it follows that \( \hat{s} \) is continuous. It is easy to check that \( \Phi \circ \hat{s} \) is the identity map on \( \text{Teich}(E^c) \). \( \square \)

**Definition 3.2.** The map \( \hat{s} \) from \( \text{Teich}(E^c) \) to \( M(E^c) \) is called the **Douady-Earle section** of \( \Phi \) for the product Teichmüller space \( \text{Teich}(E^c) \).

### 3.3. **Douady-Earle section for the Teichmüller space** \( T(E) \). Finally, we introduce the Douady-Earle section for the Teichmüller space of the closed set \( T(E) \) defined in \( \S \S 2.1 \) and \( 2.2 \).

**Proposition 3.3.** There is a continuous basepoint preserving map \( s \) from \( T(E) \) to \( M(\mathbb{C}) \) such that \( P_E \circ s \) is the identity map on \( T(E) \).

**Proof.** By Proposition 3.1, there is a continuous basepoint preserving map \( \hat{s} \) from \( \text{Teich}(E^c) \) to \( M(E^c) \) such that \( \Phi \circ \hat{s} \) is the identity map on \( \text{Teich}(E^c) \). Let \( \hat{s} \) be the map from \( \text{Teich}(E^c) \times M(E) \) to \( M(\mathbb{C}) \) such that \( \hat{s}(\tau, \nu) = \hat{s}(\tau) \) equals \( \hat{s}(\tau) \) in \( E^c \) and equals \( \nu \) in \( E \) for each \( (\tau, \nu) \) in \( \text{Teich}(E^c) \times M(E) \). Clearly, \( \hat{P}_E \circ \hat{s} \) is the identity map on \( \text{Teich}(E^c) \times M(E) \). We define \( s = \hat{s} \circ \Theta \), where \( \Theta \) is the biholomorphic map from \( T(E) \) to \( \text{Teich}(E^c) \times M(E) \) given in Remark 2.2. It follows that \( s : T(E) \to M(\mathbb{C}) \) is a continuous basepoint preserving map such that \( P_E \circ s \) is the identity map on \( T(E) \). \( \square \)

**Definition 3.4.** The map \( s \) from \( T(E) \) to \( M(\mathbb{C}) \) is called the **Douady-Earle section** of \( P_E \) for the Teichmüller space \( T(E) \).

Since \( M(\mathbb{C}) \) is contractible, we have the following

**Corollary 3.5.** The Teichmüller space \( T(E) \) is contractible.

**Remark 3.6.** Let \( t \in T(E) \) and \( P_E(\mu) = t \) for \( \mu \in M(\mathbb{C}) \). By Remark 2.2, we have

\[
\Theta(t) = \Theta(P_E(\mu)) = \hat{P}_E(\mu) = (\Phi(\mu|E^c), \mu|E).
\]

Let \( \Phi(\mu|E^c) \) be denoted by \( \tau \). By Proposition 3.3, \( s(t) = \hat{s}(\Theta(t)) \), which equals \( \hat{s}(\tau) \) on \( E^c \), and equals \( \mu \) on \( E \). By Proposition 3.1, we have \( \hat{s}(\tau) = \hat{s}(\Phi(\mu|E^c)) = \sigma(\mu) \) on \( E^c \). Thus, for \( t = P_E(\mu) \) in \( T(E) \), \( s(t) \) equals \( \sigma(\mu) \) on \( E^c \) and equals \( \mu \) on \( E \). If \( ||\mu||_\infty = k \), then \( ||s(t)||_\infty \leq \max(k, c(k)) \) where \( c(k) \) depends only on \( k \), and \( 0 \leq c(k) < 1 \).

**Corollary 3.7.** For \( t \) in \( T(E) \), \( ||s(t)||_\infty \) is bounded above by a number between 0 and 1, that depends only on \( d_{T(E)}(0,t) \).

**Proof.** Given \( t \) in \( T(E) \), choose an extremal \( \mu \) in \( M(\mathbb{C}) \) so that \( P_E(\mu) = t \). Then

\[
d_{T(E)}(0,t) = \frac{1}{2} \log K \text{ where } K = \frac{1+k}{1-k} \text{ and } k = ||\mu||_\infty.
\]

By Remark 3.6 we have \( ||s(t)||_\infty \leq \max(c(k), k) \). \( \square \)

### 3.4. **Conformal naturality of the Douady-Earle section for** \( T(E) \). Let \( G \) be a group of Möbius transformations that map \( E \) onto itself. For each \( g \in G \), there exists a biholomorphic map \( \rho_g : T(E) \to T(E) \) which is defined as follows: for each \( \mu \) in \( M(\mathbb{C}) \),

\[
(3.1) \quad \rho_g([w^\mu]_E) = [\hat{g} \circ w^\mu \circ g^{-1}]_E
\]
where \( \hat{g} \) is the unique Möbius transformation such that \( \hat{g} \circ w^\mu \circ g^{-1} \) fixes the points 0, 1, and \( \infty \). See Remark 3.4 in [10] for a discussion on “geometric isomorphisms” of \( T(E) \).

It follows from the definition that, for each \( g \) in \( G \), \( \rho_g \) is basepoint preserving.

**Definition 3.8.** We define \( M(\mathbb{C})^G \) and \( T(E)^G \) as follows:

\[
M(\mathbb{C})^G := \{ \mu \in M(\mathbb{C}) : (\mu \circ g) \frac{g'}{g} = \mu \text{ a.e. on } \mathbb{C} \text{ for each } g \in G \}
\]

and

\[
T(E)^G := \{ t \in T(E) : \rho_g(t) = t \text{ for each } g \in G \}.
\]

The next proposition shows the conformal naturality of the Douady-Earle section \( s : T(E) \to M(\mathbb{C}) \); (see [26]). We include the proof for the reader’s convenience.

**Proposition 3.9.** If \( t \in T(E)^G \), then \( s(t) \in M(\mathbb{C})^G \).

**Proof.** Let \( t \in T(E)^G \) where \( t = [w^\mu]_E \). Then, \( \rho_g(t) = t \), which, by (3.1), implies that

\[
\rho_g([w^\mu]_E) = [\hat{g} \circ w^\mu \circ g^{-1}]_E = [w^\mu]_E
\]

where \( \hat{g} \) is the unique Möbius transformation such that \( \hat{g} \circ w^\mu \circ g^{-1} \) fixes 0, 1, and \( \infty \). Let \( w^\mu := \hat{g} \circ w^\mu \circ g^{-1} \). So, we have

\[
P_E(\mu) = P_E(\bar{\mu}) \text{ where } \mu = (\bar{\mu} \circ g) \frac{g'}{g}.
\]

By Lemma 7.16 in [15], (see also the first part in §7.10 of [15]), it follows that \( \mu = \bar{\mu} \) almost everywhere in \( E \).

Let \( \{X_n\} \) be the set of connected components of \( E^c \) and let \( \mu|X_n := \mu_n \), \( \bar{\mu}|X_n := \bar{\mu}_n \) for each \( n \geq 1 \). Since \( P_E(\mu) = P_E(\bar{\mu}) \), we have \( \Phi_n(\mu_n) = \Phi_n(\bar{\mu}_n) \) for each \( n \geq 1 \). It follows that \( \sigma_n(\mu_n) = \sigma_n(\bar{\mu}_n) \) for each \( n \geq 1 \).

Now, for each \( n \geq 1 \), we have

\[
\sigma_n(\bar{\mu}_n) = \sigma_n(\mu_n) = \sigma_n((\bar{\mu}_n \circ g) \frac{g'}{g}) = \left( \sigma_n(\bar{\mu}_n) \circ g \right) \frac{g'}{g'}
\]

where the last equality holds because of the conformal naturality of the map \( \sigma_n \) (see the Corollary to Lemma 4 in [8]). (We mean that the values of the last two terms are equal.) Since \( \sigma_n(\mu_n) = \sigma_n(\bar{\mu}_n) \) for each \( n \geq 1 \), we have

\[
\left( \sigma_n(\mu_n) \circ g \right) \frac{g'}{g'} = \sigma_n(\mu_n) \text{ for each } n \geq 1.
\]

Recall that \( \sigma(\mu|E^c) = \{ \sigma_n(\mu_n) \} \). Therefore, we have

\[
\left( \sigma(\mu|E^c) \circ g \right) \frac{g'}{g'} = \sigma(\mu|E^c).
\]

Finally, since \( s(t) \) equals \( \mu \) on \( E \) and equals \( \sigma(\mu) \) on \( E^c \) (by Remark 3.4), we conclude that \( s(t) \) belongs to \( M(\mathbb{C})^G \). \( \square \)

**Remark 3.10.** More details and deeper properties of Douady-Earle section, with applications, will be given in the forthcoming paper [16].
4. Universal holomorphic motion of a closed set in \( \hat{C} \)

**Definition 4.1.** The universal holomorphic motion \( \Psi_E : T(E) \times E \to \hat{C} \) is defined as follows:

\[
\Psi_E(P_E(\mu), z) = w^\mu(z) \text{ for } \mu \in M(\mathbb{C}) \text{ and } z \in E.
\]

It is clear from the definition of \( P_E \) in §2.2 that the map \( \Psi_E \) is well-defined. It is a holomorphic motion because \( P_E \) is a holomorphic split submersion and \( \mu \mapsto w^\mu(z) \) is a holomorphic map from \( M(\mathbb{C}) \) to \( \hat{C} \) for every fixed \( z \) in \( \hat{C} \), by Theorem 11 in [1].

This holomorphic motion is “universal” in the following sense:

**Theorem 4.2.** Let \( \phi : V \times E \to \hat{C} \) be a holomorphic motion. If \( V \) is a simply connected complex Banach manifold with a basepoint, there is a unique basepoint preserving holomorphic map \( f : V \to T(E) \) such that \( f^*(\Psi_E) = \phi \).

For a proof see Section 14 in [32].

**Remark 4.3.** By Remark 2.4, when \( E = \hat{C} \), \( T(\hat{C}) \) is biholomorphically identified with \( M(\mathbb{C}) \). We have the holomorphic motion \( \Psi_{\hat{C}} : M(\mathbb{C}) \times \hat{C} \to \hat{C} \) defined as follows:

\[
\Psi_{\hat{C}}(\mu, z) = w^\mu(z) \quad \text{for all } (\mu, z) \in M(\mathbb{C}) \times \hat{C}.
\]

This is the universal holomorphic motion of \( \hat{C} \). In [9], Earle proved:

**Theorem 4.4.** If \( \phi : V \times \hat{C} \to \hat{C} \) is a holomorphic motion, where \( V \) is a connected complex Banach manifold with a basepoint, there exists a unique basepoint preserving holomorphic map \( f : V \to M(\mathbb{C}) \) such that \( f^*(\Psi_{\hat{C}}) = \phi \).

### 4.1. Two lemmas

The following two lemmas were proved in [32]. We include the proofs for the convenience of the reader. In particular, Lemma 4.6 is an important application of the Douady-Earle section (given in Definition 3.4). In fact, this lemma is a main step in proving the “universal property” in Theorem 4.2.

Let \( B \) be a path-connected Hausdorff space and \( \mathcal{H}(\hat{C}) \) be the group of homeomorphisms of \( \hat{C} \) onto itself, with the topology of uniform convergence in the spherical metric. With this topology, \( \mathcal{H}(\hat{C}) \) is a topological group. As usual, \( E \) is a closed set in \( \hat{C} \), and 0, 1, \( \infty \) belong to \( E \).

**Lemma 4.5.** Let \( h : B \to \mathcal{H}(\hat{C}) \) be a continuous map such that \( h(t)(z) = z \) for all \( t \) in \( B \), and for all \( z \) in \( E \). If \( h(t_0) \) is isotopic to the identity rel \( E \) for some fixed \( t_0 \) in \( B \), then \( h(t) \) is isotopic to the identity rel \( E \) for all \( t \) in \( B \).

**Proof.** Let \( t \) be any point in \( B \). Choose a path \( \gamma : [0, 1] \to B \) such that \( \gamma(0) = t_0 \) and \( \gamma(1) = t \). The map \( (s, z) \mapsto h(\gamma(s))(z) \) from \( [0, 1] \times \hat{C} \) to \( \hat{C} \) is an isotopy rel \( E \) between \( h(t_0) \) and \( h(t) \).

**Lemma 4.6.** Let \( f : B \to T(E) \) and \( g : B \to T(E) \) be two continuous maps such that:

1. \( \Psi_E(f(t), z) = \Psi_E(g(t), z) \) for all \( z \) in \( E \) and
2. \( f(t_0) = g(t_0) \) for some \( t_0 \) in \( B \).

Then \( f(t) = g(t) \) for all \( t \) in \( B \).
PROOF. By Proposition 3.3, there exists a basepoint preserving continuous map \( s : T(E) \to M(\mathbb{C}) \) such that \( P_E \circ s \) is the identity map on \( T(E) \). For each \( t \) in \( B \), define \( \mu(t) = s(f(t)) \) and \( \nu(t) = s(g(t)) \). Note that \( f(t) = g(t) \) if and only if the quasiconformal map \( h(t) = (w^{\mu(t)})^{-1} \circ w^{\nu(t)} \) is isotopic to the identity rel \( E \).

Since \( \mu \) and \( \nu \) are continuous maps of \( B \) into \( M(\mathbb{C}) \) and \( \mathcal{H}(\hat{\mathbb{C}}) \) is a topological group, Lemma 17 of [1] implies that \( h \) is a continuous map of \( B \) into \( \mathcal{H}(\hat{\mathbb{C}}) \).

Condition (i) and Definition (4.1) imply that
\[
w^{\mu(t)}(z) = \Psi_E(f(t), z) = \Psi_E(g(t), z) = w^{\nu(t)}(z)
\]
for all \( t \) in \( B \) and \( z \) in \( E \). Therefore \( h(t) \) fixes the set \( E \) pointwise for each \( t \) in \( B \).

By condition (ii), \( h(t_0) \) is isotopic to the identity rel \( E \). Hence, by Lemma 4.5, \( h(t) \) is isotopic to the identity rel \( E \) for all \( t \) in \( B \).

Let \( G \) be a group of Möbius transformations that map the closed set \( E \) onto itself (as usual, 0, 1, and \( \infty \) belong to \( E \)). Recall the definitions of \( G \)-equivariant holomorphic motion, and \( T(E)^G \) from Definitions 1.8 and 3.8.

PROPOSITION 4.7. Let \( \phi : V \times E \to \hat{\mathbb{C}} \) be a holomorphic motion, where \( V \) is a connected complex Banach manifold with a basepoint. Suppose there exists a basepoint preserving holomorphic map \( f : V \to T(E) \) such that \( f^*(\Psi_E) = \phi \). Then \( \phi : V \times E \to \hat{\mathbb{C}} \) is \( G \)-equivariant if and only if \( f \) maps \( V \) into \( T(E)^G \).

The proof crucially uses Lemma 4.6, and the conformal naturality of the Douady-Earle section (Proposition 3.9); see [26] (or [36]) for a complete proof.

REMARK 4.8. If \( E = \hat{\mathbb{C}} \), then \( T(E) \) and \( \Psi_E \) can be identified with \( M(\mathbb{C}) \) and \( \Psi_{\hat{\mathbb{C}}} \) respectively; see Remark 4.3. If \( \phi : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is a holomorphic motion, by Theorem 4.4, there exists a basepoint preserving holomorphic map \( f : V \to M(\mathbb{C}) \) such that \( f^*(\Psi_{\hat{\mathbb{C}}}) = \phi \). By Proposition 4.7, \( \phi : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is \( G \)-equivariant if and only if \( f \) maps \( V \) into \( M(\mathbb{C})^G \).

5. Extensions of holomorphic motions

DEFINITION 5.1. Let \( E \) and \( \hat{E} \) be two closed subsets of \( \hat{\mathbb{C}} \) (as usual, the points 0, 1, and \( \infty \) belong to both \( E \) and \( \hat{E} \)), and \( E \subset \hat{E} \). If \( \phi : V \times E \to \hat{\mathbb{C}} \) and \( \hat{\phi} : V \times \hat{E} \to \hat{\mathbb{C}} \) are two holomorphic motions, we say that \( \hat{\phi} \) extends \( \phi \) if \( \hat{\phi}(x, z) = \phi(x, z) \) for all \( (x, z) \in V \times E \).

Since its inception, a fundamental topic in the study of holomorphic motions has been the question of extensions. In particular, given a holomorphic motion \( \phi : V \times E \to \hat{\mathbb{C}} \), where \( E \) is a finite set consisting of \( n \) points, if \( a \in \hat{\mathbb{C}} \setminus E \), does there exist a holomorphic motion \( \hat{\phi} : V \times (E \cup \{a\}) \to \hat{\mathbb{C}} \) such that \( \hat{\phi} \) extends \( \phi \)? In their famous paper [39], Sullivan and Thurston called this the “holomorphic axiom of choice.”

5.1. Two questions on extending holomorphic motions. Let \( E \) be a closed set in \( \hat{\mathbb{C}} \); as usual, 0, 1, and \( \infty \) belong to \( E \). In [39], Sullivan and Thurston asked two fundamental questions that can be expressed as follows: (i) if \( \phi : \Delta \times E \to \hat{\mathbb{C}} \) is a holomorphic motion, does there exist a holomorphic motion \( \hat{\phi} : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that \( \hat{\phi} \) extends \( \phi \)? (ii) if \( G \) is a group of Möbius transformations, and \( E \) is invariant under \( G \) and \( \phi : \Delta \times E \to \hat{\mathbb{C}} \) is a \( G \)-equivariant holomorphic motion (see
Definition 1.8), does there exist a holomorphic motion \( \tilde{\phi} : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that \( \tilde{\phi} \) extends \( \phi \) and is also \( G \)-equivariant?

In his paper [35], Slodkowski gave an affirmative answer to (i). The crucial step in the proof is to show that the holomorphic axiom of choice holds for holomorphic motions over \( \Delta \). See the book [22] for a complete proof. For other approaches, see [3], [6], [7], and [19]. In [12], Earle, Kra, and Krushkal gave an affirmative answer to (ii).

We use Douady-Earle section to study the above two questions for holomorphic motions over any simply connected complex Banach manifold.

We need the following proposition. Let \( E_1 \) and \( E_2 \) be two closed subsets of \( \hat{\mathbb{C}} \) such that \( E_1 \subset E_2 \) (the points 0, 1, and \( \infty \) belong to both \( E_1 \) and \( E_2 \)). In §2.4, we defined the forgetful map \( p_{E_2,E_1} : T(E_2) \to T(E_1) \), such that \( P_{E_1} = p_{E_2,E_1} \circ P_{E_2} \). Then, \( p_{E_2,E_1} \) is a basepoint preserving holomorphic map.

**Proposition 5.2.** Let \( V \) be a connected complex Banach manifold with basepoint \( x_0 \) and let \( f \) and \( g \) be basepoint preserving holomorphic maps from \( V \) into \( T(E_1) \) and \( T(E_2) \), respectively. Then \( p_{E_2,E_1} \circ g = f \) if and only if \( g^\ast(\Psi_{E_2}) \) extends \( f^\ast(\Psi_{E_1}) \).

For a proof see [32]. The proof crucially uses Lemma 4.6.

We also note the following

**Corollary 5.3.** Let \( E \) be a closed set in \( \hat{\mathbb{C}} \), let \( V \) be a connected complex Banach manifold with basepoint \( x_0 \), and let \( f \) and \( g \) be basepoint preserving holomorphic maps from \( V \) into \( T(E) \) and \( M(\mathbb{C}) \) respectively. Then \( P_E \circ g = f \) if and only if \( g^\ast(\Psi_E) \) extends \( f^\ast(\Psi_E) \).

In fact, the corollary simply restates Proposition 5.2 for the case \( E_1 = E \) and \( E_2 = \hat{\mathbb{C}} \).

We now give an example of a holomorphic motion of a finite set over a simply connected parameter space that can be extended to a quasiconformal motion of \( \hat{\mathbb{C}} \), but the holomorphic axiom of choice does not hold. This gives a counterexample to the question in (i) for higher-dimensional parameter spaces.

**Proposition 5.4.** Let \( E = \{0, 1, \infty, \zeta_1, \zeta_2, \ldots, \zeta_n\} \) where \( \zeta_i \neq \zeta_j \) if \( i \neq j \) and \( n \geq 2 \). Consider the universal holomorphic motion \( \Psi_E : T(E) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \).

(i) There exists a quasiconformal motion \( \tilde{\Psi}_E : T(E) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that \( \tilde{\Psi}_E \) extends \( \Psi_E \).

(ii) If \( a \in \hat{\mathbb{C}} \setminus E \), there does not exist any holomorphic motion \( \tilde{\phi} : T(E) \times (E \cup \{a\}) \to \hat{\mathbb{C}} \) such that \( \tilde{\phi} \) extends \( \Psi_E \).

**Proof.** We define \( \tilde{\Psi}_E : T(E) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) as follows:

\[ \tilde{\Psi}_E(t, z) = w^{s(t)}(z) \text{ for } (t, z) \in T(E) \times \hat{\mathbb{C}} \]

where \( s \) is the Douady-Earle section in Definition 3.4. We have

\[ \Psi_E(t, z) = \Psi_E(P_E(s(t)), z) = w^{s(t)}(z) = \tilde{\Psi}_E(t, z) \]

for all \((t, z) \in T(E) \times E\). Therefore, \( \tilde{\Psi}_E \) extends \( \Psi_E \). By Remark 1.4, the map \( \Psi_E : M(\mathbb{C}) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) given in Equation (4.1) is a quasiconformal motion. Since...
s : T(E) → M(ℂ) is a continuous basepoint preserving map, it follows by Remark 1.5 that s*(Ψ̃) is a quasiconformal motion of Ĉ over T(E). Finally, note that
\[ s^*(Ψ̃)(t, z) = Ψ̃(s(t), z) = w^s(t)(z) = \Psi_E(t, z) \text{ for all } (t, z) ∈ T(E) × Ĉ. \]
This proves (i).

Let ̃E = E ∪ {a}. Consider the holomorphic motion Ψ̃ : T(E) × E → Ĉ. Let i : T(E) → T(E) be the identity map (which is obviously a basepoint preserving holomorphic map). Suppose the pullback i*(Ψ̃) (which is the same as Ψ̃) extends to a holomorphic motion ̃φ : T(E) × ̃E → Ĉ. Then, since T(E) is simply connected, it follows by Theorem 4.2 that there exists a unique basepoint preserving holomorphic map f : T(E) → T(̃E) such that f*(Ψ̃) = ̃φ, where Ψ̃ : T(̃E) → Ĉ is the universal holomorphic motion of ̃E. Since f*(Ψ̃) extends i*(Ψ̃), it follows by Proposition 5.2 that p̃E,E ◦ f = i. That means, the map p̃E,E has a holomorphic section f, which is not possible by a theorem of Earle and Kra (see [21]). That proves (ii). □

5.2. Douady-Earle section and extending holomorphic motions. The following theorem addresses the two questions of Sullivan and Thurston (given in §5.1) for infinite-dimensional parameter spaces. Part (I) appeared in [35] and Part (II) in [36]. For the reader’s convenience, we include the main outlines of the argument. We emphasize that the proof involves a direct application of Douady-Earle section (in Definition 3.4), and that we do not need any deep properties of quasiconformal motions.

**Theorem 5.5.** Let φ : V × E → Ĉ be a holomorphic motion where V is a simply connected complex Banach manifold with a basepoint x₀.

(I) There exists a quasiconformal motion ̃φ : V × Ĉ → Ĉ such that ̃φ extends φ. The map ̃φ satisfies the following properties:

(i) the map ̃φ is continuous,

(ii) for each x in V, the map ̃φₓ : Ĉ → Ĉ is quasiconformal,

(iii) if f : V → M(ℂ) denotes the map such that f(x) = μₓ, where μₓ is the Beltrami coefficient of ̃φₓ, then f is continuous, and

(iv) the L∞-norm of μₓ is bounded above by a number less than one, that depends only on ρ(x₀, x) where ρ is the Kobayashi distance from x₀ to x in V.

(II) Furthermore, if G is a group of Möbius transformations and E is invariant under G, and if φ is G-equivariant, then ̃φ can be chosen also to be G-equivariant.

**Proof.** By Theorem 4.2 there exists a unique basepoint preserving holomorphic map f : V → T(E) such that f*(Ψ̃) = φ. Recall from Definition 3.4 the Douady-Earle section s : T(E) → M(ℂ). Define ̃f : V → M(Ĉ) as ̃f = s ◦ f. Let ̃φ : V × Ĉ → Ĉ be defined as:

\[ ̃φ(x, z) = w̃f(x)(z) \]  
for all (x, z) ∈ V × Ĉ. It is easy to check that ̃φ extends φ. Furthermore, since ̃f : V → M(Ĉ) is a basepoint preserving continuous map, ̃φ is a quasiconformal
motion. The continuity of $\tilde{\phi}$ follows from the continuity of $\tilde{f}$ and Lemma 17 of [1], which says that $w^{\mu_n} \to w^{\mu}$ in the spherical metric if $\mu_n \to \mu$ in $M(\mathbb{C})$.

Finally, let $x$ be in $V$ and $x \neq x_0$. Since the Teichmüller metric on $T(E)$ is the same as its Kobayashi metric (see §2.5), we have $d_{T(E)}(0, t) \leq \rho_V(x_0, x)$ where $f(x) = t$ and $0$ denotes the basepoint in $T(E)$. Choose an extremal $\mu$ in $M(\mathbb{C})$ such that $P_E(\mu) = f(x)$. This means that $d_{T(E)}(0, P_E(\mu)) = d_M(0_M, \mu)$ where $0_M$ denotes the basepoint in $M(\mathbb{C})$. We have

$$d_{T(E)}(f(x_0), f(x)) = \frac{1}{2} \log \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} \leq \rho_V(x_0, x)$$

which gives

$$\|\mu\|_\infty \leq \frac{\exp(2\rho_V(x_0, x)) - 1}{\exp(2\rho_V(x_0, x)) + 1} < 1.$$

Since $\tilde{\phi}_x(z) = w^{\tilde{f}(x)}(z)$ for $(x, z)$ in $V \times \widehat{\mathbb{C}}$, it follows from Corollary 3.7 that $\|\tilde{f}(x)\|_\infty$ is bounded above by a number between 0 and 1, that depends only on $\rho_V(x_0, x)$.

Furthermore, if $\phi$ is $G$-equivariant (satisfying Equation 1.3), it follows from Proposition 4.7 that $f$ maps $V$ into $T(E)^G$. Then, Proposition 3.9 implies that $\tilde{f}(x)$ belongs to $M(\mathbb{C})^G$ for each $x$ in $V$. By Equation (5.1) it easily follows that the quasiconformal motion $\tilde{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is also $G$-equivariant. $\square$

5.3. **On a theorem of Bers.** Let $V$ be a connected complex manifold. In what follows, $G$ is a subgroup of $\text{PSL}(2, \mathbb{C})$, $E$ is a closed subset of $\widehat{\mathbb{C}}$ (as usual, 0, 1, and $\infty$ belong to $E$), and suppose $E$ is invariant under $G$. An isomorphism $\eta : G \to \text{PSL}(2, \mathbb{C})$ is said to be induced by an injection $f : E \to \widehat{\mathbb{C}}$ if

$$f(g(z)) = \eta(g)(f(z))$$

for all $g \in G$ and for all $z \in E$. An isomorphism induced by a quasiconformal self-map of $\widehat{\mathbb{C}}$ is called a quasiconformal deformation of $G$.

**Definition 5.6.** A holomorphic family of isomorphisms of $G$ is a family $\{\theta_x\}_{x \in V}$ such that:

(i) for each $x \in V$, $\theta_x : G \to \text{PSL}(2, \mathbb{C})$ is an isomorphism, and

(ii) for each $g \in G$, the map $x \mapsto \theta_x(g)$, for $x \in V$, is holomorphic.

An immediate consequence of (II) of Theorem 5.5 is the following theorem on holomorphic families of isomorphisms of Möbius groups. This is another application of the Douady-Earle section for $T(E)$. It proves Proposition 1 in [4] in its fullest generality.

**Theorem 5.7.** Let $V$ be a connected complex Banach manifold, and let $\{\phi_x\}_{x \in V}$ be a holomorphic family of injections of $E$ over $V$. Suppose that, for each $x$ in $V$, and for each $g$ in $G$, there exists a Möbius transformation $\theta_x(g)$ such that

$$\phi_x(g(z)) = (\theta_x(g))(\phi_x(z))$$

for all $z \in E$.

Then we have:

(i) $\{\theta_x\}_{x \in V}$ is a holomorphic family of isomorphisms of $G$, and

(ii) if $\theta_t$ is a quasiconformal deformation of $G$ for some $t$ in $V$, then $\theta_x$ is a quasiconformal deformation of $G$ for every $x$ in $V$. 

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See Section 7 in [36] for the proof and also for some related results.

An immediate consequence is the following infinite-dimensional version of the main theorem in Bers’s paper [4].

**Corollary 5.8.** Let $G$ be a non-Abelian infinite group. Let $V$ be the same as in Theorem 5.7 and let $\{\theta_x\}_{x \in V}$ be a holomorphic family of isomorphisms of $G$ defined over $V$ with $\theta_t$ a quasiconformal deformation of $G$, for some $t$ in $V$. Suppose that for all $x$ in $V$,

(i) $\theta_x(G)$ is discrete, and

(ii) $\theta_x(g)$ is parabolic if and only if $g \in G$ is parabolic.

Then, for each $x$ in $V$, $\theta_x$ is a quasiconformal deformation of $G$.

**5.4. An equivalence theorem.** The next theorem addresses the situation when the parameter space of the holomorphic motion is not simply connected. We first note the following easy lemma; recall from Definition 3.4 the Douady-Earle section.

**Lemma 5.9.** If $\psi : \hat{C} \rightarrow \hat{C}$ is any homeomorphism, there is at most one point $t$ in $T(E)$ such that $\psi$ is isotopic to $w^{s(t)} \text{ rel } E$.

**Proof.** If $w^{s(t)}$ and $w^{s(t')}$ are both isotopic to $\psi \text{ rel } E$, they are $E$-equivalent, so $t = P_E(s(t)) = P_E(s(t')) = t'$. □

**Theorem 5.10.** Let $\phi : V \times E \rightarrow \hat{C}$ be a holomorphic motion where $V$ is a connected complex Banach manifold with a basepoint $x_0$. The following are equivalent.

(i) There exists a continuous motion $\hat{\phi} : V \times \hat{C} \rightarrow \hat{C}$ such that $\hat{\phi}$ extends $\phi$.

(ii) There exists a quasiconformal motion $\bar{\phi} : V \times \hat{C} \rightarrow \hat{C}$ such that $\bar{\phi}$ extends $\phi$.

(iii) There exists a basepoint preserving holomorphic map $f : V \rightarrow T(E)$ such that $f^*(\Psi_E) = \phi$.

**Proof.** The direction (iii) $\Rightarrow$ (ii) is given in the first part of the proof of Theorem 5.5. The direction (iii) $\Rightarrow$ (i) is exactly similar.

Here is a sketch of the argument for the direction (ii) $\Rightarrow$ (i). It is proved in [34] that if $\bar{\phi} : V \times \hat{C} \rightarrow \hat{C}$ is a quasiconformal motion, there exists a (unique) basepoint preserving continuous map $f : V \rightarrow M(\mathbb{C})$ such that $f^*(\Psi_E) = \bar{\phi}$. By Lemma 17 of [1], $\Psi_E$ is a continuous motion. By Remark 1.6, it follows that $f^*(\Psi_E)$ is a continuous motion, and therefore, $\bar{\phi}$ is a continuous motion.

The direction (i) $\Rightarrow$ (iii) is difficult. Here we give an outline; this is another interesting application of the Douady-Earle section.

Let $\hat{\phi} : V \times \hat{C} \rightarrow \hat{C}$ be a continuous motion that extends $\phi$. Let $S$ be the set of points $x$ in $V$ with the following property: there exists a neighborhood $N$ of $x$ and a holomorphic map $h : N \rightarrow T(E)$ such that $w^{s(h(x'))}$ is isotopic to $\hat{\phi}_{x'} \text{ rel } E$ for all $x'$ in $N$. It is obvious that $S$ is an open set.

We claim that $x_0$ is in $S$. Choose a simply connected neighborhood $N$ of $x_0$ in $V$ and give $N$ the basepoint $x_0$. By Theorem 4.2, there exists a basepoint preserving holomorphic map $h : N \rightarrow T(E)$ such that $h^*(\Psi_E) = \phi$ on $N \times E$. Define

$$H(x) = \left(w^{s(h(x))}\right)^{-1} \circ \hat{\phi}_x.$$
for each $x$ in $N$. Clearly, $H(x_0)$ is the identity. It is easy to check that for all $x$ in $N$, and for all $z$ in $E$, $\hat{\phi}_x(z) = w^{s(h(x))}(z)$. Thus, for all $z$ in $E$, $H(x)(z) = z$. Since $H(x)$ is continuous in $x$, it follows from Lemma 4.5 that $H(x)$ is isotopic to the identity rel $E$. Hence, for each $x$ in $N$, $w^{s(h(x))}$ is isotopic to $\hat{\phi}_x$ rel $E$. Therefore $x_0$ is in $S$.

The important part is to show that $S$ is closed. That is done by changing the basepoint, and by using Proposition 2.5, and then Lemma 4.6. The reader is referred to [33] for the details.

Since $V$ is connected, it follows that $S = V$. We define the holomorphic map $f : V \rightarrow T(E)$ as follows. Given any $x$ in $V$, choose a neighborhood $N$ of $x$ and a holomorphic map $h : N \rightarrow T(E)$ such that $w^{s(h(x'))}$ is isotopic to $\hat{\phi}_x$ rel $E$ for all $x'$ in $N$. Lemma 5.9 implies that $f$ is well-defined on all of $V$. It is obviously holomorphic and $w^{s(f(x))}$ is isotopic to $\hat{\phi}_x$ rel $E$ for all $x$ in $V$. It can be checked that $f^*(\Psi_E) = \phi$; see [33] for the details. □

The following corollary is obvious. It is a direct consequence of Theorems 5.10 and 5.5, and Proposition 4.7. Here $G$ is a group of Möbius transformations such that the closed set $E$ (containing $0$, $1$, and $\infty$) is invariant under $G$.

**Corollary 5.11.** Let $\phi : V \times E \rightarrow \hat{\mathbb{C}}$ be a $G$-equivariant holomorphic motion, where $V$ is a connected complex Banach manifold with a basepoint. The following are equivalent:

(i) There exists a continuous motion $\hat{\phi} : V \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $\hat{\phi}$ extends $\phi$ and is also $G$-equivariant.

(ii) There exists a quasiconformal motion $\bar{\phi} : V \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $\bar{\phi}$ extends $\phi$ and is also $G$-equivariant.

(iii) There exists a basepoint preserving holomorphic map $f : V \rightarrow T(E)$ such that $f^*(\Psi_E) = \phi$.

In fact, $f : V \rightarrow T(E)^G$ (by Proposition 4.7).

**Remark 5.12.** The reader should note that Part(I) of Theorem 5.5 is a special case of Theorem 5.10, and Part(II) of Theorem 5.5 is a special case of Corollary 5.11. We proved Theorem 5.5 separately because its proof is a direct application of Douady-Earle section, and is independent of the properties of quasiconformal motions of $\hat{\mathbb{C}}$ proved in [34].

We conclude this section with the following proposition that has an independent interest. Recall the topological group $\mathcal{H}(\hat{\mathbb{C}})$ in Lemma 4.5.

**Proposition 5.13.** Let $\phi : V \times E \rightarrow \hat{\mathbb{C}}$ be a holomorphic motion, where $V$ is a connected complex Banach manifold with a basepoint $x_0$. Suppose that $\bar{\phi} : V \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ and $\bar{\psi} : V \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ are two continuous motions that extend $\phi$. Then, for each $x$ in $V$, the homeomorphisms $\bar{\phi}_x : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ and $\bar{\psi}_x : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ are isotopic rel $E$.

**Proof.** By Definition 1.3, $\bar{\phi}_x$ and $\bar{\psi}_x$ are both homeomorphisms of $\hat{\mathbb{C}}$ onto itself, for each $x$ in $V$. Define maps $f_1$ and $f_2$ from $V$ to $\mathcal{H}(\hat{\mathbb{C}})$ as follows: $f_1(x)(z) = \bar{\phi}(x, z)$ and $f_2(x)(z) = \bar{\psi}(x, z)$ for $x$ in $V$ and $z$ in $\hat{\mathbb{C}}$. By Theorem 5 in [2], the maps $f_1$ and $f_2$ are both continuous (since $\bar{\phi}$ and $\bar{\psi}$ are continuous maps). Therefore, the map $h : V \rightarrow \mathcal{H}(\hat{\mathbb{C}})$ defined by

$$h(x) = f_2(x)^{-1} \circ f_1(x)$$
for \( x \) in \( V \) is also continuous. Clearly, \( h(x_0) \) is the identity map of \( \hat{C} \), and for each \( x \) in \( V \), \( h(x) \) fixes \( E \) pointwise. Hence, by Lemma 4.5, it follows that \( h(x) \) is isotopic to the identity \( \text{rel} \ E \), for each \( x \) in \( V \). It follows that, for each \( x \) in \( V \), the homeomorphisms \( \tilde{\phi}_x : \hat{C} \to \hat{C} \) and \( \tilde{\psi}_x : \hat{C} \to \hat{C} \) are isotopic \( \text{rel} \ E \). \( \square \)

6. Extending holomorphic motions and lifting holomorphic maps

Let \( E_1 \) and \( E_2 \) be two closed subsets of \( \hat{C} \) such that \( E_1 \subset E_2 \) (the points \( 0, 1, \) and \( \infty \) belong to both \( E_1 \) and \( E_2 \)). In \( \S 2.4 \), we defined the forgetful map \( p_{E_2,E_1} : T(E_2) \to T(E_1) \), such that \( P_{E_1} = p_{E_2,E_1} \circ P_{E_2} \).

**Theorem 6.1.** Let \( V \) be a connected complex Banach manifold with a basepoint, such that every holomorphic motion of a closed set over \( V \) extends to a continuous motion of \( \hat{C} \) over \( V \). The following are equivalent.

(i) Every holomorphic motion \( \phi_1 : V \times E_1 \to \hat{C} \) extends to a holomorphic motion \( \phi_2 : V \times E_2 \to \hat{C} \).

(ii) For every basepoint preserving holomorphic map \( f_1 : V \to T(E_1) \), there exists a basepoint preserving holomorphic map \( f_2 : V \to T(E_2) \) such that \( p_{E_2,E_1} \circ f_2 = f_1 \).

**Proof.** (i) \( \implies \) (ii). Let \( f_1 : V \to T(E_1) \) be a basepoint preserving holomorphic map. Define \( \phi_1 := f_1^\ast(\Psi_{E_1}) \) where \( \Psi_{E_1} : T(E_1) \times E_1 \to \hat{C} \) is the universal holomorphic motion of \( E_1 \). By (i) there exists a holomorphic motion \( \phi_2 : V \times E_2 \to \hat{C} \) such that \( \phi_2 \) extends \( \phi_1 \). By hypothesis, \( \phi_2 \) extends to a continuous motion \( \hat{\phi} : V \times \hat{C} \to \hat{C} \). By Theorem 5.10, there exists a basepoint preserving holomorphic map \( f_2 : V \to T(E_2) \) such that \( f_2^\ast(\Psi_{E_2}) = \phi_2 \). Since \( \phi_2 \) extends \( \phi_1 \), it follows by Proposition 5.2 that \( p_{E_2,E_1} \circ f_2 = f_1 \).

(ii) \( \implies \) (i). Let \( \phi_1 : V \times E_1 \to \hat{C} \) be a holomorphic motion. Then, by hypothesis, \( \phi_1 \) extends to a continuous motion \( \hat{\phi} : V \times \hat{C} \to \hat{C} \). By Theorem 5.10, there exists a basepoint preserving holomorphic map \( f_1 : V \to T(E_1) \) such that \( f_1^\ast(\Psi_{E_1}) = \phi_1 \). By (ii) there exists a basepoint preserving holomorphic map \( f_2 : V \to T(E_2) \) such that \( p_{E_2,E_1} \circ f_2 = f_1 \). Define \( \phi_2 := f_2^\ast(\Psi_{E_2}) \). Since \( p_{E_2,E_1} \circ f_2 = f_1 \), it follows by Proposition 5.2 that \( \phi_2 \) extends \( \phi_1 \). \( \square \)

We say that the holomorphic map \( f_1 \) lifts to the holomorphic map \( f_2 \). The following two corollaries are obvious.

**Corollary 6.2.** Let \( V \) be a simply connected complex Banach manifold with a basepoint. The following are equivalent.

(i) Every holomorphic motion \( \phi_1 : V \times E_1 \to \hat{C} \) extends to a holomorphic motion \( \phi_2 : V \times E_2 \to \hat{C} \).

(ii) Every basepoint preserving holomorphic map \( f_1 : V \to T(E_1) \) lifts to a basepoint preserving holomorphic map \( f_2 : V \to T(E_2) \).

**Corollary 6.3.** Let \( V \) be a simply connected complex Banach manifold with a basepoint, and let \( E \) be a closed set in \( \hat{C} \) (as usual, \( E \) contains \( 0, 1, \) and \( \infty \)). The following are equivalent.

(i) Every holomorphic motion \( \phi : V \times E \to \hat{C} \) extends to a holomorphic motion \( \tilde{\phi} : V \times \hat{C} \to \hat{C} \).

(ii) Every basepoint preserving holomorphic map \( f : V \to T(E) \) lifts to a basepoint preserving holomorphic map \( \tilde{f} : V \to M(\hat{C}) \).
The next theorem gives an application of Douady-Earle section to group-equivariant holomorphic motions. In what follows, $G$ is a group of Möbius transformations such that the closed set $E$ (containing 0, 1, and $\infty$) is invariant under $G$. Recall the universal holomorphic motion $\Psi_E : T(E) \times E \to \hat{\mathbb{C}}$ in Definition 4.1, the meanings of $M(\mathbb{C})^G$ and $T(E)^G$ in Definition 3.8, and also the definition of a $G$-equivariant holomorphic motion in Definition 1.8.

**Theorem 6.4.** Let $V$ be a connected complex Banach manifold with a basepoint, such that every holomorphic motion of $E$ over $V$ extends to a continuous motion of $\hat{\mathbb{C}}$ over $V$. The following are equivalent:

(i) Every $G$-equivariant holomorphic motion $\phi : V \times E \to \hat{\mathbb{C}}$ extends to a $G$-equivariant holomorphic motion $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

(ii) Every basepoint preserving holomorphic map $f : V \to T(E)^G$ lifts to a basepoint preserving holomorphic map $\tilde{f} : V \to M(\mathbb{C})^G$.

**Proof.** (i) $\implies$ (ii). Let $f : V \to T(E)^G$ be a basepoint preserving holomorphic map. Define $\phi := f^* (\Psi_E)$. By Proposition 4.7, $\phi : V \times E \to \hat{\mathbb{C}}$ is $G$-equivariant. Therefore, there exists a $G$-equivariant holomorphic motion $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends $\phi$. By Theorem 4.4, there exists a basepoint preserving holomorphic map $\tilde{f} : V \to M(\mathbb{C})$ such that $\tilde{f}^*(\Psi_\hat{\mathbb{C}}) = \tilde{\phi}$. Since $\tilde{f}^*(\Psi_\hat{\mathbb{C}})$ extends $f^*(\Psi_E)$, it follows by Corollary 5.3 that $P_E \circ \tilde{f} = f$. Finally, since $\tilde{\phi}$ is $G$-equivariant and $\tilde{f}^*(\Psi_E) = \tilde{\phi}$, it follows by Remark 4.8 that $\tilde{f} : V \to M(\mathbb{C})^G$.

(ii) $\implies$ (i). Let $\phi : V \times E \to \hat{\mathbb{C}}$ be a $G$-equivariant holomorphic motion. By the hypothesis, there exists a continuous motion $\hat{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\hat{\phi}$ extends $\phi$. Therefore, by Theorem 5.10, there exists a basepoint preserving holomorphic map $\hat{f} : V \to T(E)$ such that $f^*(\Psi_E) = \hat{\phi}$. Furthermore, since $\phi$ is $G$-equivariant, $f : V \to T(E)^G$ (by Proposition 4.7). Therefore, there exists a basepoint preserving holomorphic map $\hat{f} : V \to M(\mathbb{C})^G$ such that $P_E \circ \hat{f} = f$. Let $\tilde{\phi} := \hat{f}^*(\Psi_\hat{\mathbb{C}})$. By Corollary 5.3, it follows that $\tilde{\phi}$ extends $\phi$. Finally, since $\tilde{\phi} = \hat{f}^*(\Psi_\hat{\mathbb{C}})$, and $\hat{f} : V \to M(\mathbb{C})^G$, it follows by Remark 4.8 that $\tilde{\phi}$ is $G$-equivariant. \hfill $\square$

**Remark 6.5.** The reader should note that the continuous motion $\hat{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ in the proof of Theorem 6.4 is not assumed to be $G$-equivariant.

**Remark 6.6.** Let $E = \{0, 1, \infty, \zeta_1, \zeta_2, \ldots, \zeta_n\}$ where $\zeta_i \neq \zeta_j$ if $i \neq j$ and $n \geq 2$. Let $a \in \hat{\mathbb{C}} \setminus E$ and $\hat{E} = E \cup \{a\}$. By Remark 2.3, the spaces $T(E)$ and $T(\hat{E})$ are identified with the classical Teichmüller spaces $Teich(\hat{\mathbb{C}} \setminus E)$ and $Teich(\hat{\mathbb{C}} \setminus \hat{E})$ respectively. Recall our discussion of holomorphic axiom of choice before §5.1. It follows from Corollary 6.2, that the holomorphic axiom of choice is equivalent to the following question: if $f$ is a basepoint preserving holomorphic map from $V$ into $T(E)$, does $f$ lift to a holomorphic map $h$ from $V$ into $T(\hat{E})$, where $V$ is a simply connected complex Banach manifold. This was already discussed in [5] when $V = \Delta$. Thus, a direct proof of this “lifting problem” (when the parameter space is $\Delta$) is an alternative method to prove Slodkowski’s extension theorem. Using some ideas of Chirka, and a result of Nag, this was proved in [27]. Proposition 5.4 gives an example where this property does not hold when the parameter space $V$ is the Teichmüller space $T(E)$. 

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Part 2. Some applications of holomorphic motions in complex analysis

In this section we focus on holomorphic motions over $\Delta$, and review some applications in complex analysis.

7. An application of Theorem 4.2

Recall from §2.5, that the Teichmüller metric on $T(E)$ is the same as its Kobayashi metric. An immediate consequence of that fact and Theorem 4.2 is the following theorem.

**Theorem 7.1.** Let $\phi : V \times E \to \hat{\mathbb{C}}$ be a holomorphic motion, where $V$ is a simply connected complex Banach manifold with a basepoint $x_0$. Then, for every $x$ in $V$, $\phi(x, \cdot)$ is the restriction to $E$ of a quasiconformal self map of $\hat{\mathbb{C}}$ with dilatation not exceeding $\exp(2 \rho_V(x, x_0))$ where $\rho_V$ is the Kobayashi distance from $x$ to $x_0$.

See §17 of [32] for a proof.

The following special case, which first appeared in [5], will be very useful in our discussions.

**Remark 7.2.** Let $\phi : \Delta \times E \to \hat{\mathbb{C}}$ be a holomorphic motion. By Slodkowski’s theorem ([38]), there exists a holomorphic motion $\tilde{\phi} : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends $\phi$. By Theorem 4.4 (or Theorem 4.2), for each $x$ in $\Delta$, $\tilde{\phi}_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map. Let $f : \Delta \to M(\mathbb{C})$ be the map such that $f$ is the Beltrami coefficient of $\tilde{\phi}_x$, for each $x$ in $\Delta$. By Theorem 4.4 (or Theorem 4.2), $f$ is holomorphic and by Theorem 7.1, the dilatation $K$ of $\tilde{\phi}_x$ satisfies

$$K \leq \frac{1 + |x|}{1 - |x|}.$$

8. Gluing germs in the Riemann sphere

A holomorphic germ $f$ is a holomorphic function defined in a neighborhood of a point $z_0$ in $\mathbb{C}$. Thus we can write $f(z)$ into the following form:

$$f(z) = z_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots, \quad z \in U$$

where $U$ is a neighborhood of $z_0$. We usually use $\lambda$ to denote $a_1$. Then it is clear that

$$\lambda = f'(z_0).$$

The following theorem was first proved in [23] Theorem 2 and Corollary 1] by using holomorphic motions. The technique in the proof of this theorem is used in all proofs of theorems in the rest of the paper. Therefore, let us first give a review of the proof.

Let $\Delta_r(z_i) = \{z \mid |z - z_i| < r\}$ be the disk of radius $r > 0$ centered at $z_i$.

**Theorem 8.1 (Gluing Theorem).** Suppose $\{f_i\}_{i=1}^k$ is a finite number of germs at distinct points $\{z_i\}_{i=1}^k$ such that $\lambda_i = f_i'(z_i) \neq 0$ for $1 \leq i \leq k$. Then for every $\epsilon > 0$ there exist a number $s > 0$ and a $(1 + \epsilon)$-quasiconformal homeomorphism $f$ of $\hat{\mathbb{C}}$ such that

$$f|_{\Delta_s(z_i)} = f_i|_{\Delta_s(z_i)}, \quad i = 1, \ldots, k.$$
We see that

\[ (8.1) \]

\[ \lambda_i = 1, \quad 1 \leq i \leq k. \]

A holomorphic germ satisfying this assumption is called a parabolic germ.

Denote

\[ B_i(r) = f_i(\Delta_r(z_i)). \]

Let \( r_0 > 0 \) be a number such that

\[ B_i(r) \cap B_j(r) = \emptyset, \quad 1 \leq i \neq j \leq k, \quad 0 < r \leq r_0. \]

Let

\[ E_r = \bigcup_{i=1}^n \Delta_r(z_i) \]

be a closed subset of \( \hat{\mathbb{C}} \).

**Step 1. Construction of a holomorphic motion.** For any \( 0 < r \leq r_0 \), write

\[ f_i(z) = z + a_{i,2}(z-z_i)^2 + \cdots + a_{i,n}(z-z_i)^n + \cdots, \quad |z-z_i| \leq r. \]

Let

\[ \eta_i(\xi) = a_{i,2}\xi^2 + \cdots + a_{i,n}\xi^n + \cdots. \]

Then

\[ f_i(z) = z + \eta_i(z-z_i), \quad |z-z_i| \leq r. \]

Let \( \phi(z) \) be defined on \( E_r \) as

\[ \phi(z) = f_i(z) = z + \eta_i(z-z_i) \quad \text{for} \quad |z-z_i| \leq r, \quad i = 1, \cdots, k. \]

We introduce a complex parameter \( c \in \Delta \) into \( \phi(z) \) as follows. Define

\[ h(c, z) = z + r(c_r_0)\eta_i \left( \frac{cr_0}{r}(z-z_i) \right), \quad |z-z_i| \leq r, \quad i = 1, \cdots, k. \]

We will show that the map \( h : \Delta \times E_r \to \hat{\mathbb{C}} \) is a holomorphic motion.

For any fixed \( c \in \Delta \), we have

\[ h'_z(c, z) = 1 + \eta'_i \left( \frac{cr_0}{r}(z-z_i) \right), \quad |z-z_i| \leq r, \quad i = 1, \cdots, k. \]

By picking \( r_0 > 0 \) small enough, we can assume

\[ |f'_i(z)| = |1 + \eta'_i(z-z_i)| \geq 1 - |\eta'_i(z-z_i)| > 0, \quad |z-z_i| < r_0, \quad i = 1, \cdots, k. \]

Thus

\[ h'_z(c, z) \neq 0 \text{ for all } |z-z_i| \leq r, \quad i = 1, \cdots, k. \]

We see that \( h_c(z) = h(x, z) \) on each \( \Delta_r(z_i) \) is injective. But images of \( \Delta_r(z_i) \) and \( \Delta_r(z_j) \), for \( 1 \leq i \neq j \leq k \), under \( h(c, z) \) are pairwise disjoint. So \( h_c(z) \) is injective on \( E_r \).

It is clear that

\[ h(0, z) = z, \quad z \in E_r. \]

For any fixed \( z \in \Delta_r(z_i), \ 1 \leq i \leq k, \)

\[ h^z(c) = h(c, z) = z + r(c_r_0)\eta_i \left( \frac{cr_0}{r}(z-z_i) \right). \]

Since

\[ \left| \frac{cr_0}{r}(z-z_i) \right| < r_0, \]

\[ \eta_i \left( \frac{cr_0}{r}(z-z_i) \right) \]
Suppose \( \Delta \) is a convergent power series of \( c \neq 0 \in \Delta \). For \( c = 0 \), \( h(0,z) = z \). So \( h(c,z) \) is holomorphic with respect to \( c \in \Delta \).

Therefore, \( h : \Delta \times E_r \to \hat{\mathbb{C}} \) is a holomorphic motion.

**Step 2. Construction of a quasiconformal extension.** Following Remark 7.2, there exists a holomorphic motion \( H : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \), such that \( H(c,z) = h(c,z) \) for all \( (c,z) \in \Delta \times E_r \). Moreover, for any \( c \in \Delta \), \( H_c(z) = H(c,z) \) is a \((1+|c|)/(1-|c|)\)-quasiconformal mapping (see Remark 7.2).

Let \( f(z) = H\left(\frac{r}{r_0}, z\right) \).

Then \( f(z) \) is a \((1+r/r_0)/(1-r/r_0)\)-quasiconformal homeomorphism. Furthermore,

\[
f|\Delta_r(z_i) = H\left(\frac{r}{r_0}, z\right)|\Delta_r(z_i) = h\left(\frac{r}{r_0}, z\right)|\Delta_r(z_i) = f|\Delta_r(z_i).
\]

Thus for any given \( \epsilon > 0 \), we take \( r = (2r_0)/(1+\epsilon) \); then \( f \) is a \((1+\epsilon)\)-quasiconformal mapping and extends \( f_i \) for all \( i = 1, 2, \cdots, k \). This completes the proof of the theorem under the assumption that \( \lambda_i = 1 \) for all \( 1 \leq i \leq k \).

For the general situation, we first suppose \( r_0 > 0 \) and suppose that

\[
f_i(z) = z_i + \lambda_i(z - z_i), \quad z \in D_{r_0}(z_i), \quad \lambda_i \neq 0, \quad 1 \leq i \leq k.
\]

Suppose

\[\overline{\mathbb{D}}_{r_0}(z_i) \cap \overline{\mathbb{D}}_{r_0}(z_j) = \emptyset, \quad \text{for all } 0 \leq i \neq j \leq k.\]

Let

\[a = \max\{|\log \lambda_i| \mid 1 \leq i \leq k\},\]

and let

\[s = r_0 e^{-\frac{a}{\epsilon}}\]

for any \( 0 < r < r_0 \).

**Step 3. Construction of another holomorphic motion.** Let \( \Delta_s(z_i) \) be defined as in Theorem 8.1, and let \( E_s = \bigcup_{i=1}^k \overline{\Delta_s(z_i)} \). Define

\[h(c,z) = z_i + e^{\frac{a}{\epsilon}} \log \lambda_i(z - z_i), \quad c \in \Delta, \quad z \in \overline{\Delta_s}(z_i).
\]

We will check that \( h : \Delta \times E_s \to \hat{\mathbb{C}} \) is a holomorphic motion.

For \( c = 0 \), we have \( \phi(0,z) = z \) for all \( z \in E_s \).

For each fixed \( c \in \Delta \), \( h_c(z) = h(c,z) \) on each \( \overline{\Delta_s}(z_i) \) is injective, but the image of \( \Delta_s(z_i) \) under \( h_c \) is contained in \( \Delta_{r_0}(z_i) \). So \( h_c \) on \( E_s \) is injective.

For fixed \( z \in E_s \), it is clear that \( h^2(c) = h(c,z) \) is holomorphic with respect to \( c \in \Delta \).

So, \( h : \Delta \times E_s \to \hat{\mathbb{C}} \) is a holomorphic motion.

**Step 4. Construction of another extension.** By Remark 7.2, there exists a holomorphic motion \( H : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that \( H(c,z) = h(c,z) \) for all \((c,z) \in \Delta \times E_s \). Moreover, for any \( c \in \Delta \), \( H_c(z) = H(c,\cdot) \) is a \((1+|c|)/(1-|c|)\)-quasiconformal homeomorphism.

Let \( f(z) = H(r,z) \). Then \( f(z) \) is a \((1+r)/(1-r)\)-quasiconformal homeomorphism. Furthermore,

\[f|\Delta_s(z_i) = H(r,z)|\Delta_s(z_i) = h(r,z)|\Delta_s(z_i) = f_i|\Delta_s(z_i).
\]
Step 5. Putting two extensions together. Now we consider the general situation,
\[ f_i(z) = z_i + \lambda_i(z - z_i) + a_{2,i}(z - z_i)^2 + \cdots, \quad z \in \Delta_{r_0}(z_i), \quad \lambda_i \neq 0, \ 1 \leq i \leq k. \]
Let
\[ g_i(z) = z_i + \lambda_i^{-1}(z - z_i), \quad 1 \leq i \leq k. \]
Then
\[ F_i(z) = f_i \circ g_i(z) = z + \frac{a_{2,i}}{\lambda_i^2}(z - z_i)^2 + \cdots, \quad 1 \leq i \leq k, \]
satisfies the assumption (6.1) in the beginning of the proof.

From the proof under the assumption (6.1), for any \( \epsilon > 0 \), we have \( 0 < s < r \leq r_0 \) and two \( \sqrt{1 + \epsilon} \)-quasiconformal homeomorphisms \( F(z) \) and \( G(z) \) of \( \hat{\mathbb{C}} \) such that
\[ F|_{\Delta_r(z_i)} = F_i|_{\Delta_r(z_i)} \quad \text{and} \quad G|_{\Delta_s(z_i)} = g_i^{-1}|_{\Delta_s(z_i)} \]
and such that
\[ G(\Delta_s(z_i)) \subset \Delta_r(z_i). \]
Then \( f(z) = F \circ G(z) \) is a \((1+\epsilon)\)-quasiconformal homeomorphism of \( \hat{\mathbb{C}} \) such that
\[ f|_{\Delta_s(z_i)} = F \circ G|_{\Delta_s(z_i)} = f_i \circ g_i \circ g_i^{-1}|_{\Delta_s(z_i)} = f_i|_{\Delta_s(z_i)}. \]
This completes the proof of the theorem. \( \square \)

We would like to mention that Theorem 8.1 has a generalized version in [28] as follows.

**Theorem 8.2.** Let \( \{z_i\}_{i=1}^k \) be a set of distinct points in the complex plane \( \mathbb{C} \) and let \( U_k \) be a neighborhoods of \( z_i \) for every \( i = 1, 2, \cdots, k \). Suppose \( \{U_i\}_{i=1}^k \) are pairwise disjoint and \( f_i(z) \) is a \( K \)-quasiconformal map defined on \( U_i \) which fixes \( z_i \) for every \( i = 1, 2, \cdots, k \). Then for every \( \epsilon > 0 \) there exist a number \( r > 0 \) and a \((K+\epsilon)\)-quasiconformal map \( f \) of \( \hat{\mathbb{C}} \) such that
\[ f|_{\Delta_r(z_i)} = f_i|_{\Delta_r(z_i)}, \]
where \( \Delta_r(z_i) \subset U_i \) is the open disk of radius \( r \) centered at \( z_i \) for \( i = 1, \cdots, k \).

9. König’s theorem, Böttcher’s theorem, and their generalizations

In this section, we first review a proof of König’s Theorem given in [24, 25] by using holomorphic motions. This method gives not only a new proof but also leads to proofs of two new theorems in [25] which generalize König’s theorem and Böttcher’s theorem.

**Theorem 9.1 (König’s Theorem).** Let \( f(z) = \lambda z + \sum_{j=2}^{\infty} a_j z^j \) be a holomorphic germ defined on \( \Delta_{r_0} \), \( r_0 > 0 \). Suppose \( 0 < |\lambda| < 1 \) or \( |\lambda| > 1 \). Then there is a conformal map \( \phi: \Delta_{\delta} \to \phi(\Delta_{\delta}) \) for some \( 0 < \delta < r_0 \) such that
\[ \phi^{-1} \circ f \circ \phi(z) = \lambda z. \]
The conjugacy \( \phi^{-1} \) is unique up to multiplication of constants.
Proof. We only need to prove it for $0 < |\lambda| < 1$. In the case of $|\lambda| > 1$, we can consider $f^{-1}$.

First, we can find a $0 < \delta < r_0$ such that

$$|f(z)| < |z|, \quad z \in \Delta_{\delta}$$

and $f$ is injective on $\Delta_{\delta}$.

Step 1. Construction of a holomorphic motion. For every $0 < r \leq \delta$, let

$$S_r = \{z \in \mathbb{C} \mid |z| = r\}$$

and

$$T_r = \{z \in \mathbb{C} \mid |z| = |\lambda|r\}.$$ 

Denote $E = S_r \cup T_r$. Define

$$\phi_r(z) = \begin{cases} z, & z \in S_r; \\ f(\frac{z}{r}), & z \in T_r. \end{cases}$$

It is clear that

$$\phi_r^{-1} \circ f \circ \phi_r(z) = \lambda z$$

for $z \in S_r$.

Now write $\phi_r(z) = \psi_r(z)$ for $z \in T_r$, where

$$\psi_r(z) = 1 + \sum_{j=1}^{\infty} \frac{a_j+1}{\lambda j+1} z^j.$$ 

Define

$$h_r(c, z) = \begin{cases} z, & z \in S_r; \\ z \psi_r(\frac{cz}{r}), & z \in T_r \quad : \Delta \times E \rightarrow \hat{\mathbb{C}}. \end{cases}$$

Note that

$$h(c, z) = z \psi_r\left(\frac{cz}{r}\right) = \frac{r}{c \delta} \phi\left(\frac{cz}{r}\right) = \frac{r}{c \delta} f\left(\frac{cz}{r \lambda}\right), \quad z \in T_r, c \neq 0.$$ 

For each fixed $z \in E$, it is clear that $h(c, z)$ is a holomorphic function of $c \in \Delta$. For each fixed $c \in \Delta$, the restriction $h(c, \cdot)$ to $S_r$ and $T_r$, respectively, are injective.

Now we claim that their images do not cross each other. That is because for any $z \in T_r$, $|z| = |\lambda|r$ and $|cz\delta^2|/r\lambda| \leq \delta$, so

$$|h(c, z)| = \left| \frac{r}{c \delta} \right| f\left(\frac{cz}{r \lambda}\right) < \left| \frac{r}{c \delta} \right| \frac{|cz\delta|}{r \lambda} = r.$$ 

Therefore, $h : \Delta \times E \rightarrow \hat{\mathbb{C}}$ is a holomorphic motion because we have $h(0, z) = z$ for all $z \in E$.

Step 2. Construction of quasiconformal conjugacies. By Remark 7.2, $h : \Delta \times E \rightarrow \hat{\mathbb{C}}$ can be extended to a holomorphic motion $H : \Delta \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, and moreover, for each fixed $c \in \Delta$, $H_c = h(c, \cdot) : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a quasiconformal homeomorphism whose quasiconformal dilatation is less than or equal to $(1 + |c|)/(1 - |c|)$. Now take $c_r = r/\delta$ and consider $H(c_r, \cdot)$. We have $H(c_r, \cdot)|E = \phi_r$. Let

$$A_{r,j} = \{z \in \mathbb{C} \mid |\lambda|^j r \leq |z| \leq |\lambda|^j r\}.$$ 

We still use $\phi_r$ to denote $H(c_r, \cdot)|A_{r,0}$.

For an integer $k > 0$, take $r = r_k = \delta|\lambda|^k$. Then

$$\Delta_{\delta} = \cup_{j=-k}^{\infty} A_{r,j} \cup \{0\}.$$
Extend $\phi_r$ to $\bar{\Delta}_\delta$, which we still denote as $\phi_r$, as follows.

$$
\phi_r(z) = f^{-j}(\phi_r((\lambda^n z)), \quad z \in A_{r,j}, \quad j = -k, \ldots, -1, 0, 1, \ldots,
$$

and $\phi_r(0) = 0$. Since $\phi_r|E$ is a conjugacy from $f$ to $\lambda z$, $\phi_r$ is continuous on $\Delta_\delta$. Since $f$ is conformal, $\phi_r$ is quasiconformal whose dilatation is the same as that of $H(e_r, \cdot)$ on $A_{r,0}$. So the dilatation of $\phi_r$ on $\Delta_\delta$ is less than or equal to $(1+r)/(1-r)$. Furthermore,

$$
f(\phi_r(z)) = \phi_r(\lambda z), \quad z \in \Delta_\delta.
$$

**Step 3. Improvement to a conformal conjugacy.** Since $f(z) = \lambda z(1 + O(z))$, $f^k(z) = \lambda^k z \prod_{i=0}^{k-1} (1 + O(\lambda^i z))$. Because $|\lambda|^{-k} r_k = \delta$, the range of $\phi_{r_k}$ on $\Delta_\delta$ is a Jordan domain bounded above and below uniformly on $k$. In addition, $0$ is fixed by $\phi_k$ and the dilatations of the $\phi_k$’s are uniformly bounded. Therefore, there exists a convergent subsequence $\{\phi_{r_k}\}_{k=1}^\infty$. Let $\phi$ be a limiting map of this family. Then we have

$$
f(\phi(z)) = \phi(\lambda z), \quad z \in \Delta_\delta.
$$

The dilatation of $\phi$ is less than or equal to $(1 + r_k)/(1 - r_k)$ for all $k > 0$. So $\phi$ is a $1$-quasiconformal map, and thus is conformal. This is the proof of the existence.

For the sake of completeness, we also provide the proof of uniqueness. Suppose $\phi_1$ and $\phi_2$ are two conjugacies such that

$$
\phi_1^{-1} \circ f \circ \phi_1(z) = \lambda z \quad \text{and} \quad \phi_2^{-1} \circ f \circ \phi_2(z) = \lambda z, \quad z \in \Delta_\delta.
$$

Then for $\Phi = \phi_2^{-1} \circ \phi_1$, we have $\Phi(\lambda z) = \lambda \Phi(z)$.

This implies that $\Phi'(\lambda z) = \lambda \Phi'(z)$ for any $z \in \Delta_\delta$. Thus $\Phi'(z) = \Phi'(\lambda^n z) = \Phi(0)$. So $\Phi$ is a constant, and $\phi_2^{-1} = k \phi_1^{-1}$ where $k$ is a constant.

Using a similar technique, we gave a new proof of the following theorem in [24].

**Theorem 9.2 (Böttcher’s Theorem).** Suppose $f(z) = \sum_{j=n}^{\infty} a_j z^j$, $a_n \neq 0$, $n \geq 2$, is holomorphic on a disk $\Delta_{\delta_0}$, $\delta_0 > 0$. Then there exists a conformal map $\phi : \Delta_{\delta} \to \phi(\Delta_{\delta})$ for some $\delta > 0$ such that

$$
\phi^{-1} \circ f \circ \phi(z) = z^n, \quad z \in \Delta_{\delta}.
$$

The conjugacy $\phi^{-1}$ is unique up to multiplication by $(n - 1)^{th}$ roots of unity.

Two new results were proved in [25] by using a similar technique for integrable asymptotically conformal fixed points as we describe below.

Let $f$ be a quasiconformal homeomorphism defined on a domain $U$ in $\mathbb{C}$. Suppose $p$ is a point in the $U$. Let $\Delta_t(p)$ denote the disk of radius $t > 0$ centered at $p$. Let $\mu_f(z) = f_\bar{z}/f_z$ be the complex dilatation of $f$ on $U$. Suppose $t_0 > 0$ is a number such that $\Delta_{t_0}(p) \subset U$. Then for any $0 < t \leq t_0$, let $\omega_{f,p}(t) = ||\mu_f|\Delta_t(p)||_\infty$. The following definition was given in [25].

**Definition 9.3.** We call $f$ asymptotically conformal at $p$ if

$$
\omega_{f,p}(t) \to 0 \quad \text{as} \quad t \to 0^+.
$$

Furthermore, we call $f$ integrable asymptotically conformal at $p$ if

$$
\int_0^{t_0} \frac{\omega_{f,p}(s)}{s} ds < \infty.
$$
If $f$ is asymptotically conformal at $p$, then $f$ maps a tiny circle centered at $p$ to an ellipse centered at $f(p)$ and, moreover, the ratio of the major axis and the minor axis tends to 1 as the radius of the tiny circle tends to 0. But the map still can fail to be differentiable at $p$. However, if $f$ is integrable asymptotically conformal at $p$, then $f$ is differentiable and conformal at $p$, i.e., the limit of $(f(z) - f(p))/(z - p)$ exists as $z$ approaches $p$. If, in addition, $p$ is a fixed point of $f$, that is, $f(p) = p$, let

$$
\lambda = \lim_{z \to p} \frac{f(z) - f(p)}{z - p}
$$

and call it the multiplier of $f$ at $p$. We call $p$

i) attracting if $0 < |\lambda| < 1$;

ii) repelling if $|\lambda| > 1$;

iii) neutral if $|\lambda| = 1$.

Correspondingly, we call $p$ an attracting, repelling, or neutral integrable asymptotically conformal fixed point of $f$. By linear changes of coordinate, we can assume that $p = f(p) = 0$. We will keep this assumption without loss of generality.

In the attracting case, we say $f$ satisfies the control condition if there are constants $\delta > 0$ and $C > 0$ such that

$$
C^{-1} \leq \left| \frac{f^n(z)}{\lambda^nz} \right| \leq C, \quad \forall \ z \in \Delta_\delta \subset U, \quad \forall \ n \geq 0.
$$

In the repelling case, $f^{-1}$ is in the attracting case, so we can define the control condition similarly.

Let $g = f \circ q_n$ where $q_n(z) = z^n$ and $f$ is a quasiconformal mapping defined in a neighborhood of 0 that fixes 0. Let $U$ be the domain of $g$. We say that $g$ is integrable asymptotically conformal at 0 if $f$ is integrable asymptotically conformal at 0 with nonzero multiplier

$$
\lambda = \lim_{z \to 0} \frac{f(z)}{z}.
$$

In this case, 0 is called a super-attracting integrable asymptotically conformal fixed point of $g$.

The following lemma will be useful in our proofs of Theorems 9.5 and 9.6.

**Lemma 9.4.** Suppose $\omega(t)$ is an increasing function of $0 < t \leq t_0$. Suppose

$$
\int_0^{t_0} \frac{\omega(s)}{s} ds < \infty.
$$

Suppose $0 < \sigma < 1$ and $C > 0$ are two constants. Let

$$
\tilde{\omega}(t) = \sum_{n=0}^{\infty} \omega(C\sigma^nt)
$$

for all $t > 0$ such that $Ct \leq t_0$. Then

$$
\tilde{\omega}(t) \leq \omega(Ct) + \frac{1}{-\log \sigma} \int_0^{Ct} \frac{\omega(s)}{s} ds.
$$

Moreover, $\tilde{\omega}(t) \to 0$ as $t \to 0^+$.

See [25] for the proof.

Using a technique similar to the proof of Theorem 9.1, we proved two new theorems in [25] as follows.
Theorem 9.5 (Generalized König’s Theorem). Let $f$ be a quasiconformal homeomorphism defined on a neighborhood about 0. Suppose 0 is an attracting or repelling integrable asymptotically conformal fixed point of $f$ with the control condition $|z^n|$. Then there is a quasiconformal homeomorphism $\phi: \Delta_\delta \to \phi(\Delta_\delta) \subset U$ from an open disk of radius $\delta > 0$ centered at 0 into $U$ which is asymptotically conformal at 0 such that

$$\phi^{-1} \circ f \circ \phi(z) = \lambda z, \quad z \in \Delta_\delta.$$  

The conjugacy $\phi^{-1}$ is unique up to multiplication of a constant.

Theorem 9.6 (Generalized Böthcher’s Theorem). Let $g(z) = f(z^n)$ be a quasiregular map defined on a neighborhood about 0 for $n \geq 2$. Suppose 0 is a super-attracting integrable asymptotically conformal fixed point of $g$. Then there is a quasiconformal homeomorphism $\phi: \Delta_\delta \to \phi(\Delta_\delta) \subset U$ from an open disk of radius $\delta > 0$ centered at 0 into $U$ which is asymptotically conformal at 0 such that

$$\phi^{-1} \circ g \circ \phi(z) = z^n, \quad z \in \Delta_\delta.$$  

The conjugacy $\phi^{-1}$ is unique up to multiplication by $(n - 1)^{\text{th}}$ roots of unity.


10. Leau-Fatou flowers and linearization

Suppose $f(z)$ is a parabolic holomorphic germ at 0. Then there is a constant $0 < r_0 < 1/2$ such that $f(z)$ is conformal with the Taylor expansion

$$f(z) = e^{\frac{2\pi i}{n}}z + a_2 z^2 + \cdots, \quad (p, q) = 1, \quad |z| < r_0.$$  

Suppose $f^m \neq id$ for all $m > 0$. Then, for appropriate $r_0$,

$$f^q(z) = z(1 + az^n + \epsilon(z)), \quad a \neq 0, \quad |z| < r_0,$$  

where $n$ is a multiple of $q$ and $\epsilon(z)$ is given by a convergent power series of the form

$$\epsilon(z) = a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots, \quad |z| < r_0.$$  

Suppose $N \subset \Delta_{r_0}$ is a neighborhood of 0. A simply connected open set $\mathcal{P} \subset N \cap f^q(N)$ with $f^q(\mathcal{P}) \subset \mathcal{P}$ and $0 \in \overline{\mathcal{P}}$ is called an attracting petal if $f^m(z)$ for $z \in \mathcal{P}$ converges uniformly to 0 as $m \to \infty$. An attracting petal $\mathcal{P}'$ for $f^{-1}$ is called a repelling petal at 0.

Theorem 10.1 (The Leau-Fatou flower). There exist $n$ attracting petals $\{\mathcal{P}_i\}_{i=0}^{n-1}$ and $n$ repelling petals $\{\mathcal{P}_j'\}_{j=0}^{n-1}$ such that

$$N_0 = \bigcup_{i=0}^{n-1} \mathcal{P}_i \cup \bigcup_{j=0}^{n-1} \mathcal{P}_j'$$  

is a neighborhood of 0.

For each attracting petal $\mathcal{P} = \mathcal{P}_i$, consider the change of coordinates

$$w = \phi(z) = \frac{d}{z^n}, \quad d = -\frac{1}{na},$$  

on $\mathcal{P}$. Suppose the image of $\mathcal{P}$ under $\phi(z)$ is a right half-plane

$$R_\tau = \{w \in \mathbb{C} \mid \Re w > \tau\}.$$  

Then

$$z = \phi^{-1}(w) = \sqrt[na]{\frac{d}{w}} : R_\tau \to \mathcal{P}.$$
is a conformal map. The form of $f^q$ in the $w$-plane is
\[ F(w) = \phi \circ f \circ \phi^{-1}(w) = w + 1 + \eta \left( \frac{1}{\sqrt[n]{|w|}} \right), \quad \Re w > \tau, \]
where $\eta(\xi)$ is a holomorphic function in a neighborhood of 0. Suppose
\[ \eta(\xi) = b_1 \xi + b_2 \xi^2 + \cdots, \quad |\xi| < r_1 \]
is a convergent power series for some $0 < r_1 \leq r_0$. Take $0 < r < r_1$ such that
\[ |\eta(\xi)| \leq \frac{1}{2}, \quad \forall |\xi| < r. \]
Then $F(R_\tau) \subset R_\tau$ for any $\tau \geq 1/r^n$ since
\[ \Re F(w) = \Re w + 1 + \Re \eta \left( \frac{1}{\sqrt[n]{|w|}} \right) \geq \Re w + \frac{1}{2}, \quad \forall \Re w \geq \tau. \]

As another application of holomorphic motions, we gave a new proof of the following theorem in [23]. Here, we include a review of this new proof.

**Theorem 10.2 (Fatou Linearization Theorem).** Suppose $\tau > 1/r^n + 1$ is a real number. Then there is a conformal map $\Psi(w) : R_\tau \to \Omega$ such that
\[ F(\Psi(w)) = \Psi(w + 1), \quad \forall w \in R_\tau. \]

**Proof.** For any $x \geq 1$, let
\[ E_{0,x} = \{ w \in \mathbb{C} \mid \Re w = x \} \]
and
\[ E_{1,x} = \{ w \in \mathbb{C} \mid \Re w = x + 1 \} \]
and let
\[ E_x = E_{0,x} \cup E_{1,x}. \]
Then $E_x$ is a subset of $\hat{\mathbb{C}}$.

**Step 1. Construction of a holomorphic motion.** Define
\[ H_x(w) = \begin{cases} 
    w, & w \in E_{0,x}; \\
    \Phi(w) = w + \eta \left( \frac{1}{\sqrt[n]{|w-1|}} \right), & w \in E_{1,x}.
\end{cases} \]
Since $H_x(w)$ on $E_{0,x}$ and on $E_{1,x}$ are injective, respectively, and since
\[ \Re(H_x(w)) \geq \Re(w) - \frac{1}{2} = x + 1 - \frac{1}{2} = x + \frac{1}{2}, \quad w \in E_{1,x}, \]
the images of $E_{0,x}$ and $E_{1,x}$ under $H_x(w)$ do not intersect. So $H_x(w)$ is injective. Moreover, $H_x(w)$ conjugates $F(w)$ to the linear map $w \mapsto w + 1$ on $E_{0,x}$, that is,
\[ F(H_x(w)) = H_x(w + 1), \quad \forall w \in E_{0,x}. \]

We first introduce a complex parameter $c \in \Delta$ into $\eta(\xi)$ as follows. Define
\[ \eta(c, \xi) = \eta(c r \xi \sqrt[n]{x-1}) = b_1(c r \xi \sqrt[n]{x-1}) + b_2(c r \xi \sqrt[n]{x-1})^2 + \cdots \]
for $|c| < 1$ and $|\xi| \leq 1/\sqrt[n]{x-1}$. Since $|c r \xi \sqrt[n]{x-1}| \leq r$, $\eta(c, \xi)$ is a convergent power series and $|\eta(c, \xi)| \leq 1/2$ for $|c| < 1$ and $|\xi| \leq 1/\sqrt[n]{x-1}$. Following this, we therefore introduce a complex parameter $c \in \Delta$ for $H_x(w)$ defined by
\[ H_x(c, w) = \begin{cases} 
    w, & (c, w) \in \Delta \times E_{0,x}; \\
    \Phi(w) = w + \eta(c, \frac{1}{\sqrt[n]{|w-1|}}), & (c, w) \in \Delta \times E_{1,x}.
\end{cases} \]
The map $H_x : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a holomorphic motion (refer to [23]).

**Step 2. Construction of quasiconformal conjugacies.** By Remark 7.2, $H_x : \Delta \times E_x \to \hat{\mathbb{C}}$ can be extended to a holomorphic motion $\tilde{H}_x : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Also, for each $c \in \Delta$, $c(w) = \tilde{H}_x(c, w) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a $(1 + |c|)/(1 - |c|)$-quasiconformal homeomorphism. When $c(x) = 1/(r \sqrt{x - 1})$, $h_c(x)$ is a quasiconformal extension of $H_x(w)$ to $\hat{\mathbb{C}}$ whose dilatation is less than or equal to

$$K(x) = 1 + \frac{1}{r \sqrt{x - 1}}.$$  

Note that $K(x) \to 1$ as $x \to \infty$.

Suppose

$$S_x = \{w \in \mathbb{C} \mid x \leq \Re w \leq x + 1\}$$

is the strip bounded by two lines $\Re w = x$ and $\Re w = x + 1$. Consider the restriction of $h_{c(x)}(w)$ on $S_x$ which we still denote as $h_c(w)$.

For any $w_0 \in R \cup E_{0, \tau}$, let $w_m = F^m(w_0)$. Since $w_m - w_{m+1}$ tends to 1 as $m$ goes to $\infty$ uniformly on $R \cup E_{0, \tau}$,

$$\frac{w_m - w_0}{m} = \frac{1}{m} \sum_{k=1}^{m} (w_k - w_{k-1}) \to 1$$

uniformly on $R \cup E_{0, \tau}$ as $m$ goes to $\infty$. So $w_m$ is asymptotic to $w$ as $m$ goes to $\infty$ uniformly in any bounded set of $R \cup E_{0, \tau}$.

Let $x_0 = \tau$ and $x_m = \Re(F^m(x_0))$. Then $x_m$ is asymptotic to $w$ as $m$ goes to $\infty$. For each $m > 0$, let

$$\Omega_m = F^{-m}(E_{0,x_m} \cup \{\infty\}).$$

It is a curve passing through $x_0 = \tau$ and $\infty$. Let

$$\Omega_m = F^{-m}(R_{x_m}).$$

It is a domain with the boundary $\Omega_m$.

Let

$$S_{i,x_m} = F^{-i}(S_{x_m}), \quad i = m, m + 1, \ldots, 1, 0, -1, \ldots, -m + 1, -m, \ldots.$$  

Then

$$\Omega_m = \cup_{i=-\infty}^{i=\infty} S_{i,x_m}.$$  

Let

$$A_m = \{w \in \mathbb{C} \mid \tau + m \leq \Re w \leq \tau + m + 1\}$$

and let $A_{i,m} = A_m - i$ for $i = m, m + 1, \ldots, 1, 0, -1, \ldots, -m + 1, -m, \ldots$. Let

$$\beta_m(w) = w + x_m - \tau - m : \mathbb{C} \to \mathbb{C}.$$  

Then it is a conformal map and

$$\beta_m(A_m) = S_{x_m}.$$  

Define

$$\psi_m(w) = h_{c(x_m)} \circ \beta_m(w).$$  

Then it is a $K(x_m)$-quasiconformal homeomorphism on $A_m$. Moreover,

$$F(\psi_m(w)) = \psi_m(w + 1), \quad \forall \Re w = m + \tau.$$
Furthermore, define
\[ \psi_m(w) = F^{-i}(\psi_{m+i}(w+i)), \quad \forall w \in A_{i,m} \]
for \( i = m, m-1, \ldots, 1, 0, -1, \ldots, -m+1, -m, \ldots \). Then it is a \( K(x_m) \)-quasiconformal homeomorphism from \( R_\tau \) to \( \Omega_m \) and
\[ F(\psi_m(w)) = \psi_m(w+1), \quad \forall w \in R_\tau. \]

**Step 3. Improvement to a conformal conjugacy.** Let \( w_0 = \tau \) and \( w_m = F^m(w_0) \) for \( m = 1, 2, \ldots \). Remember that
\[ R_{x_m} = \{ w \in \mathbb{C} \mid \Re w > x_m \} \]
where \( x_m = \Re w_m \).

For any \( \tilde{w}_0 \in R_{x_{m+1}} \), let \( \tilde{w}_m = F^m(\tilde{w}_0) \) for \( m = 1, 2, \ldots \). Since
\[ F'(w) = 1 + O\left(\frac{1}{|w|^{1+\frac{1}{n}}} \right), \quad w \in R_\tau \]
and \( \tilde{w}_m/m \to 1 \) as \( m \to \infty \) uniformly on any compact set, there is a constant \( C > 0 \) such that
\[ C^{-1} \leq \frac{|	ilde{w}_m - w_m|}{|\tilde{w}_1 - w_1|} = \prod_{k=1}^{m} \frac{|	ilde{w}_{k+1} - w_{k+1}|}{|\tilde{w}_k - w_k|} = \prod_{k=1}^{m} \left(1 + O\left(\frac{1}{k^{1+\frac{1}{n}}} \right) \right) \leq C \]
as long as \( w_1 \) and \( \tilde{w}_1 \) keep in a same compact set. Since
\[ w_{m+1} = w_m + 1 + \eta\left(\frac{1}{\sqrt{w_m}} \right) \quad \text{and} \quad \left| \eta\left(\frac{1}{\sqrt{w_m}} \right) \right| \leq \frac{1}{2}, \]

the distance between \( w_{m+1} \) and \( R_{x_m} \) is greater than or equal to \( 1/2 \). So the disk \( \Delta_{1/2}(w_{m+1}) \) is contained in \( R_{x_m} \). This implies that the disk \( \Delta_{1/(2C)}(w_1) \) is contained in \( \Omega_m \) for every \( m = 0, 1, \ldots \). Thus the sequence
\[ \psi_m(w) : R_\tau \to \Omega_m, \quad m = 1, 2, \ldots \]
has a convergent subsequence whose limit is
\[ \Psi(w) : R_\tau \to \Omega. \]

Then \( \Psi \) is \( 1 \)-quasiconformal and thus conformal and satisfies
\[ F(\Psi(w)) = \Psi(w+1), \quad \forall w \in R_\tau. \]

This completes the proof of Theorem 10.2. \( \square \)

**11. Quasiconformal rigidity for parabolic germs**

Finally, we give a review of the quasiconformal rigidity theorem for parabolic germs proved in [23] by using holomorphic motions.

**Theorem 11.1 (Quasiconformal Rigidity Theorem).** Suppose \( f \) and \( g \) are two parabolic germs at 0 and suppose \( f \) and \( g \) are topologically conjugate. Then for every \( \epsilon > 0 \) there are neighborhoods \( U_\epsilon \) and \( V_\epsilon \) about 0 such that \( f|U_\epsilon \) and \( g|V_\epsilon \) are \( (1 + \epsilon) \)-quasiconformally conjugate.
PROOF. Suppose $f$ and $g$ are two topologically conjugate parabolic germs. Suppose $f^m, g^m \neq id$ for all $m > 0$. (If some $f^m \equiv$ identity, then $g^m \equiv$ identity too.) Suppose $\lambda$ and $n + 1$ are their common multiplier and multiplicity. Suppose $0 < r_0 < 1/2$ such that both $f$ and $g$ are conformal in $\Delta_{r_0}$. Without loss of generality, we assume that $\lambda = 1$ and both of $f$ and $g$ have forms

$$f(z) = z(1 + z^n + o(z^n)) \quad \text{and} \quad g(z) = z(1 + z^n + o(z^n)), \quad |z| < r_0.$$ 

By Theorem 10.1 for any small neighborhood $N \subset \Delta_{r_0}$, there are $n$ attracting petals $\{P_{i,f}\}_{i=0}^{n-1}$ and $n$ repelling petals $\{P'_{i,f}\}_{i=0}^{n-1}$ for $f$ in $N$. Let us assume that every $P_{i,f}$ is the maximal attracting petal in $N$. Similarly, we have the same pattern of attracting petals $\{P_{i,g}\}_{i=0}^{n-1}$ and the repelling petals $\{P'_{i,g}\}_{i=0}^{n-1}$ for $g$.

By Theorem 10.2 (see also [31 page 107]), for every $0 \leq i \leq n - 1$, there is a conformal map

$$\psi_i : P_{i,g} \to P_{i,f}$$

such that

$$f(\psi_i(z)) = \psi_i(g(z)), \quad z \in P_{i,g}.$$ 

For each $0 \leq i \leq n - 1$, let

$$w = \phi(z) = -\frac{1}{nz^n}$$

be the change of coordinates. Then $f$ and $g$ in the $w$-coordinate system have forms

$$F(w) = w + 1 + \eta_f\left(\frac{1}{\sqrt[n]{w}}\right) \quad \text{and} \quad G(w) = w + 1 + \eta_g\left(\frac{1}{\sqrt[n]{w}}\right),$$

where both of

$$\eta_f(\xi) = a_1\xi + a_2\xi + \cdots \quad \text{and} \quad \eta_g(\xi) = b_1\xi + b_2\xi + \cdots,$$

are convergent power series for some number $0 < r_1 < r_0$. Take a number $0 < r < r_1$ such that

$$|\eta_f(\xi)|, |\eta_g(\xi)| \leq \frac{1}{4}, \quad |\xi| \leq r.$$ 

Without loss of generality, we assume that $\eta_g(w) \equiv 0$, that is, $G(w) = w + 1$.

Suppose both repelling petals $P'_{i,f}$ and $P'_{i,g}$ are changed to a left half-plane

$$L_{-r^n} = \{w \in \mathbb{C} \mid \Re w < -r^n\}.$$ 

Step 1. Construction of a holomorphic motion. Take $\tau_0 = r^n$. Let

$$U_{\tau_0} = \{w \in \mathbb{C} \mid \Im w > \tau_0\}$$

be an upper half-plane and let

$$D_{-\tau_0} = \{w \in \mathbb{C} \mid \Im w < -\tau_0\}$$

be a lower half-plane. Define

$$\Psi(w) = \left\{ \begin{array}{ll} \phi \circ \psi_i \circ \phi^{-1}(w), & w \in U_{\tau_0}, \\ \phi \circ \psi_{i+1} \circ \phi^{-1}(w), & w \in D_{-\tau_0}. \end{array} \right.$$ 

(If $i + 1 = n$, we consider it as 0.) Then

$$F(\Psi(w)) = \Psi(G(w)), \quad w \in U_{\tau_0} \cup D_{-\tau_0}.$$ 

We can have the property that $\Psi(w)/w \to 1$ as $w \to \infty$ (refer to [31 pp. 109]).

Let $a = e^{-2\pi r_0}$. Consider the covering map

$$\xi = \beta(w) = e^{2\pi iw} : \mathbb{C} \to \mathbb{C} \setminus \{0\}.$$
Then it maps $U_{\tau_0}$ to $\Delta_a \setminus \{0\}$ and $D_{-\tau_0}$ to $\mathbb{C} \setminus \overline{\Delta_1/a}$.

The inverse of $w = \beta^{-1}(\xi)$ is a multi-valued holomorphic function on $\mathbb{C} \setminus \{0\}$.

We take one branch as $\beta^{-1}$. Since $\Psi(w)$ is asymptotic to $w$ as $w \to \infty$, the map
\[
\theta(\xi) = \beta \circ \Psi \circ \beta^{-1}(\xi)
\]
is holomorphic in $\Delta_a$ and in $\overline{\Delta_1/a} = \hat{\mathbb{C}} \setminus \overline{\Delta_1/a}$. Suppose
\[
\theta(\xi) = \xi + a_2\xi^2 + \cdots, \quad |\xi| < a
\]
and
\[
\theta(\xi) = \xi + b_1\xi^2 + \cdots, \quad |\xi| > \frac{1}{a}
\]
are two convergent power series.

For any $\tau > \tau_0$, let $\epsilon = e^{-2\pi \tau}$. Suppose $\overline{\Delta_1/\epsilon} = \hat{\mathbb{C}} \setminus \overline{\Delta_1/\epsilon})$. Let
\[
E = \Delta_\epsilon \cup \overline{\Delta_1/\epsilon}.
\]

It is a subset of $\hat{\mathbb{C}}$. We now introduce a complex parameter $c \in \Delta$ into $\theta(\xi)$ such that it is a holomorphic motion of $E$ parametrized by $\Delta$ and with the base point 0.

Define
\[
\theta(c,\xi) = \frac{\epsilon}{ca} \theta\left(\frac{ca\xi}{\epsilon}\right) = \xi + a_2\left(\frac{ca}{\epsilon}\right)\xi^2 + \cdots, \quad |c| < 1, |\xi| \leq \epsilon.
\]
and
\[
\theta(c,\xi) = \frac{ca}{\epsilon} \theta\left(\frac{\epsilon\xi}{ca}\right) = \xi + b_1\left(\frac{ca}{\epsilon}\right)^2 + \cdots, \quad |c| < 1, |\xi| \geq \frac{1}{a}.
\]

We claim that $\theta : \Delta \times E \to \hat{\mathbb{C}}$ is a holomorphic motion.

(1) It is clear that $\theta(0,\xi) = \xi$ for all $\xi \in E$.

(2) For any fixed $c \neq 0 \in \Delta$, $\theta(c,\xi)$ on $\Delta_\epsilon$ is a conjugation map of $\theta(\xi)$ by the linear map $\xi \mapsto (ca/\epsilon)\xi$. And $\theta(c,\xi)$ on $\overline{\Delta_1/\epsilon}$ is a conjugation map of $\theta(\xi)$ by the linear map $\xi \mapsto (\epsilon/(ca))\xi$. So they are injective. Since the image $\theta(c,\Delta_\epsilon)$ is contained in $\Delta_a$ and the image $\theta(c,\overline{\Delta_1/\epsilon})$ is contained in $\overline{\Delta_1/a}$, they do not intersect. So $\theta(c,\cdot)$ on $E$ is injective.

(3) For any fixed $\xi \in \Delta_\epsilon$, since $|ca\xi/\epsilon| < a$ for $|c| < 1$, it is a convergent power series of $c$. So $\theta(\cdot,\xi)$ is holomorphic on $c$. For any fixed $\xi \in \overline{\Delta_1/\epsilon}$, since $|\epsilon\xi/(ca)| > 1/a$ for $|c| < 1$, so it is a convergent power series of $c$. So $\theta(\cdot,\xi)$ is holomorphic on $c$. We have proved the claim.

Let $E_\tau = U_{\tau} \cup D_{-\tau}$. Then $\beta(E_\tau) = E$. Since $\beta : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering map, the holomorphic motion $\theta : \Delta \times E \to \hat{\mathbb{C}}$ induces a holomorphic motion $h_0 : \Delta \times E_\tau \to \hat{\mathbb{C}}$. When $c(\tau) = \epsilon/a$, $h_0(c(\tau),w) = \Psi(w)$.

Let $w_1 = -\tau + i\tau$ and $w_2 = -\tau - i\tau$. Consider the vertical segment connecting them
\[
s_\tau = \left\{tw_1 + (1-t)w_2 \mid 0 \leq t \leq 1\right\}.
\]

Let
\[
s'_\tau = s_\tau + \frac{1}{2} = \left\{tw_1 + (1-t)w_2 + 1 \mid 0 \leq t \leq 1\right\}.
\]

Define
\[
h_1(c,tw_1 + (1-t)w_2) = th_0(c,w_1) + (1-t)h_0(c,w_2) : \Delta \times s_\tau \to \hat{\mathbb{C}}
\]
and
\[
h_2(c,tw_1 + (1-t)w_2 + 1) = F(h_1(c,tw_1 + (1-t)w_2) : \Delta \times s'_\tau \to \hat{\mathbb{C}}.
\]
Both $h_1$ and $h_2$ are holomorphic motions. Since

$$h_2(c, tw_1 + (1 - t)w_2 + 1) = th_0(c, w_1) + (1 - t)h_0(c, w_2) + 1 + \eta(th_0(c, w_1) + (1 - t)h_0(c, w_2))$$

and since

$$|\eta(w)| \leq 1/4, \quad \forall |w| \geq \tau,$$

the images of these two holomorphic motions do not intersect. Therefore, we define a holomorphic motion

$$h(c, w) = \begin{cases} h_0(c, w), & (c, w) \in \Delta \times E_\tau; \\ h_1(c, w), & (c, w) \in \Delta \times s_\tau; \\ h_2(c, w), & (c, w) \in \Delta \times s'_\tau. \end{cases}$$

of $\Sigma = E_\tau \cup s_\tau \cup s'_\tau$ parametrized by $\Delta$ and with base point 0.

**Step 2. Construction of a quasiconformal conjugacy.** For $c(\tau) = c/a$, $h(c(\tau), w)$ is a conjugacy from $F$ to $G$ on $E_\tau \cup s_\tau$, i.e.,

$$F(h(c(\tau), w)) = h(c(\tau), G(w)), \quad w \in E_\tau \cup s_\tau.$$

By Remark 7.2, $h$ extends to a holomorphic motion $H : \Delta \times \widehat{C} \to \widehat{C}$ and for each $c \in \Delta$, $H(c, \cdot)$ is a $(1 + |c|)/(1 - |c|)$-quasiconformal homeomorphism of $\widehat{C}$.

Let $H(w) = H(c(\tau), w)$ and

$$K(\tau) = \frac{1 + c(\tau)}{1 - c(\tau)},$$

Note that $K(\tau) \to 1$ as $\tau \to \infty$. Then $H(w)$ is a $K(\tau)$-quasiconformal homeomorphism of $\widehat{C}$ such that

$$H(w) = h(c(\tau), w), \quad \forall w \in \Sigma.$$

Let

$$A_0 = \{w \in C \mid -\tau \leq \Re w \leq -\tau + 1\}$$

and $A_m = A_0 - m$ for $m = 1, 2, \cdots$. Define

$$\Psi(w) = F^{-m}(H(w + m)), \quad w \in A_m, \quad m = 0, 1, \cdots.$$

Then $\Psi(w)$ is a $K(\tau)$-quasiconformal homeomorphism defined on the left half-plane

$$L_{-\tau+1} = \{w \in C \mid \Re w \leq -\tau + 1\}$$

and extends $\Psi(w)$ on $U_\tau \cup D_{-\tau}$.

Now let

$$\psi(z) = \phi^{-1} \circ \Psi \circ \phi(z).$$

It extends

$$\psi_i : P_{i,g} \to P_{i,f} \quad \text{and} \quad \psi_{i+1} : P_{i+1,g} \to P_{i+1,f}$$

in a small neighborhood $N$ to a $K(\tau)$-quasiconformal homeomorphism

$$\psi(z) : P_{i,g} \cup P'_{i,g} \cup P_{i+1,g} \to P_{i,f} \cup P'_{i,f} \cup P_{i+1,f}$$

and

$$f \circ \psi(z) = \psi \circ g(z), \quad \forall z \in P_{i,g} \cup P'_{i,g} \cup P_{i+1,g}.$$

If we work out the above for every $0 \leq i \leq n - 1$, we get that for any $\varepsilon > 0$, there is a neighborhood $U_\varepsilon$ of 0 and a $(1 + \varepsilon)/(1 - \varepsilon)$-quasiconformal homeomorphism

$$\psi(z) : U_\varepsilon \to V_\varepsilon = \psi(V_\varepsilon)$$
such that it extends every $\psi_i : P_{i,g} \to P_{i,f}$ in $U_\varepsilon$ and such that

$$f \circ \psi(z) = \psi \circ g(z), \quad \forall z \in U_\varepsilon.$$

This completes the proof of Theorem 11.1.

References


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