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Bounded Geometry and Families of Meromorphic Functions with Two Asymptotic Values

Tao Chen, Yunping Jiang¹ and Linda Keen²

Abstract

In this paper we study the topological class of universal covering maps from the plane to the sphere with two removed points; we call the elements topological transcendental maps with two asymptotic values and we denote the space by $\mathcal{AV}2$. We prove that an element $f \in \mathcal{AV}2$ with finite post-singular set is combinatorially equivalent to a meromorphic transcendental map g with constant Schwarzian derivative if and only if f satisfies an analytic condition we call *bounded geometry*. We plan to relate the bounded geometry condition to topological conditions such as Levy cycles and Thurston obstructions and to the geometric condition called a canonical Thurston obstruction in a future paper.

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Keywords and Phrases: Topological transcendental maps with two asymptotic values, meromorphic transcendental map with constant Schwarzian derivative, bounded geometry, combinatorial **equivalence asymptotic** value, post-singularly finite, topological constraint, compactness.

1 Introduction

Thurston asked the question “when can we realize a given branched covering map as a holomorphic map in such a way that the post-critical sets correspond?” and answered it for post-critically finite degree d branched covers of the sphere [T, DH].

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His theorem is that a **postcritically** finite degree $d \geq 2$ branched covering of the sphere, with hyperbolic orbifold, is either combinatorially equivalent to a rational map or there is a topological obstruction, now called a “Thurston obstruction”. The rational map is unique up to conjugation by a Möbius transformation.

Thurston’s theorem is proved by defining an appropriate Teichmüller space of rational maps and a holomorphic self map of this space. Iteration of this map converges if and only if no Thurston obstruction exists. This method does not naturally extend to transcendental maps because the proof uses the finiteness of both the degree and the post-critical set in a crucial way. Hubbard, Schleicher, and Shishikura [HSS] generalized Thurston’s theorem to a special infinite degree family they call “exponential type” maps. In that paper, the authors study the limiting behavior of quadratic differentials associated to the exponential functions with finite post-singular set. They use a Levy cycle condition (a special case of Thurston’s topological condition) to characterize when it is possible to realize a given exponential type map with finite post-singular set as an exponential map by combinatorial equivalence. The main purpose of this paper is to use a different approach based on the framework expounded in [Ji] (see also [JZ, CJ]) to understand the characterization problem for a slight generalization of this family of infinite degree maps.

In this paper we define a class of maps called topological transcendental maps with two asymptotic values which we denote by $\mathcal{AV}2$. The elements in this class are universal covering maps from the plane to the sphere with two removed points. A meromorphic transcendental map g in $\mathcal{AV}2$ is a meromorphic function with constant Schwarzian derivative and we denote the space of all meromorphic functions with constant Schwarzian derivative by $\mathcal{M}2$. Our main result in this paper is the characterization of an $f \in \mathcal{AV}2$ combinatorially equivalent to a map $g \in \mathcal{M}2$ in terms of a condition called *bounded geometry*. The main theorem is

Theorem 1 (Main Theorem). *A post-singularly finite map f in $\mathcal{AV}2$ is combinatorially equivalent to a post-singularly finite transcendental meromorphic function g with constant Schwarzian derivative if and only if it has bounded geometry. The realization is unique up to conjugation by an affine map of the plane.*

Our techniques involve adapting the Thurston iteration scheme to our situation. We work with a fixed normalization. There are two important parts to the proof of the main theorem. The first part is to prove that the bounded geometry condition implies the iterates remain in a compact subset of the Teichmüller space. This analysis depends on defining a topological condition that constrains the iterates. The second part is to use the compactness of the iterates to prove that the iteration scheme converges in the Teichmüller space. This part of the proof involves an analysis of quadratic differentials associated to our functions.

The paper is organized as follows. In §2 we review the properties of meromorphic functions with two asymptotic values that constitute the space \mathcal{M}_2 . In §3, we define the family $\mathcal{AV}2$ that consists of topological maps modeled on maps in \mathcal{M}_2 and show that $\mathcal{M}_2 \subset \mathcal{AV}2$. In §4 we define *combinatorial equivalence* between maps in $\mathcal{AV}2$ and in §5 define the Teichmüller space \mathcal{T}_f for a map $f \in \mathcal{AV}2$. In §6, we introduce the induced map σ_f from the Teichmüller space \mathcal{T}_f into itself; this is

the map that defines the Thurston iteration scheme. In §7, we define the concept of *bounded geometry* and in §8 we prove the necessity of the bounded geometry condition in the main theorem. In §9, we give the proof of the sufficiency assuming the iterates remain in a compact subset of T_f . In §10.1, we define a topological property of the post-singularly finite map f in $\mathcal{AV}2$ in terms of the winding number of a certain closed curve. We prove that the winding number is unchanged during iteration of the map σ_f and so provides a topological constraint on the iterates. Finally, in §10.2, we show how the bounded geometry condition together with this topological constraint implies the functions remain in a compact subset of T_f under the iteration to complete the proof of the main theorem.

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2 The space \mathcal{M}_2

In this section we define the space of meromorphic functions \mathcal{M}_2 . It is the model for the more general space of topological functions $\mathcal{AV}2$ that we define in the next section. We need some standard notation and definitions:

\mathbb{C} is the complex plane, $\hat{\mathbb{C}}$ is the Riemann sphere and \mathbb{C}^* is complex plane punctured at the origin.

Definition 1. *Given a meromorphic function g , the point v is a logarithmic singularity for the map g^{-1} if there is a neighborhood U_v and a component V of $g^{-1}(U_v \setminus \{v\})$ such that the map $g : V \rightarrow U_v \setminus \{v\}$ is a holomorphic universal covering map. The point v is also called an asymptotic value for g and V is called an asymptotic tract for g . A point may be an asymptotic value for more than one asymptotic tract. An asymptotic value may be an omitted value.*

Definition 2. *Given a meromorphic function g , the point v is an algebraic singularity for the map g^{-1} if there is a neighborhood U_v such that for every component V_i of $g^{-1}(U_v)$ the map $g : V_i \rightarrow U_v$ is a degree d_{V_i} branched covering map and $d_{V_i} > 1$ for finitely many components V_1, \dots, V_n . For these components, if $c_i \in V_i$ satisfies $g(c_i) = v$ then $g'(c_i) = 0$; that is c_i is a critical point of g for $i = 1, \dots, n$ and v is a critical value.*

Note that by a theorem of Hurwitz, if a meromorphic function is not a homeomorphism, it must have at least two singular points (i.e., critical points and asymptotical values) and, by the big Picard theorem, no transcendental meromorphic function $g : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ can omit more than two values.

The space \mathcal{M}_2 consists of meromorphic functions whose only singular values are its omitted values. More precisely,

Definition 3. The space \mathcal{M}_2 consists of meromorphic functions $g : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ with exactly two asymptotic values and no critical values.

2.1 Examples

Examples of functions in \mathcal{M}_2 are the exponential functions $\alpha e^{\beta z}$ and the tangent functions $\alpha \tan i\beta z = i\alpha \tanh \beta z$ where α, β are complex constants.

The asymptotic values for the exponential functions above are $\{0, \infty\}$; the half plane $\Re\beta z < 0$ is an asymptotic tract for 0 and the half plane $\Re\beta z > 0$ is an asymptotic tract for infinity. The asymptotic values for the tangent functions above are $\{\alpha i, -\alpha i\}$ and the asymptotic tract for αi is the half plane $\Im\beta z > 0$ while the asymptotic tract for $-\alpha i$ is the half plane $\Im\beta z < 0$.

2.2 Nevanlinna's theorem

To find the form of the most general function in \mathcal{M}_2 we use a special case of a theorem of Nevanlinna [N].

Theorem 2 (Nevanlinna). *Every meromorphic function g with exactly p asymptotic values and no critical values has the property that its Schwarzian derivative is a polynomial of degree $p - 2$. That is*

$$S(g) = \left(\frac{g''}{g'}\right)' - \frac{1}{2} \left(\frac{g''}{g'}\right)^2 = a_{p-2}z^{p-2} + \cdots + a_1z + a_0. \quad (1)$$

Conversely, for every polynomial $P(z)$ of degree $p-2$, the solution to the Schwarzian differential equation $S(g) = P(z)$ is a meromorphic function with p asymptotic values and no critical values.

It is easy to check that $S(\alpha e^{\beta z}) = -\frac{1}{2}\beta^2$ and $S(\alpha \tan \frac{i\beta}{2}z) = -\frac{1}{2}\beta^2$.

To find all functions in \mathcal{M}_2 , let $\beta \in \mathbb{C}$ be constant and consider the Schwarzian differential equation

$$S(g) = -\beta^2/2 \quad (2)$$

and the related second order linear differential equation

$$w'' + \frac{1}{2}S(g)w = w'' - \frac{\beta^2}{4}w = 0. \quad (3)$$

It is straightforward to check that if w_1, w_2 are linearly independent solutions to equation (3), then $g_\beta = w_2/w_1$ is a solution to equation (2).

Normalizing so that $w_1(0) = 1, w_1'(0) = -1, w_2(0) = 1, w_2'(0) = 1$ and solving equation (3), we have $w_1 = e^{-\frac{\beta}{2}z}, w_2 = e^{\frac{\beta}{2}z}$ as linearly independent solutions and $g_\beta(z) = e^{\beta z}$ as the solution to equation (2). An arbitrary solution to equation (2) then has the form

$$\frac{Aw_2 + Bw_1}{Cw_2 + Dw_1}, \quad A, B, C, D \in \hat{\mathbb{C}}, \quad AD - BC = 1 \quad (4)$$

and its asymptotic values are $\{A/C, B/D\}$.

Remark 1. *The asymptotic values are distinct and omitted.*

Remark 2. *If $B = C = 0, AD = 1, A = \sqrt{\alpha}$ we obtain the exponential family $\{\alpha e^{\beta z}\}$ with asymptotic values at 0 and ∞ . If $A = -B = \sqrt{\frac{\alpha i}{2}}, C = D = \sqrt{-\frac{i}{2\alpha}}$ we obtain the tangent family $\{\alpha \tan \frac{i\beta}{2} z\}$ whose asymptotic values $\{\pm \alpha i\}$ are symmetric with respect to the origin.*

Remark 3. *Note that in the solutions of $S(g) = -\beta^2/2$ what appears is e^β , not β ; this creates an ambiguity about which branches of the logarithm of e^β correspond to the solution of equation (2). In Section 10.1 we address this ambiguity in our situation. We show that the topological map we start with determines a topological constraint which in turn, defines the appropriate branch of the logarithm for each of the iterates in our iteration scheme.*

Remark 4. *One of the basic features of the Schwarzian derivative is that it satisfies the following cocycle relation: if f, g are meromorphic functions then*

$$S(g \circ f(z)) = S(g(f(z)))f'(z)^2 + S(f(z)).$$

In particular, if T is a Möbius transformation, $S(T(z)) = 0$ and $S(T \circ g(z)) = S((g(z)))$ so that post-composing by T doesn't change the Schwarzian.

In our dynamical problems the point at infinity plays a special role and the dynamics are invariant under post-composition by an affine map. Thus, we may assume that all the solutions have one asymptotic value at 0 and that they take the value 1 at 0.

Since this is true for $g_\beta(z) = e^{\beta z}$, any solution with this normalization has the form³

$$g_{\alpha,\beta}(z) = \frac{\alpha g_\beta(z)}{(\alpha - \frac{1}{\alpha})g_\beta(z) + \frac{1}{\alpha}} \tag{5}$$

where α is an arbitrary value in \mathbb{C}^* . The second asymptotic value is $\lambda = \frac{\alpha}{\alpha - \frac{1}{\alpha}}$. It takes values in $\mathbb{C} \setminus \{0, 1\}$. The point at infinity is an essential singularity for all these functions.

The parameter space \mathcal{P} for these functions is the two complex dimensional space

$$\mathcal{P} = \{\alpha, \beta \in \mathbb{C}^*\}.$$

The parameters define a natural complex structure for the space \mathcal{M}_2 . The subspace of *entire* functions in \mathcal{M}_2 is the one dimensional subspace of \mathcal{P} defined by fixing $\alpha = 1$ and varying β ;

$$g_\beta(z) = e^{\beta z}.$$

The tangent family has symmetric asymptotic values. Renormalized, it forms another one dimensional subspace of \mathcal{P} . This is defined by fixing $\alpha = \sqrt{2}$ and varying β ;

$$g_{\frac{1}{\sqrt{2}},\beta}(z) = 1 + \tanh \frac{\beta}{2} z = \frac{\sqrt{2}e^{\beta z}}{\frac{1}{\sqrt{2}}e^{\beta z} + \frac{1}{\sqrt{2}}}.$$

³Notice that $g_{\alpha,\beta}$ is obtained from g_β by a Möbius transformation with determinant 1.

These functions have asymptotic values at $\{0, 2\}$ and $g_{\frac{1}{\sqrt{2}}, \beta}(0) = 1$.

Definition 4. For $g_{\alpha, \beta}(z) \in \mathcal{M}_2$, the set $\Omega = \{0, \lambda\}$ of asymptotic values is the set of singular values. The post-singular set P_g is defined by

$$P_g = \overline{\bigcup_{x \in \Omega} \bigcup_{n \geq 0} g^n(x)} \cup \{\infty\}.$$

Note that we include the point at infinity separately in P_g because whether or not it is an asymptotic value, it is an essential singularity and its forward orbit is not defined. The asymptotic values are in P_g and, since 0 and λ are omitted and $g_{\alpha, \beta}(0) = 1 \in P_g$, $\#P_g \geq 3$.

3 The space $\mathcal{AV}2$

We now want to consider the topological structure of functions in \mathcal{M}_2 and define $\mathcal{AV}2$ to be the set of maps with the same topology.

Definition 5. Let X be a simply connected open surface and let S^2 be the 2-sphere. Let $f_{a,b} : X \rightarrow S^2 \setminus \{a, b\}$ be an unbranched covering map; that is, a universal covering map. If Y is also a simply connected open surface we say the pair $(X, f_{a,b}^1)$ is equivalent to the pair $(Y, f_{c,d}^2)$ if and only if there is a homeomorphism $h : X \rightarrow Y$ such that $f_{c,d}^2 \circ h = f_{a,b}^1$. An equivalence class of such classes is called a 2-asymptotic value map and the space of these pairs is denoted by $\mathcal{AV}2$.

Let $(X, f_{a,b})$ be a representative of a map in $\mathcal{AV}2$. By abuse of notation, we will often suppress the dependence on the equivalence class and identify X with $S^2 \setminus \{\infty\}$ and refer to $f_{a,b}$ as an element of $\mathcal{AV}2$.

By definition $f_{a,b}$ is a local homeomorphism and satisfies the following conditions:

For $v = a$ or $v = b$, let $U_v \subset X$ be a neighborhood of v whose boundary is a simple closed curve that separates a from b and contains v in its interior.

1. $f_{a,b}^{-1}(U_v \setminus \{v\})$ is connected and simply connected.
2. The restriction $f_{a,b} : f_{a,b}^{-1}(U_v \setminus \{v\}) \rightarrow (U_v \setminus \{v\})$ is a regular covering of a punctured topological disk whose degree is infinite.
3. $f^{-1}(\partial U_v)$ is an open curve extending to infinity in both directions.

In analogy with meromorphic functions we say

Definition 6. v is called a logarithmic singularity of $f_{a,b}^{-1}$ or, equivalently, an asymptotic value of $f_{a,b}$. The domain $V_v = f_{a,b}^{-1}(U_v \setminus \{v\})$ is called an asymptotic tract for v .

Definition 7. $\Omega_f = \{a, b\}$ is the set of singular values of $f_{a,b}$.

Endow S^2 with the standard complex structure so that it is identified with $\hat{\mathbb{C}}$. By the classical uniformization theorem, for any pair $(X, f_{a,b})$, there is a map $\pi : \mathbb{C} \rightarrow X$ such that $g_{a,b} = f_{a,b} \circ \pi$ is meromorphic. It is called *the meromorphic function associated to $f_{a,b}$* .

By Nevanlinna's theorem $S(g(z))$ is constant and moreover,

Proposition 1. *If $g(z) \in \mathcal{M}_2$ with $\Omega_g = \{a, b\}$ then $g(z) = g_{a,b}(z) \in \mathcal{AV}2$ and, conversely, if $g_{a,b} \in \mathcal{AV}2$ is meromorphic then $g_{a,b} \in \mathcal{M}_2$.*

Proof. Any $g(z) \in \mathcal{M}_2$ is a universal cover $g : \mathbb{C} \rightarrow \hat{\mathbb{C}} \setminus \Omega_g$ and so belongs to $\mathcal{AV}2$. Conversely, if $g_{a,b} \in \mathcal{AV}2$, it is meromorphic and its only singular values are the omitted values $\{a, b\}$; it is thus in \mathcal{M}_2 . □

We define the post-singular set for functions in $\mathcal{AV}2$ just as we did for functions in \mathcal{M}_2 .

Definition 8. *For $f = f_{a,b} \in \mathcal{AV}2$, the post-singular set P_f is defined by*

$$P_f = \overline{\bigcup_{n \geq 0} f^n(\Omega_f)} \cup \{\infty\}.$$

Note that under the identification of S^2 with the Riemann sphere and X with the complex plane, $S^2 \setminus X$ is the point at infinity and it has no forward orbit although it may be an asymptotic value. We therefore include it in P_f .

Post-composition of $f_{a,b}$ with an affine transformation T results in another map in $\mathcal{AV}2$. In what follows, therefore, we will always assume $a = 0$ and the second asymptotic value, λ , depending on T and b , is determined by the condition $f(0) = 1$.

We will be concerned only with functions in $\mathcal{AV}2$ such that P_f is finite. Such functions are called *post-singularly finite*.

4 Combinatorial equivalence

In this section we define combinatorial equivalence for functions in $\mathcal{AV}2$. Choosing representatives $(X, f_{a,b})$ of the $\mathcal{AV}2$ -equivalence classes, we may assume X is always $S^2 \setminus \{\infty\}$ and $\{0, b\}$ are the singular points for all the functions so we will omit the subscripts denoting the omitted points in the definitions below.

Definition 9. *Suppose $(X, f_1), (X, f_2)$ are representatives of two post-singularly finite functions in $\mathcal{AV}2$, chosen as above. We say that they are combinatorially equivalent if there are two homeomorphisms ϕ and ψ of S^2 onto itself fixing $\{0, \infty\}$ such that $\phi \circ f_2 = f_1 \circ \psi$ on X and $\phi^{-1} \circ \psi$ is isotopic to the identity of S^2 rel P_{f_1} .*

The commutative diagram for the above definition is

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X \\ \downarrow f_1 & & \downarrow f_2 \\ S^2 & \xrightarrow{\phi} & S^2 \end{array}$$

5 Teichmüller space T_f .

Let $\mathbf{M} = \{\mu \in L^\infty(\hat{\mathbb{C}}) \mid \|\mu\|_\infty < 1\}$ be the unit ball in the space of all measurable functions on the Riemann sphere. Each element $\mu \in \mathbf{M}$ is called a Beltrami coefficient. For each Beltrami coefficient μ , the Beltrami equation,

$$w_{\bar{z}} = \mu w_z$$

has a unique quasiconformal solution w^μ which maps $\hat{\mathbb{C}}$ to itself and fixes $0, 1, \infty$. Moreover, w^μ depends holomorphically on μ .

Let f be a post-singularly finite function in $\mathcal{AV}2$ with singular set $\Omega_f = \{0, \lambda\}$ and postsingular set P_f . By definition, we have $\#(\Omega_f) = 2$ and $\#(P_f) > 2$. Since post-composition by an affine map is in the equivalence class of f we may always choose a representative such that $\{f(0) = 1\} \subset P_f$; we assume we have always made this choice. It follows that $\lambda \neq 1$ so we always have $\{0, 1, \lambda, \infty\} \subset P_f$.

The Teichmüller space $T(P_f)$ is defined as follows. Given Beltrami differentials $\mu, \nu \in \mathbf{M}$ we say that μ and ν are equivalent, and denote this by $\mu \sim \nu$, if w^μ and w^ν fix $0, 1, \infty$ and $(w^\mu)^{-1} \circ w^\nu$ is isotopic to the identity map of $\hat{\mathbb{C}}$ rel P_f . We set $T_f = T(P_f) = \mathbf{M} / \sim = \{[\mu]\}$.

There is an obvious isomorphism between T_f and the classical Teichmüller space $Teich(R)$ of Riemann surfaces with basepoint $R = \hat{\mathbb{C}} \setminus P_f$. It follows that T_f is a finite-dimensional complex manifold so that the Teichmüller distance d_T and the Kobayashi distance d_K on T_f coincide. It also follows that there are always locally quasiconformal maps in the equivalence class of f ; we always assume we have chosen one such as our representative.

6 Induced holomorphic map σ_f .

For any post-singularly finite f in $\mathcal{AV}2$, there is an induced map $\sigma = \sigma_f$ from T_f into itself given by

$$\sigma([\mu]) = [f^* \mu],$$

where

$$\tilde{\mu}(z) = f^* \mu(z) = \frac{\mu_f(z) + \mu((f(z))\theta(z))}{1 + \overline{\mu_f(z)}\mu(f(z))\theta(z)}, \quad \mu_f = \frac{f_{\bar{z}}}{f_z}, \quad \theta(z) = \frac{\overline{f_z}}{f_z}. \quad (6)$$

Because σ is a holomorphic map we have

Lemma 1. *For any two points τ and τ' in T_f ,*

$$d_T(\sigma(\tau), \sigma(\tau')) \leq d_T(\tau, \tau').$$

The next lemma follows directly from the definitions.

Lemma 2. *A post-singularly finite f in $\mathcal{AV}2$ is combinatorially equivalent to a meromorphic map in \mathcal{M}_2 if and only if σ has a fixed point in T_f .*

Remark 5. *If $\#(P_f) = 3$, then T_f consists of a single point. This point is trivially a fixed point for σ so the main theorem holds. We therefore assume that $\#(P_f) \geq 4$ in the rest of the paper.*

7 Bounded geometry

Let the base point of T_f be the hyperbolic Riemann surface $R = \hat{\mathbb{C}} \setminus P_f$ equipped with the standard complex structure $[0] \in T_f$. For τ in T_f , denote by R_τ the hyperbolic Riemann surface R equipped with the complex structure τ .

A simple closed curve $\gamma \subset R$ is called non-peripheral if each component of $\hat{\mathbb{C}} \setminus \gamma$ contains at least two points of P_f . Let γ be a non-peripheral simple closed curve in R . For any $\tau = [\mu] \in T_f$, let $l_\tau(\gamma)$ be the hyperbolic length of the unique closed geodesic homotopic to γ in R_τ .

For any $\tau_0 \in T_f$, let $\tau_n = \sigma_f^n(\tau_0)$, $n \geq 1$.

Definition 10 (Hyperbolic Version). *We say f has bounded geometry if there is a constant $a > 0$ and a point $\tau_0 \in T_f$ such that $l_{\tau_n}(\gamma) \geq a$ for all $n \geq 0$ and all non-peripheral simple closed curves γ in R .*

The iteration sequence $\tau_n = \sigma_f^n \tau_0 = [\mu_n]$ determines a sequence of subsets of $\hat{\mathbb{C}}$

$$P_n = w^{\mu_n}(P_f), \quad n = 0, 1, 2, \dots$$

Since all the maps w^{μ_n} fix $0, 1, \infty$, it follows that $0, 1, \infty \in P_n$.

Definition 11 (Spherical Version). *We say f has bounded geometry if there is a constant $b > 0$ and a point $\tau_0 \in T_f$ such that*

$$d_{sp}(p_n, q_n) \geq b$$

for all $n \geq 0$ and $p_n, q_n \in P_n$, where

$$d_{sp}(z, z') = \frac{|z - z'|}{\sqrt{1 + |z|^2} \sqrt{1 + |z'|^2}}$$

is the spherical distance on $\hat{\mathbb{C}}$.

Note that $d_{sp}(z, \infty) = \frac{|z|}{\sqrt{1 + |z|^2}}$. Away from infinity the spherical metric and Euclidean metrics are equivalent. Precisely, in any bounded $K \subset \mathbb{C}$, there is a constant $C > 0$ which depends only on K such that

$$C^{-1}d_{sp}(x, y) \leq |x - y| \leq Cd_{sp}(x, y), \quad \forall x, y \in K.$$

The following simple lemma justifies using the term “bounded geometry” in both of the definitions above for f .

Lemma 3. *Consider the hyperbolic Riemann surface $\hat{\mathbb{C}} \setminus X$ equipped with the standard complex structure where X is a finite subset such that $0, 1, \infty \in X$. Let $m = \#(X) \geq 3$. Let $a > 0$ be a constant. If every simple closed geodesic in $\hat{\mathbb{C}} \setminus S$ has hyperbolic length greater than a , then there is a constant $b = b(a, m) > 0$ such that the spherical distance between any two distinct points in S is bounded below by b .*

Proof. If $m = 3$ there are no non-peripheral simple closed curves so in the following argument we always assume that $m \geq 4$. Let $X = \{x_1, \dots, x_{m-1}\}$ and $x_m = \infty$ and let $|\cdot|$ denote the Euclidean metric on \mathbb{C} .

Suppose $0 = |x_1| \leq \dots \leq |x_{m-1}|$. Let $M = |x_{m-1}|$. Then $|x_2| \leq 1$, and we have

$$\prod_{2 \leq i \leq m-2} \frac{|x_{i+1}|}{|x_i|} = \frac{|x_{m-1}|}{|x_2|} \geq M.$$

Hence

$$\max_{2 \leq i \leq m-2} \left\{ \frac{|x_{i+1}|}{|x_i|} \right\} \geq M^{\frac{1}{m-3}}.$$

Let

$$A_i = \{z \in \mathbb{C} \mid |x_i| < z < |x_{i+1}|\}$$

and let $\text{mod}(A_i) = \frac{1}{2\pi} \log \frac{|x_{i+1}|}{|x_i|}$ be its modulus. Then for some integer $2 \leq i_0 \leq m_0 - 2$ it follows that

$$\text{mod}(A_{i_0}) \geq \frac{\log M}{2\pi(m-3)}.$$

Denote the minimum length of closed curves γ in A_{i_0} , measured with respect to the hyperbolic metric on A_{i_0} , by $\|\gamma\|_{A_{i_0}}$. Because A_{i_0} is a round annulus, the core curve realizes this minimum and we can compute its hyperbolic length as $\|\gamma\|_{A_{i_0}} = \frac{\pi}{\text{mod}(A_{i_0})}$.

Since $A_{i_0} \subset \hat{\mathbb{C}} \setminus S$, the hyperbolic density on A_{i_0} is smaller than the hyperbolic density on $\hat{\mathbb{C}} \setminus S$. Therefore, if $l_{\tau_n}(\gamma)$ denotes the length of the shortest geodesic in the homotopy class of γ with respect to the hyperbolic metric on $\hat{\mathbb{C}} \setminus S$, we have $l_{\tau_n}(\gamma) \leq \|\gamma\|_{A_{i_0}}$. This implies that

$$\frac{\pi}{l_{\tau_n}(\gamma)} \geq \text{mod}(A_{i_0}) \geq \frac{\log M}{2\pi(m-3)}.$$

Thus

$$\log M \leq \frac{2\pi^2(m-3)}{l_{\tau_n}} \leq \frac{2\pi^2(m-3)}{a}.$$

This implies that

$$M \leq M_0 = e^{\frac{2\pi^2(m-3)}{a}}.$$

Thus the spherical distance between ∞ and any finite point in X has a positive lower bound M_0 which depends only on a and m .

Next we show that the spherical distance between any two finite points in X has a positive lower bound dependent only on a and m . By the equivalence of the spherical and Euclidean metrics in a bounded set in the plane, it suffices to prove that $|x - y|$ is greater than a constant b for any two finite points in X .

First consider $J_0(z) = 1/z$ which preserves hyperbolic length with $0, 1, \infty \in J_0(X)$. The above argument implies that $1/|x_i| \leq M_0$ for any $2 \leq i \leq m-1$. This implies that $|x_i| \geq 1/M_0$ for any $2 \leq i \leq m-1$. Similarly, for any $x_i \in X$ for $2 \leq i \leq m-1$, consider $J_i(z) = z/(z-x_i)$ which again preserves hyperbolic length.

The above argument implies that $|x_j|/|x_j - x_i| \leq M_0$ for any $2 \leq j \neq i \leq m - 1$. This in turn implies that $|x_j - x_i| \geq 1/M_0^2$ for any $2 \leq j \neq i \leq m - 1$ which proves the lemma. □

8 Necessity

If f is combinatorially equivalent to $g \in \mathcal{M}2$, then σ_f has a unique fixed point τ_0 so that $\tau_n = \tau_0$ for all n . The complex structure on $\hat{\mathbb{C}} \setminus P_f$ defined by τ_0 induces a hyperbolic metric on it. The shortest geodesic in this metric gives a lower bound on the lengths of all geodesics so that f satisfies the hyperbolic definition of bounded geometry.

9 Sufficiency assuming compactness

The proof of sufficiency of our main theorem (Theorem 1) is more complicated and needs some preparatory material. There are two parts: one is a compactness argument and the other is a fixed point argument. From a conceptual point of view, the compactness of the iterates is very natural and simple. From a technical point of view, however, it is not at all obvious. Once one has compactness, the proof of the fixed point argument is quite standard (see [Ji]) and works for much more general cases. We postpone the compactness proof to the next two sections and here give the fixed point argument.

Given $f \in \mathcal{AV}2$ and given any $\tau_0 = [\mu_0] \in T_f$, let $\tau_n = \sigma^n(\tau_0) = [\mu_n]$ be the sequence generated by σ . Let w^{μ_n} be the normalized quasiconformal map with Beltrami coefficient μ_n . Then

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in \mathcal{M}2$$

since it preserves μ_0 and hence is holomorphic. Thus iterating σ , the ‘‘Thurston iteration’’, determines a sequence $\{g_n\}_{n=0}^\infty$ of maps in $\mathcal{M}2$ and a sequence of subsets $P_{f,n} = w^{\mu_n}(P_f)$. Note that $P_{f,n}$ is not, in general, the post-singular set P_{g_n} of g_n . If it were, we would have a fixed point of σ .

Suppose f is a post-singularly finite map in $\mathcal{AV}2$. For any $\tau = [\mu] \in T_f$, w^μ denotes a representative normalized quasiconformal map fixing $0, 1, \infty$ with Beltrami differential μ ; let T_τ and T_τ^* denote the respective tangent space and cotangent space of T_f at τ . Then T_τ^* coincides with the space \mathcal{Q}_μ of integrable meromorphic quadratic differentials $q = \phi(z)dz^2$ on $\hat{\mathbb{C}}$. Integrability means that the norm of q , defined as

$$\|q\| = \int_{\hat{\mathbb{C}}} |\phi(z)| dz d\bar{z}$$

is finite. The finiteness implies that q may only have poles at points of $w^\mu(P_f)$ and that these poles are simple.

Set $\tilde{\tau} = \sigma(\tau) = [\tilde{\mu}]$. By abuse of notation, we write $f^{-1}(P_f)$ for $f^{-1}(P_f \setminus \{\infty\}) \cup \{\infty\}$. We have the following commutative diagram:

$$\begin{CD} \hat{\mathbb{C}} \setminus f^{-1}(P_f) @>w^{\tilde{\mu}}>> \hat{\mathbb{C}} \setminus w^{\tilde{\mu}}(f^{-1}(P_f)) \\ @VfVV @VVg_{\mu, \tilde{\mu}}V \\ \hat{\mathbb{C}} \setminus P_f @>w^{\mu}>> \hat{\mathbb{C}} \setminus w^{\mu}(P_f). \end{CD}$$

By the definition of σ , $\tilde{\mu} = f^*\mu$ so that the map $g = g_{\mu, \tilde{\mu}} = w^{\mu} \circ f \circ (w^{\tilde{\mu}})^{-1}$ defined on $\hat{\mathbb{C}}$ is meromorphic. By Remark 1, $g(z)$ is in \mathcal{M}_2 and in particular, $g'(z) \neq 0$.

Let $\sigma_* = d\sigma : T_{\tau} \rightarrow T_{\tilde{\tau}}$ and $\sigma^* : T_{\tilde{\tau}}^* \rightarrow T_{\tau}^*$ be the respective tangent and co-tangent maps of σ . Let η be a tangent vector at τ so that $\tilde{\eta} = \sigma_*\eta$ is the corresponding tangent vector at $\tilde{\tau}$. These tangent vectors can be pulled back to vectors $\xi, \tilde{\xi}$ at the origin in T_f by maps

$$\eta = (w^{\mu})^*\xi \quad \text{and} \quad \tilde{\eta} = (w^{\tilde{\mu}})^*\tilde{\xi}.$$

This results in the following commutative diagram,

$$\begin{CD} \tilde{\eta} @<(w^{\tilde{\mu}})^*<< \tilde{\xi} \\ @Vf^*VV @VVg^*V \\ \eta @<(w^{\mu})^*<< \xi \end{CD}$$

Now suppose \tilde{q} is a co-tangent vector in $T_{\tilde{\tau}}^*$ and let $q = \sigma^*\tilde{q}$ be the corresponding co-tangent vector in T_{τ}^* . Then $\tilde{q} = \tilde{\phi}(w)dw^2$ is an integrable quadratic differential on $\hat{\mathbb{C}}$ whose only poles can be simple and occur at the points in $w^{\tilde{\mu}}(P_f)$; $q = \phi(z)dz^2$ is an integrable quadratic differential on $\hat{\mathbb{C}}$ whose only poles can be simple and occur at the points in $w^{\mu}(P_f)$. This implies that $q = \sigma_*\tilde{q}$ is also the push-forward integrable quadratic differential

$$q = g_*\tilde{q} = \phi(z)dz^2$$

of \tilde{q} by g . This follows from the fact that $w^{\tilde{\mu}}$ takes the tessellation of fundamental domains for f to a tessellation of fundamental domains for g and on each fundamental domain g is a homeomorphism onto $\hat{\mathbb{C}} \setminus \{0, \lambda\}$ since $0, \lambda$ are the two asymptotic values of g . The coefficient $\phi(z)$ of q is therefore given by the standard transfer operator \mathcal{L}

$$\phi(z) = (\mathcal{L}\tilde{\phi})(z) = \sum_{g(w)=z} \frac{\tilde{\phi}(w)dw^2}{(g'(w))^2}. \tag{7}$$

Since $g'(w) \neq 0$, equation (7) implies the poles of q occur only at the images of the poles of \tilde{q} ; the integrability implies these poles can only be simple. Therefore, as a meromorphic quadratic differential defined on $\hat{\mathbb{C}}$, q satisfies

$$\|q\| \leq \|\tilde{q}\|. \tag{8}$$

By formula (7) we have

$$\langle \tilde{q}, \tilde{\xi} \rangle = \langle q, \xi \rangle$$

which, together with inequality (8), implies

$$\|\tilde{\xi}\| \leq \|\xi\|.$$

This gives another proof that σ is weakly contracting. We can, however, prove strong contraction.

Lemma 4.

$$\|q\| < \|\tilde{q}\|$$

and

$$\|\tilde{\xi}\| < \|\xi\|.$$

Proof. Suppose there is a \tilde{q} such that $\|q\| = \|\tilde{q}\|$ and that Z is the set of poles of \tilde{q} . Then, since g has no critical points, the poles of q must be contained in $g(Z)$. Using a change of variables on each fundamental domain we obtain the equalities

$$\int_{\hat{\mathbb{C}}} |\phi(z)| dz d\bar{z} = \int_{\hat{\mathbb{C}}} |\tilde{\phi}(w)| dw d\bar{w} = \int_{\hat{\mathbb{C}}} \left| \frac{\tilde{\phi}(w)}{(g'(w))^2} \right| dz d\bar{z}.$$

The triangle inequality then implies that at every point z the argument of $\frac{\tilde{\phi}(w)}{(g'(w))^2}$ is the same; that is, for each pair w, w' with $g(w) = g(w') = z$, there is a positive real number a_z such that

$$\frac{\tilde{\phi}(w)}{(g'(w))^2} = a_z \frac{\tilde{\phi}(w')}{(g'(w'))^2}.$$

Thus, by formula (7) we see that $\|q\| = \|\tilde{q}\|$ implies $\phi(z) = \infty$ giving us a contradiction. □

Remark 6. *What we have shown is that $\|q\| = \|\tilde{q}\|$ implies*

$$g_*q = \phi(g(w)) = a\tilde{q}(w)$$

and therefore all the pre-images of all the poles are poles. That is,

$$g^{-1}(g(Z)) \subset Z \cup \Omega_g.$$

But this is a contradiction because $g^{-1}(g(Z))$ is an infinite set and $Z \cup \Omega_g$ is a finite set.

As an immediate corollary we have strong contraction.

Corollary 1. *For any two points τ and τ' in T_f ,*

$$d_T(\sigma(\tau), \sigma(\tau')) < d_T(\tau, \tau').$$

Furthermore,

Proposition 2. *If σ has a fixed point in T_f , then this fixed point must be unique. This is equivalent to saying that a post-singularly finite f in $AV2$ is combinatorially equivalent to at most one map $g \in \mathcal{M}_2$.*

We can now finish the proof of the sufficiency under the assumptions that f has bounded geometry and that the meromorphic maps defined by

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \tag{9}$$

remain inside a compact subset of $\mathcal{M}2$.

Note that if $P_f = \{0, 1, \infty\}$, then f is a universal covering map of \mathbb{C}^* and is therefore combinatorially equivalent to $e^{2\pi iz}$. Thus in the following argument, we assume that $\#(P_f) \geq 4$. Then, given our normalization conventions and the bounded geometry hypothesis we see that the functions $g_n, n = 0, 1, \dots$ satisfy the following conditions:

- 1) $m = \#(w^{\mu_n}(P_f)) \geq 4$ is fixed.
- 2) $0, 1, \infty, g_n(1) \in w^{\mu_n}(P_f)$.
- 3) $\{0, 1, \infty\} \subseteq g_n^{-1}(w^{\mu_n}(P_f))$.
- 4) There is a $b > 0$ such that $d_{sp}(p_n, q_n) \geq b$ for any $p_n, q_n \in w^{\mu_n}(P_f)$.

Any integrable quadratic differential $q_n \in T_{\tau_n}^*$ has, at worst, simple poles in the finite set $P_{n+1, f} = w^{\mu_{n+1}}(P_f)$. Since $T_{\tau_n}^*$ is a finite dimensional linear space, there is a quadratic differential $q_{n, max} \in T_{\tau_n}^*$ with $\|q_{n, max}\| = 1$ such that

$$0 \leq a_n = \sup_{\|q_n\|=1} \|(g_n)_*q_n\| = \|(g_n)_*q_{n, max}\| < 1.$$

Moreover, by the bounded geometry condition, the possible simple poles of $\{q_{n, max}\}_{n=1}^\infty$ lie in a compact set and hence these quadratic differentials lie in a compact subset of the space of quadratic differentials on $\hat{\mathbb{C}}$ with, at worst, **simple** poles at $m = \#(P_f)$ points.

Let

$$a_{\tau_0} = \sup_{n \geq 0} a_n.$$

Let $\{n_i\}$ be a sequence of integers such that the subsequence $a_{n_i} \rightarrow a_{\tau_0}$ as $i \rightarrow \infty$. By our assumption of compactness, $\{g_{n_i}\}_{i=0}^\infty$ has a convergent subsequence, (for which we use the same notation) that converges to a holomorphic map $g \in \mathcal{M}2$. Taking a further subsequence if necessary, we obtain a convergent sequence of sets $P_{n_i, f} = w^{\mu_{n_i}}(P_f)$ with limit set X . By bounded geometry, $\#(X) = \#(P_f)$ and $d_{sp}(x, y) \geq b$ for any $x, y \in X$. Thus we can find a subsequence $\{q_{n_i, max}\}$ converging to an integrable quadratic differential q of norm 1 whose only poles lie in X and are simple. Now by Lemma 4,

$$a_{\tau_0} = \|g_*q\| < 1.$$

Thus we have proved that there is an $0 < a_{\tau_0} < 1$, depending only on b and f and τ_0 such that

$$\|\sigma_*\| \leq \|\sigma^*\| \leq a_{\tau_0}.$$

Let l_0 be a curve connecting τ_0 and τ_1 in T_f and set $l_n = \sigma_f^n(l_0)$ for $n \geq 1$. Then $l = \cup_{n=0}^\infty l_n$ is a curve in T_f connecting all the points $\{\tau_n\}_{n=0}^\infty$. For each point $\tilde{\tau}_0 \in l_0$, we have $a_{\tilde{\tau}_0} < 1$. Taking the maximum gives a uniform $a < 1$ for all points in l_0 . Since σ is holomorphic, a is an upper bound for all points in l . Therefore,

$$d_T(\tau_{n+1}, \tau_n) \leq a d_T(\tau_n, \tau_{n-1})$$

for all $n \geq 1$. Hence, $\{\tau_n\}_{n=0}^\infty$ is a convergent sequence with a unique limit point τ_∞ in T_f and τ_∞ is a fixed point of σ .

10 Compactness

The final step in the proof of the main theorem is to show the compactness assumption is valid. In the case of rational maps where the map is a branched covering of finite degree, the bounded geometry condition guarantees compactness, in the case of $f \in \mathcal{AV}2$, however, because the map is a branched covering of infinite degree, we need a further discussion of the topological properties of post-singular maps. We will show that for these maps there is a topological constraint that, together with bounded geometry condition guarantees compactness under the iteration process. The point is that this constraint and the bounded geometry condition together control the size of the fundamental domains so that they are neither too small nor too big.

10.1 A topological constraint

We start with $f \in \mathcal{AV}2$; recall $\Omega_f = \{0, \lambda\}$ is the set of asymptotic values of f and that we have normalized so that $f(0) = 1$. Suppose that this f is post-singularly finite; that is, P_f is finite so that the orbits $\{c_k = f^k(0)\}_{k=0}^\infty$ and $\{c'_k = f^k(\lambda)\}_{k=0}^\infty$ are both finite, and thus, preperiodic. Note that neither can be periodic because the asymptotic values are omitted. Consider the orbit of 0. Preperiodicity means there are integers $k_1 \geq 0$ and $l \geq 1$ such that $f^l(c_{k_1+1}) = c_{k_1+1}$. That is,

$$\{c_{k_1+1}, \dots, c_{k_1+l}\}$$

is a periodic orbit of period l . Set $k_2 = k_1 + l$.

Let γ be a continuous curve connecting c_{k_1} to c_{k_2} in \mathbb{R}^2 which is disjoint from P_f , except at its endpoints. Because $f(c_{k_1}) = f(c_{k_2}) = c_{k_1+1}$, the image curve $\delta = f(\gamma)$ is a closed curve. We can choose γ once and for all such that δ separates 0 and λ ; that is, so that δ is a non-trivial curve closed curve in $\hat{\mathbb{C}} \setminus \{0, \lambda\}$. The fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda\}) = \mathbb{Z}$ so the homotopy class $\eta = [\delta]$ in the fundamental group is an integer which essentially counts the number of fundamental domains between c_{k_1} and c_{k_2} and defines a “distance” between the fundamental domains. The integer η depends only on the choice of γ and since γ is fixed, so is η .

We now show that η is an invariant of the Thurston iteration procedure and is thus a topological constraint on the iterates.

Lemma 5. *Given $\tau_0 = [\mu_0] \in T_f$, let $\tau_n = \sigma^n(\tau_0) = [\mu_n]$ be the sequence generated by σ . Let w^{μ_n} be the normalized quasiconformal map with Beltrami coefficient μ_n . Let $\gamma_{n+1} = w^{\mu_{n+1}}(\gamma)$, $\delta_n = w^{\mu_n}(\delta)$ and $\lambda_n = w^{\mu_n}(\lambda)$. Then $[\delta_n] \in \pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda_n\}) = \eta$ for all n .*

Proof. The iteration defines the map

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in \mathcal{AV}2$$

which is holomorphic since it preserves μ_0 . The continuous curve

$$\gamma_{n+1} = w^{\mu_{n+1}}(\gamma)$$

goes from $c_{k_1, n+1} = w^{\mu_{n+1}}(c_{k_1})$ to $c_{k_2, n+1} = w^{\mu_{n+1}}(c_{k_2})$. The image curve

$$\delta_n = g_n(\gamma_{n+1}) = w^{\mu_n}(f((w^{\mu_{n+1}})^{-1}(\gamma_{n+1}))) = w^{\mu_n}(f(\gamma)) = w^{\mu_n}(\delta)$$

is a closed curve through the point $c_{k_1+1, n} = w^{\mu_n}(c_{k_1+1})$.

From our normalization, it follows that

$$g_n(z) = g_{\alpha_n, \beta_n}(z) = \frac{\alpha_n e^{\beta_n z}}{(\alpha_n - \frac{1}{\alpha_n})e^{\beta_n z} + \frac{1}{\alpha_n}} \tag{10}$$

and 0 is an omitted value for g_n . Since $\lambda_n = w^{\mu_n}(\lambda)$, it is also omitted for g_n and

$$\lambda_n = \frac{\alpha_n}{\alpha_n - \frac{1}{\alpha_n}} \in P_n. \tag{11}$$

Because

$$w^{\mu_n} : \hat{\mathbb{C}} \setminus \{0, \lambda\} \rightarrow \hat{\mathbb{C}} \setminus \{0, \lambda_n\}$$

is a normalized homeomorphism, it preserves homotopy classes and $\eta = [\delta_n] \in \pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda_n\}) = \mathbb{Z}$. Thus the homotopy class of δ_n in the space $\hat{\mathbb{C}} \setminus \{0, \lambda_n\}$ is the same throughout the iteration. □

10.2 Bounded geometry implies compactness

By hypothesis f has bounded geometry and by the normalization of f , $\Omega_f = \{0, \lambda\}$, $f(0) = 1$ so that $\{0, 1, \lambda, \infty\} \subset P_f$. Moreover the iterates

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1}$$

belong to \mathcal{M}_2 .

Recall that $P_n = w^{\mu_n}(P_f)$ and because w^{μ_n} fixes $\{0, 1, \infty\}$ for all $n \geq 0$, $\{0, 1, \infty\} \subset P_n$. By equation (10),

$$g_n(1) = w^{\mu_n}(f(1)) = \frac{\alpha_n e^{\beta_n}}{(\alpha_n - \frac{1}{\alpha_n})e^{\beta_n} + \frac{1}{\alpha_n}} \in P_n,$$

so that

$$\{0, 1, \lambda_n, g_n(1), \infty\} \subseteq P_n.$$

If $\#(P_f) = 3$, then $\lambda = \infty$ and $f(1) = 1$. In this case, $\lambda = \lambda_n = \infty$, $g_n(1) = 1$ for all $n \geq 0$ and $\#(P_n) = 3$ so that $g_n(z) = e^{\beta_n z}$. The homotopy class of δ_n is always η , which is its winding number about the origin in the complex analytic sense. Thus $\beta_n = 2\pi i \eta$ for all n and $g_n = e^{2\pi i \eta z}$, which is the fixed under Thurston iteration and trivially lies in a compact subset in $\mathcal{M}2$.

From now on we assume that $\#(P_f) \geq 4$. We first prove the compactness of the iterates in the case that $\lambda = \infty$. By normalization, $\lambda_n = \infty$ and

$$g_n(z) = e^{\beta_n z}$$

for all $n \geq 0$.

Because f has bounded geometry, $g_n(1) \neq 1$ has a definite spherical distance from 1 and the sequence $\{|\beta_n|\}$ is bounded from below; that is, there is a constant $k > 0$ such that

$$k \leq |\beta_n|, \quad \forall n > 0.$$

Now we use the topological constraint to prove that the sequence $\{|\beta_n|\}$ is also bounded from above. We have $g'_n(z) = \beta_n g_n(z)$ and the homotopy class of δ_n is the winding number about the origin in the complex analytic sense, thus

$$\begin{aligned} \eta &= \frac{1}{2\pi i} \oint_{\delta_n} \frac{dw}{w} = \frac{1}{2\pi i} \int_{\gamma_n} \frac{g'_n(z)}{g_n(z)} dz \\ &= \frac{\beta_n}{2\pi i} (c_{k_2, n+1} - c_{k_1, n+1}). \end{aligned}$$

Both $c_{k_2, n+1}, c_{k_1, n+1} \in P_{n+1}$, so bounded geometry implies the constant $k > 0$ above can be chosen so that

$$|c_{k_2, n+1} - c_{k_1, n+1}| \geq k.$$

Combining this with the formula for η we get

$$|\beta_n| \leq \frac{\eta}{2\pi |c_{k_2, n+1} - c_{k_1, n+1}|} \leq \frac{\eta}{2\pi k}$$

and thus deduce that $\{g_n(z) = e^{\beta_n z}\}$ forms a compact subset in $\mathcal{M}2$.

Now let us prove compactness of the iterates when $\lambda \neq \infty$. In this case, since

$$\lambda_n = \frac{\alpha_n}{\alpha_n - \frac{1}{\alpha_n}} \in P_{n+1}$$

has a definite spherical distance from 0, 1, and ∞ , bounded geometry implies there are two constants $0 < k < K < \infty$ such that

$$k \leq |\alpha_n|, \quad |\alpha_n - 1| \leq K, \quad \forall n \geq 0.$$

In this case, we have that $g_n(1) \neq 1$ too. Since $g_n(1) \in P_{n+1}$, bounded geometry implies that the constant k can be chosen such that

$$k \leq |\beta_n|, \quad \forall n > 0.$$

Again we use the topological constraint to prove that $\{|\beta_n|\}$ is also bounded from above. Let

$$M_n(z) = \frac{\alpha_n z}{\left(\alpha_n - \frac{1}{\alpha_n}\right)z + \frac{1}{\alpha_n}}$$

so that $g_n(z) = M_n(e^{\beta_n z})$. The map $M_n : \hat{\mathbb{C}} \setminus \{0, \infty\} \rightarrow \hat{\mathbb{C}} \setminus \{0, \lambda_n\}$ is a homeomorphism so it induces an isomorphism from the fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \infty\})$ to the fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda_n\})$. Thus, η is the homotopy class $[\tilde{\delta}_n]$ where $\tilde{\delta}_n = M_n^{-1}(\delta_n)$.

Note that $\tilde{\delta}_n$ is the image of γ_{n+1} under $\tilde{g}_n(z) = e^{\beta_n z}$. Since $\tilde{\delta}_n$ is a closed curve in $\hat{\mathbb{C}} \setminus \{0, \infty\}$, η is the winding number of $\tilde{\delta}_n$ about the origin in the complex analytic sense, and we can compute

$$\eta = \frac{1}{2\pi i} \oint_{\tilde{\delta}_n} \frac{dw}{w} = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \frac{\tilde{g}'_n(z)}{\tilde{g}_n(z)} dz = \frac{\beta_n}{2\pi i} (c_{k_2, n+1} - c_{k_1, n+1}).$$

As above, $c_{k_2, n+1}, c_{k_1, n+1} \in P_{n+1}$, and by bounded geometry there is a constant $k > 0$ such that

$$|c_{k_2, n+1} - c_{k_1, n+1}| \geq k,$$

so that

$$|\beta_n| \leq \frac{\eta}{2\pi |c_{k_2, n+1} - c_{k_1, n+1}|} \leq \frac{\eta}{2\pi k}.$$

This inequality proves that $\{g_n(z) = g_{\alpha_n, \beta_n}(z)\}$ forms a compact subset in $\mathcal{M}2$.

Finally, we have shown that in all cases the sequence $\{g_n\}$ is a compact subset in $\mathcal{M}2$. This combined with the proof in section 9 completes the proof of sufficiency in our main theorem.

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