An Introduction to Geometric Gibbs Theory

Yunping Jiang

Abstract This is an article I wrote for Dynamics, Games, and Science. In Dynamics, Game, and Science, one of the most important equilibrium states is a Gibbs state. The deformation of a Gibbs state becomes an important subject in these areas. An appropriate metric on the space of underlying dynamical systems is going to be very helpful in the study of deformation. The Teichmüller metric becomes a natural choice. The Teichmüller metric, just like the hyperbolic metric on the open unit disk, makes the space of underlying dynamical systems a complete space. The Teichmüller metric precisely measures the change of the eigenvalues at all periodic points which are essential data needed to obtain the Gibbs state for a given dynamical system. In this article, I will introduce the Teichmüller metric and, subsequently, a generalization of Gibbs theory which we call geometric Gibbs theory.

1 Introduction

The mathematical theory of Gibbs states, an important idea originally from physics, is a beautiful mathematical theory starting from the celebrated work of Sinai [23, 24] and Ruelle [20, 21]. It leads to the study of SRB-measures in Anosov dynamical systems and, more generally, Axiom A dynamical systems due to the further work of Sinai, Ruelle, Bowen, and many other people (see [2]). A very important feature of a Gibbs state is that it is an equilibrium state. This equilibrium state plays an important role in mathematics, as well as many other areas such as physics, chemistry, biology, economy, and game theory.

An important topic in the current study of Gibbs states (in mathematics, we also call them Gibbs measures) is to study the deformation of a Gibbs state. For example, how does a Gibbs measure (or a SRB-measure) changes when the

Y. Jiang (✉)
Department of Mathematics, Queens College of the City University of New York, Flushing, NY 11367-1597, USA

Department of Mathematics, Graduate School of the City University of New York, 365 Fifth Avenue, New York, NY 10016, USA
e-mail: yunping.jiang@qc.cuny.edu

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Yunping.Jiang@qc.cuny.edu

underlying dynamical system changes? How does the density of a Gibbs measure (or SRB-measure) with respect to the Lebesgue measure changes when the underlying dynamical system changes? To study the deformation of a Gibbs measure, an appropriate metric on the space of underlying dynamical systems will be very helpful. Ruelle has proposed to use the Whitney theory (see [16, 22]). In this note I would like to introduce another metric from Teichmüller theory and, subsequently, a generalized Gibbs theory which we call geometric Gibbs theory. The Teichmüller metric closely relates to the eigenvalues at all periodic points: Given a one-dimensional (expanding) smooth dynamical system f, the essential data needed to determine the Gibbs measure is the set of eigenvalues

$$\left\{\log \lambda_f(p) = \log |(f^n)'(p)| \ ; \ p \text{ a periodic point of } f \text{ of period } n\right\}.$$

Given two topologically conjugate dynamical systems f and g, with

$$g \circ h = h \circ f,$$

where h is the conjugacy, how can we measure the geometric difference between f and g? The answer is the set of ratios

$$\left\{0 < \alpha(p) = \min\left\{\frac{\log \lambda_f(p)}{\log \lambda_g(h(p))}, \frac{\log \lambda_g(h(p))}{\log \lambda_f(p)}\right\} \leq 1\right\}.$$

Actually, h is locally $\alpha(p)$-Hölder continuous near p but the exponent changes with p. These exponents can be measured precisely by using so-called "quasiconformal dilatation" from complex analysis (refer to [9]), that is, the Teichmüller metric. The Teichmüller metric, just like the hyperbolic metric (or Lobachevsky metric or Poincaré metric) on the open unit disk, makes the space a complete space.

This article intends to give a summary of our work in this direction. A more complete version of our work with more detailed proofs will be available in [15]. I first give a brief review of classical Gibbs theory in Sect. 2. Then, following the traditional terminology in dynamical systems, I introduce a circle g-function in Sect. 3. In Sect. 4, I give the definition of a geometric Gibbs measure associated to a circle g-function. In the same section, I show the existence of a geometric Gibbs measure for any circle g-function and the uniqueness for the constant g-function. I further show that a geometric Gibbs measure is an equilibrium state. Finally, I introduce the Teichmüller metric on the space of all circle g-functions in Sect. 5. The Teichmüller metric makes the space of all circle g-functions a complete space. I expect this new metric will play an important role in the study of deformations of geometric Gibbs measures. In particular, when a circle g-function is Hölder continuous, the corresponding geometric Gibbs measure is absolutely continuous with respect to the Lebesgue measure. Therefore, we have a density function.

Yunping.Jiang@qc.cuny.edu

We would like to study the derivative of a density function with respect to a Hölder continuous g-function and connect this derivative with the susceptibility function

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_T \rho(dx) X(x) \frac{d}{dx}(A(f^n x))$$

at $\lambda = 1$, which is formally the derivative of the density with respect to a Hölder continuous g-function, as described in [16, 22].

2 Classical Gibbs Theory

Suppose $d \geq 2$ is a positive integer. Consider the symbolic dynamical system $\sigma : \Sigma \to \Sigma$, where

$$\Sigma = \{w = \cdots i_{n-1} \cdots i_1 i_0 \mid i_{n-1} \in \{0, \cdots, d-1\},\ n = 1, 2, \cdots\}$$

and

$$\sigma : w = \cdots i_{n-1} \cdots i_1 i_0 \to \sigma(w) = \cdots i_{n-1} \cdots i_1$$

is the shift map. The space $\Sigma = \prod_{\infty}^{0} \{0, 1, \cdots, d-1\}$ is a compact topological space with the product topology. *We purposely write*

$$w = \cdots i_{n-1} \cdots i_1 i_0$$

from the right to left because we will later use

$$v = j_0 j_1 \cdots j_{n-1} \cdots$$

to represent a point on the unit circle. An n-cylinder $[w]_n$ containing $w = \cdots i_{n-1} \cdots i_1 i_0$ is the subset of all elements $w' = \cdots i'_{n+m} \cdots i'_n i_{n-1} \cdots i_0$ for $i'_{n+m} \in \{0, \cdots, d-1\}$ and $m = 0, 1, \cdots$.

A real function $\phi : \Sigma \to \mathbf{R}$ is called Hölder continuous if there are two constants $C > 0$ and $0 < \tau < 1$ such that $|\phi(w) - \phi(w')| \leq C\tau^n$ as long as w and w' are in the same n-cylinder. We use $\mathscr{C}^H$ to denote the space of all Hölder continuous real functions on Σ. We call a positive Hölder continuous function ψ on Σ a Hölder potential. We also use $\mathscr{C}$ to denote the space of all continuous real functions on Σ and $\mathscr{M}$ to denote the space of all finite Borel measures on Σ, which is the dual space of $\mathscr{C}$. Then $\mathscr{M}(\sigma)$ means the space of all σ-invariant probability measures in $\mathscr{M}$, that is, the space of measures with total measure 1 and satisfying $\mu(\sigma^{-1}(A)) = \mu(A)$ for all Borel subsets A of Σ.

Yunping.Jiang@qc.cuny.edu

The classical Gibbs theory ensures that associated to each Hölder potential ψ, there is a number $P = P(\log\psi)$, called the pressure, and unique σ-invariant probability measure $\mu = \mu_\psi$, called a Gibbs measure, such that

$$C^{-1} \leq \frac{\mu([w]_n)}{\exp(-Pn + \sum_{i=0}^{n-1} \log\psi(\sigma^i(w)))} \leq C \tag{1}$$

for any n-cylinder $[w]_n$, where C is a fixed constant. A Gibbs measure depends only on a cohomologous equivalence class and is an equilibrium state in the sense that

$$P(\log\psi) = ent_\mu(\sigma) + \int_\Sigma \log\psi \, d\mu = \sup_{\nu\in\mathscr{M}(\sigma)} \left\{ent_\nu(\sigma) + \int_\Sigma \log\psi \, d\nu\right\}$$

where $ent_\nu(\sigma)$ is the measure-theoretical entropy of σ with respect to ν.

In a proof of the existence and uniquess of the Gibbs measure μ for a given Hölder potential ψ, we use the Ruelle-Perron-Frobenius operator

$$\mathscr{L}_\psi\phi(w) = \sum_{\sigma(w')=w} \psi(w')\phi(w') : \mathscr{C}^H \to \mathscr{C}^H. \tag{2}$$

The Ruelle-Perron-Frobenius theorem (refer to [10] for a proof) says that there is a positive real number λ and a positive Hölder function $\rho \in \mathscr{C}^H$ such that $\mathscr{L}_\psi\rho = \lambda\rho$. Here λ is the unique, maximal, positive, simple eigenvalue of $\mathscr{L}_\psi$. Note that in this case, the pressure $P = \log\lambda$. If we consider a new Hölder potential

$$\tilde{\psi} = \frac{\psi\cdot\rho}{\lambda\cdot\rho\circ\sigma},$$

then we get a normalized Ruelle-Perron-Frobenius operator $\mathscr{L}_{\tilde{\psi}}$, that is, $\mathscr{L}_{\tilde{\psi}}1 = 1$. Let $\mathscr{L}^*_{\tilde{\psi}} : \mathscr{M} \to \mathscr{M}$ be its dual operator. Then the Gibbs measure $\mu_{\tilde{\psi}}$ is just the unique fixed point of $\mathscr{L}^*_{\tilde{\psi}}$ in this case. (The Gibbs measure $\mu_\psi = \rho\cdot\mu_{\tilde{\psi}}$.) This leads to the study of g-measure theory in Keane's paper [17] as follows.

A non-negative continuous real function g defined on Σ is called a g-function if

$$\sum_{i=0}^{d-1} g(wi) = 1. \tag{3}$$

For a g-function g, define the transfer operator

$$\mathscr{L}_g\phi(w) = \sum_{\sigma(w')=w} g(w')\phi(w') : \mathscr{C} \to \mathscr{C}.$$

It is a normalized Ruelle-Perron-Frobenius operator, that is, $\mathscr{L}_g 1 = 1$. Let $\mathscr{L}^*_g : \mathscr{M} \to \mathscr{M}$ be its dual operator. Every fixed point μ of $\mathscr{L}^*_g$ is called a g-measure associated with g. Here we always assume that μ is a probability measure.

Yunping.Jiang@qc.cuny.edu

Any $\mu \in \mathscr{M}(\sigma)$ is absolutely continuous with respect to $\tilde{\mu} = \mathscr{L}_1^* \mu$. So we have the Radon-Nikodým derivative

$$RND_{\mu,\tilde{\mu}}(w) = \frac{d\mu}{d\tilde{\mu}}(w), \quad \tilde{\mu} - a.e.\ w.$$

It is a $\tilde{\mu}$-measurable function. The following theorem is in Leddrapier's paper [18] and is used in Walters' paper [25] for the study of a generalized version of Ruelle's theorem. Let $\mathscr{B}$ be the Borel σ-algebra on Σ.

Theorem 1 *Suppose g is a g-function and $\mu \in \mathscr{M}$ is a probability measure. The following statements are equivalent:*

(i) *μ is a g-measure for g, i.e., $\mathscr{L}_g^* \mu = \mu$.*
(ii) *$\mu \in \mathscr{M}(\sigma)$ and $RND_{\mu,\tilde{\mu}}(w) = g(w)$ for $\tilde{\mu}$-a.e. w.*
(iii) *$\mu \in \mathscr{M}(\sigma)$ and*

$$E[\phi|\sigma^{-1}(\mathscr{B})](w) = \mathscr{L}_g \phi(\sigma(w)) = \sum_{\sigma(w')=\sigma(w)} g(w')\phi(w'), \text{ for } \mu\text{-a.e. } w,$$

where $E[\phi|\sigma^{-1}(\mathscr{B})]$ is the conditional expectation of ϕ with respect to $\sigma^{-1}(\mathscr{B})$.
(iv) *$\mu \in \mathscr{M}(\sigma)$ and is an equilibrium state for g in the sense that*

$$0 = ent_\mu(\sigma) + \int_\Sigma \log g \, d\mu = \sup_{\nu \in \mathscr{M}(\sigma)} \left\{ ent_\nu(\sigma) + \int_\Sigma \log g \, d\nu \right\}.$$

Furthermore, if g is a positive Hölder continuous g-function, then $RND_{\mu,\tilde{\mu}}$ is actually a Hölder continuous function and

$$RND_{\mu,\tilde{\mu}}(w) \equiv g(w). \tag{4}$$

However, this fact may not be true in general for a merely continuous g-function. One of our goals in generalized Gibbs theory is to associate with a circle g-function (see the definition in the next section) a g-measure, which we will call a geometric Gibbs measure, such that the equality (4) holds. We will also study the uniqueness of a geometric Gibbs measure for any given circle g-function.

3 Circle g-Functions

We use

$$T = \{z \in \mathbf{C} \mid |z| = 1\}$$

Yunping.Jiang@qc.cuny.edu

to denote the unit circle in the complex plane $\mathbf{C}$. The universal cover of T is the real line $\mathbf{R}$ with the covering map

$$z = \pi(x) = e^{2\pi i x} : \mathbf{R} \to T,$$

where $2\pi x$ is the angle of z.

Consider an orientation-preserving covering map $f : T \to T$ of degree d. We normalize it by assuming that $f(1) = 1$. We use F to denote the lift of f to $\mathbf{R}$ such that $F(0) = 0$. We have that $F(x+1) = F(x) + d$.

We use h to denote a circle homeomorphism and assume that $h(1) = 1$. Let H be the lift of h such that $H(0) = 0$. We also have $H(x+1) = H(x) + 1$.

A C^1 circle endomorphism f is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C\lambda^n, \quad x \in \mathbf{R}, \quad n = 1, 2, \cdots.$$

A circle endomorphism f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if f' is α-Hölder continuous.

A circle homeomorphism h is called quasisymmetric if there is a constant $M \geq 1$ such that

$$M^{-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq M, \quad \forall x \in \mathbf{R},\ \forall t > 0.$$

In particular, if we can take $M = 1 + \varepsilon(t)$ for some bounded positive real function $\varepsilon(t)$ with $\varepsilon(t) \to 0^+$ as $t \to 0^+$, then h is called symmetric. For example, a C^1-diffeomorphism of T is symmetric. But, we would like to remind the reader, a symmetric homeomorphism may be totally singular, that is, it may map a positive Lebesgue measure subset to a zero Lebesgue measure subset, and vice versa.

A circle endomorphism f is called uniformly symmetric if there is a bounded real function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that

$$\frac{1}{1+\varepsilon(t)} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1 + \varepsilon(t),\ \forall x \in \mathbf{R},\ \forall t > 0,\ n = 1, 2, \cdots.$$

A $C^{1+\alpha}$, for some $0 < \alpha \leq 1$, circle expanding endomorphism f is uniformly symmetric. Again we would like to remind the reader that a uniformly symmetric circle endomorphism may be totally singular.

In the terms of complex analysis, the reader can find descriptions of quasisymmetric circle homeomorphisms in [1], symmetric circle homeomorphisms in [7], and uniformly symmetric circle endomorphisms in [5].

Consider the space

$$\Sigma^+ = \{v = j_0 j_1 \cdots j_{n-1} \cdots \mid j_{n-1} \in \{0, 1, \cdots, d-1\},\ n = 1, \cdots\}$$

Yunping.Jiang@qc.cuny.edu

and the shift map

$$\sigma^+(v) = j_1 \cdots j_{n-1} j_n \cdots : \Sigma^+ \to \Sigma^+.$$

The space $\Sigma^+ = \prod_0^\infty \{0, 1, \cdots, d-1\}$ is a compact topological space with the product topology. An n-cylinder $[v]_n^+$ containing $v = j_0 j_1 \cdots j_{n-1} \cdots$ is the subset of all points $v' = j_0 \cdots j_{n-1} j'_n j'_{n+1} \cdots$ for $j'_{n+m} \in \{0, 1, \cdots, d-1\}$ and $m = 0, 1, \cdots$. The set of all cylinders forms a topological basis of Σ^+ such that it is a compact topological space. The space Σ^+ with this topology is called the symbolic representation of the unit circle T. More precisely, for any $z = e^{2\pi i x} \in T$, we have that

$$x = x(v) = \sum_{k=0}^{\infty} \frac{j_k}{d^{k+1}}, \quad v = j_0 j_1 \cdots j_{n-1} \cdots \in \Sigma^+. \tag{5}$$

The Lebesgue metric $|v - v'| = |x(v) - x(v')|$ induces the Lebesgue measure m_0 on Σ^+.

Every uniformly symmetric circle endomorphism f is semi-conjugate to σ^+, that is, we have a projection $\pi_f : \Sigma^+ \to T$, which is 1-1 except for a countable set, such that

$$\pi_f \circ \sigma^+(v) = f \circ \pi_f(v), \quad v \in \Sigma^+.$$

This implies that any two uniformly symmetric circle endomorphisms f and g of the same degree are topologically conjugate, that is, there is a circle homeomorphism h of T such that

$$f \circ h = h \circ g.$$

Furthermore, h is a quasisymmetric circle homeomorphism (refer to [9]).

Now let us return to the space Σ. Suppose f is a uniformly symmetric circle endomorphism. For any $w = \cdots i_{n-1} \cdots i_1 i_0 \in \Sigma$, let $v_n = j_0 j_1 \cdots j_{n-2} j_{n-1}$ and $v_{n-1} = \sigma(v_n) = j_0 j_1 \cdots j_{n-2}$ where $j_0 = i_{n-1}, \cdots, j_{n-2} = i_1, j_{n-1} = i_0$. Consider two intervals on T,

$$I_{v_n} = \pi_f([v]_n^+) \subset I_{v_{n-1}} = \pi_f([v']_{n-1}^+)$$

where $v = v_n \cdots$ and $v' = v_{n-1} \cdots$ and the ratio

$$g_n(w) = \frac{|I_{v_n}|}{|I_{v_{n-1}}|}.$$

Yunping.Jiang@qc.cuny.edu

We have that

Theorem 2 (Circle g-Function [15]) *Suppose f is a uniformly symmetric circle endomorphism. Then the limiting function*

$$g(w) = \lim_{n\to\infty} g_n(w) : \Sigma \to \mathbf{R}$$

exists and is a continuous positive function. The convergence is uniform. Furthermore, if f is $C^{1+\alpha}$, then $g(w)$ is a Hölder continuous positive function and the convergence is exponential. The function $g(w)$ is called a circle g-function since it satisfies the condition (3).

Furthermore, when $d = 2$, $g(w)$ also satisfies a compatibility condition

$$\prod_{n=0}^{\infty} \frac{g(w0\overbrace{1\cdots1}^{n})}{g(w1\underbrace{0\cdots0}_{n})} = const, \quad \forall w \in \Sigma, \tag{6}$$

where the convergence in the formula is uniform on Σ. The conditions (3) and (6) give a complete characterization of a circle g-function as proved in [3, 4, 11]. That is, a continuous positive g-function is a circle g-function if and only if it satisfies the conditions (3) and (6). For a Hölder continuous positive g-function, it is a circle g-function if and only if it satisfies the conditions (3) and (6) and the convergence in (6) is exponential. Furthermore, the realized uniformly symmetric circle endomorphism f is $C^{1+\alpha}$. Note that, from our proof of Katok's conjecture in [12], any C^1 uniformly symmetric circle endomorphism is expanding.

We use $\mathscr{G}$ to denote the space of all circle g-functions on Σ and $\mathscr{HG}$ to denote the space of all Hölder continuous circle g-functions on Σ.

4 Geometric Gibbs Measures

For any $v = j_0 \cdots j_{n-1} \cdots \in \Sigma^+$, let $x = x(v)$ in (5). Then $z = e^{2\pi i x} \in T$. Consider the cylinder $[v]_n^+$ in Σ^+. Let $v_n = j_0 \cdots j_{n-1}$ and $w_n = i_{n-1} \cdots i_0$ where $i_0 = j_{n-1}, \cdots, i_{n-1} = j_0$. Then we have the n-cylinder $[w]_n$ in Σ where $w = \cdots w_n \in \Sigma$.

Suppose $\mu \in \mathscr{M}(\sigma)$ is non-atomic and does not take zero on any cylinders $[w]_n$. For each $n \geq 0$, take d^{n+1} intervals labeled $I_{\underbrace{00\ldots0}_{n}}, \cdots, I_{v_n}, \cdots, I_{\underbrace{(d-1)(d-1)\ldots(d-1)}_{n}}$, each with angle length

$$|I_{v_n}| = 2\pi\mu([w]_n).$$

Yunping.Jiang@qc.cuny.edu

Arrange them counter-clockwise in numerical order of the angle of z on the unit circle T beginning at 1. Since μ is σ-invariant, we have that

$$I_{v_n} = \cup_{j=0}^{d-1} I_{v_n j}$$

and that

$$\cdots \subset I_{v_n} \subset I_{v_{n-1}} \subset \cdots \subset I_{v_1}.$$

Since μ is non-atomic and does not take zero on any cylinders $[w]_n$, $\cap_{k=1}^{\infty} I_{v_n}$ contains only one point which we denote as $h_\mu(z)$. This defines a homeomorphism $h_\mu(z) : T \to T$.

Definition 1 Suppose g is a circle g-function. A non-atomic σ-invariant probability measure μ is a geometric Gibbs measure associated with g if $f_\mu = h_\mu \circ q_d \circ h_\mu^{-1}$ is a uniformly symmetric circle endomorphism and

$$\lim_{n\to\infty} \frac{\mu([w]_n)}{\mu([\sigma(w)]_{n-1})} = g(w) \tag{7}$$

uniformly on $w \in \Sigma$.

We have proved the following theorem.

Theorem 3 (Existence [15]) *For any circle g-function g, we can find a geometric Gibbs measure $\mu = \mu(g)$ associated with it. Furthermore, if g is a Hölder continuous circle g-function, then the measure μ found in the first part of this theorem is the Gibbs measure in the classical sense.*

For the measure μ in the above theorem, we have that

$$RND_{\mu,\tilde{\mu}}(w) \equiv g(w),$$

as we expected in Sect. 2.

An important question to ask at this point is whether a geometric Gibbs measure is unique. Even when g is a Hölder continuous circle g-function, although we know it has a unique Gibbs measure in the classical sense, it may still have more than one geometric Gibbs measures. However, we have the following theorem.

Theorem 4 (Uniqueness [13, 15]) *The constant circle g-function $g(w) = 1/d$ has only one geometric Gibbs measure associated with it.*

We would like to note that a general circle g-function is very non-trivial. This makes the study of uniqueness more difficult but interesting.

Another important topic to study at this point is the ergodicity of a geometric Gibbs measure. For a Hölder continuous circle g-function g, we know that the Gibbs measure $\mu = \mu(g)$ in the classical sense is ergodic, that is, for any Borel subset A of Σ, if $\sigma^{-1}(A) = A$ and if $\mu(A) > 0$, then $\mu(A) = 1$. We would like to know the

Yunping.Jiang@qc.cuny.edu

ergodicity for any geometric Gibbs measure associated with a circle g-function. We expect that the answer is affirmative.

Furthermore, we know that a geometric Gibbs measure is an equilibrium state.

Theorem 5 (Equilibrium [6, 15]) *Suppose g is a circle g-function. Every geometric Gibbs measure μ corresponding to g is an equilibrium state in the following sense,*

$$0 = ent_\mu(\sigma) + \int_\Sigma \log g \, d\mu = \sup_{\nu \in \mathscr{M}(\sigma)} \left\{ ent_\nu(\sigma) + \int_\Sigma \log g \, d\nu \right\}.$$

5 Teichmüller Metric

We introduce a Teichmüller metric on $\mathscr{G}$ and show that it is a complete metric. Under this metric, $\mathscr{HG}$ is dense in $\mathscr{G}$. We use [1, 7, 19] as references for the Teichmüller theory and for the quasiconformal mapping theory.

Suppose $\mathscr{US}$ is the space of all uniformly symmetric circle endomorphisms of degree d. Let $q_d(z) = z^d$ be the basepoint in $\mathscr{US}$. We first define the Teichmüller space for $\mathscr{US}$ as follows. For any $f \in \mathscr{US}$, let h_f be the conjugacy from f to q_d, i.e.,

$$f \circ h_f = h_f \circ q_d.$$

We know that h_f is quasisymmetric. Thus we can think of $\mathscr{US}$ as the space of marking pairs (f, h_f). We define an equivalence relation $\sim_T$: Two pairs $(f, h_f) \sim_T (g, h_g)$ if $h_f \circ h_g^{-1}$ is symmetric. The Teichmüller space

$$\mathscr{TUS} = \{[(f, h_f)] \mid (f, h_f) \in \mathscr{US}, \text{ with the basepoint } [(q_d, id)]\}$$

is the space of all $\sim_T$-equivalence classes $[(f, h_f)]$. We have a one-to-one correspondence between $\mathscr{G}$ and $\mathscr{TUS}$ (refer to [15]). Thus we have that

$$\mathscr{G} = \mathscr{TUS}.$$

By using $\mathscr{TUS}$, we define a Teichmüller metric on $\mathscr{G}$.

Let $\mathscr{QS}$ be the set of all quasisymmetric homeomorphisms of T. Let $\mathscr{S}$ be the subset of $\mathscr{QS}$ consisting of all symmetric homeomorphisms of T. For any $h \in \mathscr{QS}$, let $\mathscr{E}_h$ be the set of all quasiconformal extensions of h into the unit disk. For each $\tilde{h} \in \mathscr{E}_h$, let $\mu_{\tilde{h}} = \tilde{h}_{\bar{z}}/\tilde{h}_z$ be its complex dilatation. Let

$$k_{\tilde{h}} = \|\mu(z)\|_\infty \quad \text{and} \quad K_{\tilde{h}} = \frac{1 + k_{\tilde{h}}}{1 - k_{\tilde{h}}}.$$

Yunping.Jiang@qc.cuny.edu

Here $K_{\tilde{h}}$ is called the quasiconformal dilatation of $\tilde{h}$. Using quasiconformal dilatation, we can define a pseudo-distance in $\mathscr{QS}$ by

$$d(h_1, h_2) = \frac{1}{2} \inf\{\log K_{\tilde{h}_1 \tilde{h}_2^{-1}} \mid \tilde{h}_1 \in \mathscr{E}_{h_1}, \tilde{h}_2 \in \mathscr{E}_2\}.$$

It will induce a distance in the space $\mathscr{UT}$ of $\mathscr{QS}$ modulo the space of Möbius transformation preserving T, which is the universal Teichmüller space and which is a complete metric space and a complex manifold with complex structure compatible with the Hilbert transform. Now consider the space

$$\mathscr{AUT} = \mathscr{QS} \text{ modulo } \mathscr{S}.$$

It is called an asymptotical universal Teichmüller space. Given two cosets $\mathscr{S}h_1$ and $\mathscr{S}h_2$ in this factor space, define

$$\overline{d}(\mathscr{S}h_1, \mathscr{S}h_2) = \inf_{A,B \in \mathscr{S}} d(Ah_1, Bh_2).$$

It defines a distance on $\mathscr{AUT}$. The asymptotical Teichmüller space $(\mathscr{AUT}, \overline{d}(\cdot,\cdot))$ is a complete metric space and a complex manifold. The topology on $(\mathscr{AUT}, \overline{d}(\cdot,\cdot))$ is the finest topology which makes the projection $\pi : \mathscr{UT} \to \mathscr{AUT}$ continuous, and π is also holomorphic. Refer to [7].

An equivalent topology on the quotient space $\mathscr{AUT}$ can be defined as follows. For any $h \in \mathscr{QS}$, let $\tilde{h}$ be a quasiconformal extension of h to a small neighborhood of T in the complex plane. Suppose U is the domain of $\tilde{h}$. Let

$$\mu_{\tilde{h}}(z) = \frac{\tilde{h}_{\bar{z}}(z)}{\tilde{h}_z(z)},\ z \in U, \quad k_{\tilde{h}} = \|\mu_{\tilde{h}}(z)\|_{\infty,U}, \quad \text{and} \quad B_{\tilde{h}} = \frac{1 + k_{\tilde{h}}}{1 - k_{\tilde{h}}}.$$

Then the boundary dilatation h is defined as

$$B_h = \inf_{\tilde{h}} B_{\tilde{h}},$$

where the infimum is taken over all quasiconformal extensions $\tilde{h}$ of h in a neighborhood of T. It is known that h is symmetric if and only if $B_h = 1$. Define

$$\tilde{d}(h_1, h_2) = \frac{1}{2} \log B_{h_2^{-1} h_1}.$$

Then it is a distance on $\mathscr{AUT}$. The two distances $\overline{d}$ and $\tilde{d}$ on $\mathscr{AUT}$ are equal.

There is a natural embedding from

$$\mathscr{G} = \mathscr{TUS} \ni g = [f, h_f] \hookrightarrow [h_f] \in \mathscr{AUT}.$$

Yunping.Jiang@qc.cuny.edu

Thus, the restriction of $\tilde{d}(\cdot,\cdot)$ on $\mathscr{G} = \mathscr{TUS}$ gives a distance which we denote as $d_{\mathscr{G}}(\cdot,\cdot)$. We call the space

$$\left(\mathscr{G}, d_{\mathscr{G}}(\cdot,\cdot)\right)$$

the Teichmüller space of circle g-functions.

Theorem 6 (Completeness [15]) *The Teichmüller space* $\left(\mathscr{G}, d_{\mathscr{G}}(\cdot,\cdot)\right)$ *is a complete metric space and* $\mathscr{HG}$ *is dense in this space.*

Moreover, the space $\left(\mathscr{G}, d_{\mathscr{G}}(\cdot,\cdot)\right)$ has a complex Banach manifold structure (refer to [8, 15]). We would like to point out that there is a maximal norm

$$\|g\| = \max_{w\in\Sigma} |g(w)|$$

on the space $\mathscr{G}$. This also introduces a metric $d_{max}(\cdot,\cdot)$ on $\mathscr{G}$. But this metric is not complete. Moreover, this metric will not measure the change of a geometric Gibbs measure in a good sense. For example, even if $d_{max}(\cdot,\cdot)$ is small, the change of a geometric Gibbs measure could be big. So it is just like the Euclidean metric on the open unit disk. The Teichmüller metric we have introduced is precisely like the hyperbolic metric (or Lobachevsky metric or Poincaré metric) on the open unit disk. Topologies induced from both metrics are the same (refer to [14, 15]).

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Yunping.Jiang@qc.cuny.edu

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Yunping.Jiang@qc.cuny.edu