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INFIMUM OF THE METRIC ENTROPY OF VOLUME PRESERVING ANOSOV SYSTEMS

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Abstract. In this paper we continue our study [9] of the infimum of the metric entropy of the SRB measure in the space of hyperbolic dynamical systems on a smooth Riemannian manifold of higher dimension. We restrict our study to the space of volume preserving Anosov diffeomorphisms and the space of volume preserving expanding endomorphisms. In our previous paper, we use the perturbation method at a hyperbolic periodic point. It raises the question whether the volume can be preserved. In this paper, we answer this question affirmatively. We first construct a smooth path starting from any point in the space of volume preserving Anosov diffeomorphisms such that the metric entropy tends to zero as the path approaches the boundary of the space. Similarly, we construct a smooth path starting from any point in the space of volume preserving expanding endomorphisms with a fixed degree greater than one such that the metric entropy tends to zero as the path approaches the boundary of the space. Therefore, the infimum of the metric entropy as a functional is zero in both spaces.

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1. **Introduction.** For a uniformly hyperbolic dynamical system f on a compact smooth Riemannian manifold M possessing an SRB measure μ_f, it is well known that the metric entropy of f with respect to μ_f is positive. The metric entropy changes in general when we perturb the dynamical system f in a neighborhood. Such change is in fact differentiable with respect to f under appropriately chosen differentiable structures [21]. In a recent paper [9], we have shown that while the entropy remains positive, it can be made as small as possible by making successive perturbations to f along a C^1-path, still preserving the uniform hyperbolicity.

The question is raised whether there is some obstacle that might prevent the entropy from approaching zero while the dynamical system is perturbed along a path within a given subspace of uniformly hyperbolic dynamical systems. In this paper, we explore this question within the space of all volume preserving uniformly hyperbolic dynamical systems.

We will construct a C^1-path in the space of all volume preserving uniformly hyperbolic dynamical systems and show that there is no obstacle to reduce the entropy while keeping the volume form preserved. The metric entropy along this path can be made as small as possible by making successive perturbations.

The metric entropy of an Anosov map f with respect to its SRB measure ν can be calculated by the formula

$$h_\nu(f) = \int_M \log |\det Df|_{E^u_x(f)}| d\nu$$

according to the entropy formula given in [17]. Therefore, to have $h_\mu(f)$ arbitrarily small, we need to construct f such that $|\det Df|_{E^u_x(f)}|$ is sufficiently close to 1 on a set with volume sufficiently close to 1. The complement of this set has small volume and its contribution to the value of h_ν is small even though $|\det Df|_{E^u_x(f)}|$ is large. This is the central idea to construct a path such that the metric entropy can be arbitrarily small. There are two possible approaches to realize this idea. One approach is to make the motion of f very slow in a small neighborhood of a fixed point, then the mass of the SRB measure with respect to f will mostly concentrate in this neighborhood. So the measure of this neighborhood respect to the SRB measure is close to 1 and $|\det Df|_{E^u_x(f)}|$ in this neighborhood is also close to 1. In the complement of this neighborhood, $|\det Df|_{E^u_x(f)}|$ may be large but the measure of this complement with respect to the SRB measure is small, so it contributes little to the value of $h_\nu(f)$. Such a map is close to some almost Anosov system that has infinite SRB measures ([7]) and the limiting map along this path is this almost Anosov system. This is the approach we have used in the previous paper ([9]). However, such a path cannot be made as a path of volume preserving Anosov systems. In this paper, we develop the other approach. We first reduce the speed of the motion of a Anosov system f in an entire neighborhood of a fixed point by embedding it into a flow. This forces the SRB measure again to concentrate in this neighborhood. However, we make sure that every Anosov system in the path has a smooth invariant measure. This enables us to apply Moser's Theorem to smoothly conjugate this path to another path consisting of all volume preserving Anosov systems. The smooth conjugacy will help spread this slow motion to nearly the entire manifold. Consequently, this Anosov system has a strong expansion and a strong contraction on a set with a small Lebesgue volume. The path $\{f_t\}_{0<t\leq 1}$ we will construct is chosen in such a way that f_t may converge to the identity point-wise, although the C^1-norm of an Anosov system in the path may become increasingly large. We would like to emphasize the main difference of these two

approaches: In the first approach, the path $\{f_t\}_{0<t\leq 1}$ we constructed in [9] has a limiting map as an almost Anosov system (i.e., degenerate one); In the second approach, the path $\{f_t\}_{0<t\leq 1}$ we construct may lead to the identity map where all Anosov properties will disappear in the limiting map (i.e., totally degenerate one).

The method in the proof of Theorem 1 is different from the method used in [9]. In [9], we construct a path in the space of all Anosov diffeomorphisms tending to the boundary of the space as an almost Anosov diffeomorphism. It is analogous to a path in the Teichmüller space of quasi-Fuchsian groups tending to the boundary of Teichmüller space as a Fuchsian group with a parabolic element. Thus, if we view this path as a path of Riemann surfaces up to homotopy, it tends to a Riemann surface with a node. However, in the proof of Theorem 1 in this paper, we construct a path without any concrete limiting point at one end. It is analogous to a path in the Teichmüller space of quasi-Fuchsian groups tending to the boundary of Teichmüller space as a totally degenerate Fuchsian group, that is, if we view this path as a path of Riemann surfaces up to homotopy, one can see no limiting object. To view the second statement more clearly, in Appendix B, we adapt the example from [12, Theorem 22] to construct a C^1 path of C^r expanding maps $\{f_t : 0 < t \leq \frac{1}{6}\}$ preserving the Lebesgue measure m such that as $t \to 0$, $h_m(f_t) \to 0$ and $f_t(x) \to x$ for all x except for one point at which the derivative of f tends to infinite.

We organize the paper as follows. In Section 2, we state our main results, Theorems 1 and 2. In Section 3, we prove our first main result, Theorem 1. We divide our proof into three main steps. In the first step, we linearize the given map in a neighborhood of a fixed point (in general, a periodic point). In the second step, we show that while we linearize the map, we can also preserve the volume form. We complete the proof in the third step by first embedding the linearized map into a flow and then using an idea of Katok [13] to slow down the flow to reduce the metric entropy while preserving the volume form. In Section 4, we prove our second main result for endomorphisms, Theorem 2. Since the proof is very similar to the proof of Theorem 1, we only give detailed proof for those parts that are different from the proof of Theorem 1. Finally in Appendix A, we state the Dacorogna-Moser Theorem that is used in proofs of theorems in the paper. A sketch of its proof is also provided since we need an additional statement on differentiable dependence on parameters. Appendix A also provides additional details omitted in the proofs of 1 and 2. Appendix B contains a one-dimensional example constructed explicitly where the C^1 path leads to the identity except for one point.

2. **Statements of main results.** Let M be a compact C^∞ Riemannian manifold of dimension $n \geq 2$. Recall that a diffeomorphism f of M is called Anosov if the tangent bundle TM admits an exponential splitting $TM = E^s \otimes E^u$ where E^s and E^u are invariant under the differential operator df and there exist constants $0 < \lambda < 1 < \mu$ and $C > 0$ such that $\|Df^n v\| \leq C\lambda^n \|v\|$ and $\|Df^n u\| \geq C\mu^n \|u\|$ for all $v \in E^s_x$, $u \in E^u_x$, and $x \in M$. Let μ be the volume form (that is, Lebesgue measure) on M. Let $\mathrm{Diff}^r(M)$ denote the space of all $C^r, 1 < r \leq \infty$, diffeomorphisms of M. Using the C^r-norm on the tangent bundle, we can equip $\mathrm{Diff}^r(M)$ with a differentiable Banach manifold structure. Let $\mathrm{Diff}^r_\mu(M)$ denote the space of all volume preserving C^r-diffeomorphisms of M, that is, for any Borel set $A \subset M$, $f \in \mathrm{Diff}^r_\mu(M)$,

$$\mu(f(A)) = \mu(A).$$

Let $\mathcal{A}^r_\mu(M) \subset \mathrm{Diff}^r_\mu(M)$ denote the space of all volume preserving Anosov diffeomorphisms.

A C^s, $s \geq 1$, path in $\mathrm{Diff}^r(M)$ is a one parameter family $\{f_t\}_{0<t\leq 1}$ such that $t \to f_t$ is C^s with respect to the usual Euclidean topology and the Banach manifold structure on $\mathrm{Diff}^r(M)$.

For every $f \in \mathrm{Diff}^r_\mu(M)$, μ is f-invariant. Hence we can define the metric entropy (measure-theoretic entropy) $h_\mu(f)$. Our first main result states that the infimum of $h_\mu(f)$ over $\mathcal{A}^r_\mu(M)$ is zero.

Theorem 1. *For any $f \in \mathcal{A}^r_\mu(M), r > 2$, there exists a C^1 path in $\mathcal{A}^r_\mu(M)$,*

$$H = \{f_t \in \mathcal{A}^r_\mu(M) \mid 0 < t \leq 1\},$$

such that $f_1 = f$ and

$$\lim_{t\to 0^+} h_\mu(f_t) = 0.$$

Our second main result is for endomorphisms. Let $\mathrm{End}^r(M)$ be the space of C^r, $1 < r \leq \infty$, endomorphisms of M equipped with a Banach manifold structure induced by the C^r-norm in the tangent bundle of M. Let $\mathrm{End}^r_\mu(M)$ denote the space of all volume preserving endomorphisms $f \in \mathrm{End}^r(M)$, that is, for any Borel set $A \subset M$,

$$\mu(f^{-1}(A)) = \mu(A).$$

Let $\mathcal{E}^r_\mu(M) \subset \mathrm{End}^r_\mu(M)$ denote the space of all volume preserving expanding endomorphisms, that is, in addition to preserve the volume μ, it also satisfies

$$\|Df^n v\| \geq C\lambda^n \|v\|, \quad \forall\ v \in T_xM, \quad \forall\ n \geq 1,$$

where $C > 0$ and $\lambda > 1$ are two fixed constants. Our second main result states that the infimum of $h_\mu(f)$ over $\mathcal{E}^r_\mu(M)$ is zero.

Theorem 2. *For every $f \in \mathcal{E}^r_\mu(M), r > 2$, there exists a C^1 path in $\mathcal{E}^r_\mu(M)$*

$$H = \{f_t \in \mathcal{E}^r_\mu(M) \mid 0 < t \leq 1\}$$

such that $f_1 = f$ and

$$\lim_{t\to 0^+} h_\mu(f_t) = 0.$$

Remark 1. The condition $r > 2$ is used in our proofs for the purpose of easy explanation and is not necessary, and can be replaced by the condition $r > 1$.

Remark 2. The map f is topologically conjugate to f_t for any $t \in (0, 1]$ due to the structural stability of Anosov dynamical systems and expanding endomorphisms. Hence, all maps $f_t, 0 < t \leq 1$, have the same topological entropy.

Remark 3. The path $t \in (0, 1] \to f_t$ can be constructed to have higher order differentiability. For simplicity of the proofs, we have not included this part of discussion in this paper.

3. **Proof of Theorem 1.** We first give the strategy of the proof of our first main result. The proof has three steps. In the first step, we perturb the map f in its C^1 neighborhood to linearize f at a given hyperbolic periodic point. For the simplicity, we use a fixed point. In this perturbation, the resulting map may be no longer volume-preserving but still remaining as an Anosov diffeomorphism. In the second step, we use the Dacorogna-Moser theorem (Theorem A, we give a precise statement in Appendix A) to make the perturbed map volume-preserving within its C^1 neighborhood. We state the local version called *the pasting lemma* of the Dacorogna-Moser theorem as Lemma 2, whose proof can be found in [1]. In this

step, we keep the perturbed map homotopic to f via a C^1 path. In the third step, we complete our proof by embedding the local linear map in a flow and slow down the flow to reduce the entropy. This is a technique also used by Katok in his work [13] in dimension two. We adapted his idea and show that it works for any dimension ≥ 2.

We start our first step: linearization at a periodic point. In this step, we do not need f preserves the volume form.

Suppose f is a C^r, $r > 1$, Anosov diffeomorphism on M. Then it always has a periodic point. For simplicity, we assume it has fixed point p. (Otherwise, we consider f^n where n is the period of f at this periodic point.) With a fixed local chart $\phi : M \to \mathbb{R}^n$ at p with $\phi(p) = 0$, we can assume f is a local diffeomorphism on the Euclidean space $\mathbb{R}^n$ with the fixed point p identified with the origin.

Lemma 1. *For a sufficiently small $\tau_0 > 0$, there exists a one-parameter family of Anosov diffeomorphisms $f_\tau, 0 < \tau < \tau_0$ satisfying the following properties:*

(1) The C^1 distance between f and f_τ is bounded by $C_0\tau$ for a constant C_0 depending only on f.

(2) $f_\tau(x) = f(x)$ for $x \in M \setminus O_\tau(p)$ and $f_\tau(x) = Df(0)x$ for $x \in \phi(O_{\frac{\tau}{4}}(p))$, where $O_\tau(p)$ and $O_{\frac{\tau}{4}}(p)$ are open neighborhoods of p with radii τ and $\frac{\tau}{4}$, respectively, and $x \to Df(0)x$ is the linear map induced by the derivative operator $Df(0) = Df_p$ in $\mathbb{R}^n$.

(3) f_τ depends on τ, $0 < \tau < \tau_0$, smoothly: $\tau \to f_\tau \in \mathrm{Diff}^r(M)$ is C^∞.

Proof. This lemma is similar to Lemma 3.1 in [9]. The small difference is that we now construct a smooth path f_τ, $\tau \in (0, \tau_0)$, such that the C^1 distance between f and f_τ is bounded by $C_0\tau$. We construct f_τ by perturbing f locally near the fixed point p.

Let $\eta(t)$ be a C^∞ real function defined on $[0, \infty)$ satisfying the following conditions.

1. $0 \leq \eta(t) \leq 1$, $-2 \leq \eta'(t) \leq 0$;
2. $\eta(t) = 1, t \in [0, \frac{1}{4}]$ and $\eta(t) = 0$, for all $t \geq 1$.

We define a map f_τ by

$$f_\tau(x) = f(x) + \eta\Big(\frac{|x|}{\tau}\Big)(Df(0)x - f(x)), \quad |x| < \tau, \tag{3.1}$$

where $|x|$ denotes the Euclidean norm of x. The map is well-defined as long as τ is sufficiently small. For x with $|x| \geq \tau$, $f_\tau(x) = f(x)$. f_τ is a C^r map since f is C^r $(r \geq 2)$.

Furthermore, the C^1 norm of $f_\tau - f$ is bounded by $C_0\tau$ for some constant C_0 independent of τ. To see this, we note that $|f(x) - Df(0)x| \leq C_1|x|^2$ for some constant C_1 and $|x| \leq \tau$ since $f(x)$ is at least C^2. For the derivative, we have

$$\|Df_\tau(x) - Df(x)\| \leq C_2\|(f(x) - Df(0)x)\| + \|Df(x) - Df(0)\| \leq C_3|x|$$

for other constants $C_2, C_3 > 0$.

We now get that f_τ is C^1 close to f with the C^1-distance bounded by $C_0\tau$ and thus, $\lim_{\tau\to 0} f_\tau = f$ in C^1-distance. For $|x| \leq \dfrac{\tau}{4}$, $f_\tau(x) = Df(0)x$ is linear and for $|x| \geq \tau$, $f_\tau = f$. When $0 < \tau < \tau_0$ for sufficiently small τ_0, f_τ clearly depends on τ smoothly since $\eta(t)$ is C^∞ and f_τ is also Anosov since f is Anosov. □

Now we move to the second step: making f_τ volume preserving. In this step, we assume that f preserves the volume form.

The map f_τ in the first step may not preserve the volume form. Since τ is small, the perturbation is limited in one local coordinate chart U. Thus, we can assume f preserves the volume form $\Omega_0 = \Omega_0(x) = dx_1 \wedge \cdots \wedge dx_n,\ x \in U$. Let

$$\Omega_1 = \Omega_1(x) = \left|\det Df_\tau|_{f_\tau^{-1}(x)}\right| dx_1 \wedge \cdots \wedge dx_n,\ x \in U.$$

Note that $\Omega_1(x) = \Omega_0(x)$ if $|x| \le \frac{\tau}{4}$ and $|x| \ge \tau$ and Ω_1 is a C^{r-1} volume form. Let $A = \{x : \frac{\tau}{4} \le |x| \le \tau\}$ be the annulus. Since both f and f_τ are diffeomorphisms on a compact manifold, we must have

$$\int_A \Omega_1 = \int_A \Omega_0 = \mu(A).$$

We now need the following lemma, which is derived from the Dacorogna-Moser theorem in [5] as a local version. The theorem is also used to prove the pasting lemma in [1]. We will use this lemma twice in our proof: making both the map after the linearization of the map near the fixed point and the time-one map of the slowed-down flow volume preserving. A simplified version of Theorem 5 of [5] and its proof are provided in Appendix A.

Lemma 2. *Assume that Ω_0 and Ω_1 are two $C^r, r \ge 1$, volume forms in an open neighborhood U of the origin of $\mathbb{R}^n$. Assume that, $\Omega_1 = a_\tau(x)\Omega_0$, where $a_\tau(x) > 0$ is a C^r function on U for each parameter $\tau \in (0, \tau_0)$, $a_\tau(x) = 1$ for $x \notin A = \{x : l_1(\tau) < |x| < l_2(\tau)\} \subset U$, and*

$$\int_A \Omega_1 = \int_A a_\tau(x)\Omega_0 = \int_A \Omega_0 = \mu(A),$$

where $l_1(\tau), l_2(\tau) \to 0$ as $\tau \to 0$. Then, there exists a C^r diffeomorphism T_τ of U such that T_τ maps the volume form Ω_1 to Ω_0: $T_\tau^*\Omega_1 = \Omega_0$, i.e,

$$\det DT_\tau(x) = [a_\tau(x)]^{-1}$$

and T_τ is the identity outside the annulus A. Assume further that the relative density function $a_\tau(x)$ and two radii $l_1(\tau)$ and $l_2(\tau)$ depend on τ differentiably. Then, T_τ depends on τ differentiably. In particular, for each τ, there exists a constant κ such that

$$\|T_\tau - \mathrm{id}\,\|_{C^r} \le \kappa \|a_\tau(x) - 1\|_{C^r}.$$

The lemma can be obtained by applying Theorem A of the Appendix A to the open annulus A. Since the diffeomorphism constructed has support inside A, it can be extended to be a diffeomorphism of U.

Applying Lemma 2 to Ω_1 and Ω_0 defined for the map f_τ, we have T_τ such that $T_\tau^*\Omega_1 = \Omega_0$ i.e,

$$\det DT_\tau(x) = \left|\det Df_\tau|_{f_\tau^{-1}(x)}\right|^{-1}.$$

This implies that map $T_\tau f_\tau$ is volume preserving and linear inside the $\frac{\tau}{4\sigma}$-neighborhood of the origin where σ denote the norm of $Df(0)$. It is Anosov when τ is small.

By the mean value theorem, we conclude that the path $\tau \in (0, \tau_0) \to f_\tau$ is a C^1-path in $\mathcal{A}_\mu^r(M)$. With a little abuse of notation, we again denote $T_\tau f_\tau$ by f_τ.

In the third step, we complete the proof of Theorem 1. The main technique is to embed the linearization near the fixed point into a flow and then follow an idea of Katok in [13] to slow down the flow. The flow in [13] is of dimension two but we will show that the construction is completely parallel in the higher dimension case.

We start with a volume preserving Anosov diffeomorphism f that is linear in a r_2-neighborhood of a fixed point p under a given system of coordinate charts. Since the perturbation of f will be restricted to this r_2-neighborhood, we can assume that we are working in an r_2-neighborhood of the Euclidean space with $p = 0$. Thus, Df_p is an $n \times n$ hyperbolic matrix with $|\det(Df_p)| = 1$.

We now write $Df_p = EA$, where A is a matrix whose eigenvalues are positive real numbers, and E is an isometry: a product of rotations and/or reflections restricted to each eigenspace of Df_p (see Remark 5 in Appendix A). By choosing an appropriate basis, both E and A are block diagonal matrices:

$$E = \begin{bmatrix} E^u & 0 \\ 0 & E^s \end{bmatrix}, \quad A = \begin{bmatrix} A^u & 0 \\ 0 & A^s \end{bmatrix},$$

where E^u and E^s are both isometries and A^u, A^s are expanding and contracting linear maps, respectively. Note that both maps E and A preserve the stable and unstable subspaces of the linear map Df_p.

Since the eigenvalues of A are all positive numbers, the linear map $x \to Ax$ can be embedded in a flow $g^t(x)$ [18] with $g(x) = g^1(x) = Ax$. Let $\mathcal{X}(x)$ denote the vector field of the flow in $B(p, r_2)$. Indeed, $\mathcal{X}(x) = Lx$ where $L = \ln(A)$. Note that L is also a block-diagonal matrix.

We first choose $s_1, 0 < s_1 < r_2$ such that $|g^t(x)| \geq s_1$ for all $|x| \geq \frac{7}{8}r_2$ and all $t \in [0, 1]$.

We now define a C^∞ (target) function $\psi_0(s) : [0, r_2] \to [0, 1]$ that will be used to slow down the flow in $B(p, r_2)$. The function $\psi_0(s)$ satisfies the following conditions:

(1) $\psi_0(0) = \psi_0'(0) = 0$; $\psi_0(s) = 1$ for $s \in [s_1, r_2]$;
(2) $\psi_0(s) > 0$, for $s \in (0, r_2]$; $\psi_0'(s) \geq 0$, for $s \in [0, r_2]$.

We then define a one-parameter family of C^∞ positive functions $\psi_w(s) : [0, r_2] \to [0, 1], w \in [0, r_2]$ such that $\psi_w(s)$ converges to $\psi_0(s)$ as $w \to 0$ with the following additional properties:

(1) $\psi_w(s) = 1$ for $s \in [0, r_2]$ when $w \geq \frac{7r_2}{8}$.
(2) $\psi_w(s) = 1$ when $s \geq s_1$ for all $w \in [0, r_2]$.
(3) $\psi_w(s) = \psi_0(s)$ for $s \geq w$,
(4) $\psi_w(s) = \psi_0(\frac{w}{4})$, a constant, for $0 \leq s \leq w/4$,
(5) $\psi_w'(s) \geq 0$ for all $s \in [0, r_2]$,
(6) $w \to \psi_w(s)$ is differentiable in $w \in (0, r_2]$ in C^r-topology, $r \geq 2$.

For each $w \in [0, r_2]$, we define a vector field $\mathcal{X}_w$ by

$$\mathcal{X}_w(x) = \psi_w(|x|^n)\mathcal{X}(x).$$

Let g_w be the time one map of the flow defined by the vector field $\mathcal{X}_w$. Since $\psi_w(s) = 1$ for $s \geq s_1$, the choice of $\psi_0(s)$ implies $g_w(x) = g(x)$ for any $x \in B(p, r_2) \backslash B(p, \frac{7r_2}{8})$. Now we define

$$\tilde{f}_w(x) = \begin{cases} Eg_w(x) & x \in B(p, r_2); \\ f(x) & x \notin B(p, r_2). \end{cases}$$

Since on $B(p, r_2) \setminus B(p, \frac{7}{8}r_2)$, $Eg_w = Eg = E \cdot A = f$, we know that $\tilde{f}_w$ is a C^r diffeomorphism for any $w \in [0, r_2]$ and it depends on w in C^r topology when $w \in (0, r_2]$. Furthermore, since both stable and unstable subspaces of Df_p are preserved by $\tilde{f}_w$, $\tilde{f}_w$ is Anosov by Lemma 3.2 [9].

For $w \in (0, r_2]$, we define $\rho_w(x) = [\psi_w(|x|^n)]^{-1}$ if $x \in B(p, r_2)$ and $\rho_w(x) = 1$ otherwise.

For any two given differentiable volume forms ω_1 and $\omega_2 = \rho(x)\omega_1$, $\rho > 0$, and a differentiable vector field X on M, the divergences of X with respect to the volume forms ω_1 and ω_2 satisfy the equation [11]

$$\mathrm{div}_{\omega_2} X = \mathrm{div}_{\omega_1} X + \langle \mathrm{grad} \ln \rho(x), X \rangle. \tag{3.2}$$

It can be directly verified that the divergence of the vector field $\mathcal{X}_w$ is zero with respect to the volume form defined by the measure $\rho_w d\mu$. Thus, the time-one map g_w preserves the volume ρ_w. So does the isometry E as the function $\psi_w(|x|^n)$ is invariant under E:

$$\psi_w(|x|^n) = \psi_w(|Ex|^n).$$

It means that the probability measure

$$\nu_w = \frac{\rho_w d\mu}{\int_M \rho_w d\mu}$$

is invariant under the map $\tilde{f}_w$.

Since $\psi_0(0) = \psi'(0) = 0$, and ψ_0 is C^∞, we know that the Taylor expansion of $\psi_0(s)$ has the form $a_2 s^2 + o(s^2)$ near $s = 0$. It follows that

$$\int_{B(p,r_2)} \big(\psi_w(|x|^n)\big)^{-1} d\mu(x) < \infty$$

for $w > 0$ and

$$\int_{B(p,r_2)} \big(\psi_0(|x|^n)\big)^{-1} d\mu(x) = \infty \quad \text{and} \quad \lim_{w\to 0} \int_M \rho_w(x) d\mu(x) = \infty.$$

Hence, for any fixed neighborhood $U \ni p$, $U \subset B(p, r_2)$,

$$\lim_{w\to 0} \nu_w(M \setminus U) = \lim_{w\to 0} \int_{M\setminus U} \frac{\rho_w d\mu}{\int \rho_w d\mu} = \lim_{w\to 0} \frac{\int_{M\setminus U} \rho_w d\mu}{\int_M \rho_w d\mu} = 0, \tag{3.3}$$

and

$$\lim_{w\to 0} \nu_w(U) = 1. \tag{3.4}$$

It is well known that by the entropy formula [17], if ν is absolutely continuous with respect to the Lebesgue measure, then the entropy for an Anosov map f is given by

$$h_\nu(f) = \int_M \log |\det Df|_{E^u_x(f)}| d\nu.$$

Since $D\tilde{f}_w$ is close to the identity map near p, we can find a small neighborhood $B(p, \delta)$ such that $\log |\det D\tilde{f}_w|_{E^u_x(\tilde{f}_w)}| < \frac{\epsilon}{2}$ for all $|x| \le \delta$ when w is sufficiently small. On the other hand, the function $\log |\det D\tilde{f}_w|_{E^u_x(\tilde{f}_w)}|$ is always bounded above. Thus, we can choose w small enough such that

$$\int_{M\setminus B(p,\delta)} \log |\det D\tilde{f}_w|_{E^u_x(\tilde{f}_w)}| d\nu_w < \frac{\epsilon}{2}.$$

So we have $h_{\nu_w}(f) \to 0$ when $w \to 0$.

Finally, we use the Dacorogna-Moser theorem (Theorem A) again. There is a C^∞ diffeomorphism Ψ_w that maps (M, ν_w) to (M, μ) satisfying $\Psi_w^* \nu_w = \mu$. If we take $f_w = \Psi_w \tilde{f}_w \Psi_w^{-1}$, $w \in (0, r_2]$. then f_w is a Anosov diffeomorphism and preserves μ. Since smooth conjugacy does not change the metric entropy, $\{f_w\}$ are the desired diffeomorphisms in Theorem 1. We have completed our proof of Theorem 1.

Remark 4. By (3.3) and (3.4) we see that $\tilde{f}_w$ has the absolutely continuous invariant measure ν_w that is close to 1 for any neighborhood U of the fixed point p at which $|\det D\tilde{f}|_{E^u_x(f)}|$ is close to 1 if w is close to 0. Hence $\tilde{f}_w$ have small metric entropy with respect to ν_w. On the other hand, since f_w preserves the volume μ, we can see that the volume of $\Psi_w(U)$ is close to 1, and $|\det Df|_{E^u_x(f)}|$ is close to 1 for $x \in \Psi_w(U)$. The latter implies that $\|Df|_{E^u_x(f)}\|$ is close to 1. Since $|\det Df|_{T_xM}| = 1$, it implies $\|Df|_{E^u_x(f)}\|$ is close to 1 as well.

4. **Proof of Theorem 2.** The proof of Theorem 2 is similar to that of Theorem 1. So we only give a description of changes needed.

Take a fixed point p of f. Let $p_0 = p$ and $p_1, \cdots, p_{k-1}$ be the other preimages of p under f:

$$f^{-1}(p) = f^{-1}(p_0) = \{p_0, p_1, \cdots, p_{k-1}\}$$

where k is the degree of the expanding map f. At each point p_i, $i = 0, 1, \cdots, k-1$, take a local chart $\phi_i : M \to \mathbb{R}^n$ as in the first step in the proof of Theorem 1 such that $\phi_i(p_i) = 0 \in \mathbb{R}^n$, and $y \to Df|_p y$ is the linear map induced by the derivative operator $Df|_p$ in $\mathbb{R}^n$. By taking a local chart at each p_i, $i = 0, 1, \cdots, k-1$, we can assume that p_i has a neighborhood U_i that is a subset of the Euclidean space $\mathbb{R}^n$. We assume further that $\{U_i\}$ are pairwise disjoint and that each U_i contains a ball $B(p_i, \tau^*)$ of radius $\tau^* > 0$ about p_i respectively. Take $\tau_0 \in (0, \tau^*)$ such that $f(B(p_i, \tau_0)) \subset B(p, \tau^*)$ for any $i = 0, 1, \cdots, k-1$.

On each $B(p_i, \tau_0)$, we define f_τ as in (3.1):

$$f_\tau(x) = f(x) + \eta\Big(\frac{|x|}{\tau}\Big)(Df(0)x - f(x)), \quad |x| < \tau,$$

Clearly, the map is well-defined as long as τ is sufficiently small. For x with $|x| \geq \tau$, $f_\tau(x) = f(x)$. f_τ is a C^r map whenever f is C^r $(r \geq 2)$.

As in the first step in the proof of Theorem 1, we can also see that the C^1 norm of $f_\tau - f$ is bounded by $C_0\tau$ for some constant C_0 independent of τ, and therefore f_τ is C^1 close to f with the C^1-distance bounded by $C_0\tau$ and thus, $\lim_{\tau\to 0} f_\tau = f$ in C^1-distance. For $x \in U_i$ with $|x| \leq \dfrac{\tau}{4}$, $f_\tau(x) = Df(0)x$ is linear; and $f_\tau = f$ if $|x| \geq \tau$. When $0 < \tau < \tau_0$, f_τ clearly depends on τ smoothly since $\eta(t)$ is C^∞ and f_τ is also uniformly expanding since f is.

As in the second step in the proof of Theorem 1, On the annulus $A = \{x : \frac{\tau}{4} \leq |x| \leq \tau\}$, choose a volume form Ω_0 and define

$$\Omega_1(x) = \sum_{y \in f_\tau^{-1}(x)} |\det Df_\tau(y)|^{-1}\Omega_0(y).$$

Then we have

$$\int_A \Omega_1 = \int_A \Omega_0 = \mu(A).$$

We can now apply Lemma 2 to make f_τ volume preserving on M as well as linear in $B(p, \frac{\tau}{4})$.

Now we deform f_τ by slowing down its dynamics near p. Denote again for simplicity $f = f_\tau$ for a fixed $\tau \in (0, \tau_0)$. We write $Df_p = EA$ as in the third step in the proof of Theorem 1, that is, A is a matrix whose eigenvalues are real numbers greater than 1, and E is an isometry: a product of rotations and/or reflections restricted to each eigenspace of Df_p (see Remark 5 in Appendix A).

Since the eigenvalues of A are positive, the linear map $x \to Ax$ can be embedded in a flow $g^t(x)$ [18]. The local linear vector field $\mathcal{X}(x)$ is given by $(\ln A)x$ and the flow's time one map is given by $x \to g(x) = g^1(x) = Ax$ on $B(p, \frac{r}{4})$. In this case the divergence of the vector field is $\mathrm{div}\mathcal{X} = \mathrm{tr}\ln A > 0$. We construct a family of functions $\{\psi_w(s)\}$ and a vector field $\mathcal{X}_w(x)$ as in the third step in the proof of Theorem 1 and let $g_w = g_w^1$ be the time one map of $\mathcal{X}_w(x)$. Then we take $\tilde{f}_w = Eg_w$ on $B(p, r_2)$, and $\tilde{f}_w = f$ on the complement of $B(p, r_2)$. By the same reasons, $\tilde{f}_w$ is a C^r map for any $w \in [0, r_2]$ and it depends on w in C^r topology when $w \in (0, r_2]$. Furthermore, $\tilde{f}_w$ is expanding except $D\tilde{f}_0(p) = \mathrm{id}$.

For $w \in (0, r_2]$, let $\rho_w(x) = [\psi_w(|x|^n)]^{-1}$ if $x \in B(p, r_2)$, and $\rho_w(x) = 1$ otherwise. Note that volume preserving of f implies that

$$\sum_{y \in f^{-1}(x)} |\mathrm{det} Df|^{-1} = 1.$$

We now verify the map $\tilde{f}_w$ preserves the volume ρ_w. We only need to show that for any point $x \in B(p, r_2)$,

$$\sum_{y \in \tilde{f}_w^{-1}(x)} |\mathrm{det}_{\rho_w} D\tilde{f}_w|^{-1} = 1,$$

where the determinant is calculated with respect to the volume $\rho_w d\mu$.

Since $\tilde{f}_w(x) = f(x)$ and $\rho_w(x) = 1$ when $x \notin B(p, r_2)$, we have $\mathrm{det}_{\rho_w} D\tilde{f}_w(y) = \mathrm{det} Df(y)$ if $y \notin B(p, r_2)$.

If $x \in B(p, r_2)$, $\mathrm{det}(\tilde{f}_w) = \mathrm{det}(E)\mathrm{det}(g_w)$. $|\mathrm{det}(E)| = 1$ because of the symmetry of the function $\psi_w(|x|^n)$ and $\ln \mathrm{det}(g_w) = \mathrm{div}_{\rho_w}(\psi_w(|x|^n)\mathcal{X})$, where the divergence div_{ρ_w} is calculated with respect to the volume $\rho_w d\mu$. By the divergence formula (3.2), we have

$$\mathrm{div}_{\rho_w}(\psi_w(|x|^n)\mathcal{X}) = \mathrm{div}_{\rho_w}(\psi_w(|x|^n)(\ln A)x) = \mathrm{tr}(\ln A).$$

Thus, $\mathrm{det}(g_w) = \mathrm{det}(A) = \mathrm{det} Df$ and therefore, $\tilde{f}_w$ preserves the volume $\rho_w d\mu$.

Normalize the volume, we have that the probability measure

$$\nu_w = \frac{\rho_w d\mu}{\int \rho_w d\mu}$$

is invariant under the map $\tilde{f}_w$.

The rest of the proof is identical to that of Theorem 1. Hence we have $\lim_{w \to 0} h_\mu(\tilde{f}_w) = 0$.

Appendix A. **Dacorogna-Moser theorem and some remarks.** In this part we provide a statement and a sketch of proof of a version of Moser's Theorem, given by Dacorogna and Moser in 1990, which directly gives Lemma 2 in Section 3. Also, we include here three remarks, two of which provide more details for the decomposition $Df_p = EA$, and equation (3.2), used in Step 3 in the proof of Theorem 1, and the other gives explanation how to carry out the perturbation arguments along a periodic orbit instead of a fixed point.

In the following theorem, the differentiable dependence of the diffeomorphism on parameters is not explicitly mentioned in the original paper. Nevertheless, it can be derived from the proof of the theorem. We provide here a slightly modified version of this theorem with a differentiable dependence statement. A sketch of the proof

is also provided. We refer readers interested in additional details to the original papers [5, 19].

Theorem (Dacorogna and Moser [5])**.** *Let $r \geq 1$ be an integer and A be a bounded connected open set of $\mathbb{R}^n$ with C^r boundary ∂A. Let $f(x), g(x) > 0$ be two C^r functions over the closure $\bar{A} = A \cup \partial A$. Assume that* $\mathrm{supp}(f-g) \subset A$ *and*

$$\int_A [f(x) - g(x)]dx = 0.$$

Then, the following statements hold true.

(1) There exists a diffeomorphism $\varphi = \varphi(f,g) \in \mathrm{Diff}^r(\bar{A})$ with $\mathrm{supp}(\varphi - \mathrm{id}) \subset A$ *such that*

$$\int_E f(x)dx = \int_{\varphi(E)} g(x)dx$$

for every open subset $E \subset A$, where id is the identity map. This implies

$$g(\varphi(x))\det(D\varphi(x)) = f(x)$$

for all $x \in \bar{A}$.

(2) The map $(f,g) \to \varphi = \varphi(f,g)$ is C^∞ from $C^r(\bar{A}) \times C^r(\bar{A})$ to $\mathrm{Diff}^r(\bar{A})$.

Lemma 2 follows when we identify $f(x)dx$ and $g(x)dx$ with volume forms Ω_0 and $\Omega_1 = a_\tau(x)\Omega_0$. Thus, $T_\tau = \varphi$ is the desired diffeomorphism. Since $\varphi = \mathrm{id}$ when $f(x) = g(x)$, the inequality $\|T_\tau - \mathrm{id}\|_{C^r} \leq \kappa\|1 - a_\tau(x)\|_{C^r}$ follows from the last statement of the theorem.

Details of the proof of part (1) of the theorem can be found in [19] and [5]. The results in [19] and [5] are more general. They include the case when $\mathrm{supp}(f-g) \subset \bar{A}$. For Lemma 2, we only need the theorem in the case when $\mathrm{supp}(f-g) \subset A$. We provide a sketch of the proof appeared in [19, 5] and explain why the statement (2) is true. We use essentially the same notation from [19, 5] for objects involved in the proof.

Step one. The first step is to reduce the case for arbitrarily open set A to that when $\bar{A}$ is C^k diffeomorphic to a unit cube $\bar{Q}$. This is based on the following statement.

Lemma 3 ([19], Lemma 1)**.** *Assume that $\{U_j\}_{j=0}^m$ is an open cover of a manifold M, including the case when $M = A$ is an open set of $\mathbb{R}^n$, and g is a C^r function on M satisfying the condition $\int_M g dx = 0$. Then, there exist C^r functions g_j, $j = 0, 1, 2, \cdots, m$ such that*

$$g = \sum_{j=0}^m g_j, \ \mathrm{supp}(g_j) \in U_j, \quad \text{and} \quad \int_M g_j dx = 0, \ j = 0, 1, 2, \cdots, m.$$

Moreover, for each j, the map $g \to g_j$ is linear and $\|g_j\|_{C^r} \leq c\|g\|_{C^r}$, where constant c depends on M and the covering $\{U_j\}_{j=1}^m$ only.

Indeed, let $\{\phi_j\}_{j=0}^m$ be a C^r partition of unity for the open covering $\{U_j\}_{j=0}^m$. For each $k = 1, 2, \cdots, m$, let $\rho(k) < k$ denote the smallest integer such that $U_k \cap U_{\rho(k)} \neq \emptyset$ and for each $j = 1, 2, \cdots, m$, let η_k be a C^r function such that $\mathrm{supp}(\eta_k) \subset U_k \cap U_{\rho(k)}$ and $\int \eta_k dx = 1$. Then, g_j can be defined by

$$g_j = g\phi_j - \sum_{k=1}^m \lambda_k \alpha_{jk} \eta_k,$$

where $\alpha_{jk} = 1, \text{if} j = k; -1, \text{if} j = \rho(k); 0, \text{otherwise}$ and $\lambda_1, \lambda_2, \cdots, \lambda_m$ is the solution to the nonsingular linear system

$$\sum_{k=1}^{m} \alpha_{jk}\lambda_k = \int g\phi_j dx, \ j = 1, 2, \cdots, m.$$

We may regard each set U_j in the lemma as the unit cube Q, and construct the diffeomorphism φ on cube that satisfies the requirements of the theorem.

Step two. The construction is based on the implicit function theorem and induction on the dimension of the unit cube. We give an outline of this construction in the case when the dimension of the manifold is 2 to illustrate the C^∞ dependence of φ on the functions f and g.

Assume that $A = Q = \{x = (x_1, x_2) : \ 0 < x_1, x_2 < 1\}$ and $0 < f, g \in C^r(\bar{Q})$ with $\text{supp}(f - g) \subset Q$ and $\int_Q (f - g)dx = 0$. Take $\epsilon > 0$ such that

$$\epsilon \max g < \min \{g, \ \frac{1}{2} f \ \}.$$

Then choose a number $\delta > 0$ sufficiently small and a 'cut-off' function $\zeta(t) \in C^\infty(0, 1)$ satisfying the following conditions:

(1) $\text{supp}(f - g) \subset [\delta, 1 - \delta] \times [\delta, 1 - \delta]$;
(2) $\text{supp}(\zeta) = [\delta, 1 - \delta]$ and $0 \le \zeta(t) \le 1 + \epsilon, \ \forall t \in [0, 1]$;
(3) $\int_0^1 \zeta(t)dt = 1$ and $\int |\zeta(t) - 1|dt < \epsilon$.

Set $u(0) = 0$ and let $u = u(x_2)$ be the solution to the equation

$$G_2(x_2, u, f, g) := \int_0^1 \left[\int_0^{x_2+\zeta(x_1)u} g(x_1, s)ds - \int_0^{x_2} f(x_1, s)ds \right] dx_1 = 0$$

for all $x_2 \in (0, 1]$. Then define

$$\phi_2(x_1, x_2) = (x_1, x_2 + \zeta(x_1)u(x_2)),$$

and take $g_1(x) = g(\phi_2(x))\det(D\phi_2(x)) \in C^r(\bar{Q})$, where $x = (x_1, x_2)$.

It can be proved that $\psi_2(x_1, x_2)$ is unique and of C^r, $\text{supp}(\psi_2 - \text{id}) \subset Q$, and $\int_0^1 g_1(x_1, x_2)dx_1 = \int_0^1 f(x_1, x_2)dx_1$ for all $x_2 \in [0, 1]$.

Further, since the function G is affine in f and g, the function $u(x_2)$ depends on f and g smoothly, that is, the map $(f, g) \to u(x_2)$ is C^∞.

Similarly, let $v(x_1, x_2)$ be that function satisfies $v(0, x_2) = 0$ and

$$G_1(x_1, x_2, v, f, g_1) := \int_0^{v(x_1,x_2)} g_1(s, x_2)ds - \int_0^{x_1} f(s, x_2)ds = 0$$

for all $x_1, x_2 \in [0, 1]$.

Then take

$$\phi_1(x_1, x_2) = (v(x_1, x_2), x_2).$$

We can get $\text{supp}(\phi_1 - \text{id}) \subset Q$. Again, by Implicit Function Theorem, ϕ_1 depends on f and g_1 smoothly.

Let $\varphi(x) = \phi_2 \circ \phi_1$, which is the desired diffeomorphism that depends on f and g smoothly.

Remark 5. Given a nonsingular real matrix A, there exists a real matrix P such that $P^{-1}AP$ consists of real Jordan blocks with each block corresponding to an eigenspace. Let $B = P^{-1}AP$. For each Jordan block B_i, if the eigenvalue λ_i is real, we multiply the block by matrix $E_i = I$ if $\lambda > 0$ and by $E_i = -I$ if $\lambda < 0$. If the

eigenvalue is a complex number $\lambda_i = \alpha + i\beta$, then we have matrix $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ on the diagonal, and let $E_i = \dfrac{1}{\sqrt{\alpha^2+\beta^2}} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, which is a rotation in the eigenspace. In each case we have $E_i^{-1} = E_i$, $B_i = E_i E_i B_i$, and $E_i B_i$ has positive real eigenvalues. We now have $B = EE^{-1}B$ such that $E^{-1}B$ has positive eigenvalues and E is either the identity, a reflection or a rotation. Since $P^{-1}AP = B = EE^{-1}B$, $A = PBP^{-1} = PEP^{-1}PE^{-1}BP^{-1}$. The fact $|\det(PBP^{-1})| = 1$ implies that PBP^{-1} preserves volume. Hence $PE^{-1}BP^{-1}$ has positive real eigenvalues and preserves volume.

Remark 6. For a C^1 vector field X on a Riemannian manifold M with a metric $< \cdot, \cdot >$, by Divergence Theorem, we only need to show that for any C^∞ test function $g(x)$ with its support in a small neighborhood of x, the following equation holds:

$$\int_M g(\mathrm{div}_{\omega_1} X + < \mathrm{grad} \ln \rho(x), X >) d\omega_2 = \int_M g(\mathrm{div}_{\omega_2} X) d\omega_2,$$

where two volume forms satisfy the relation $d\omega_2 = \rho(x) d\omega_1$.

Indeed, since for any C^1 function on M, $\mathrm{div}(\rho X) = \rho \mathrm{div} X + < \mathrm{grad}\rho, X >$, we have

$$\begin{aligned}
& \int_M g(\mathrm{div}_{\omega_1} X + < \mathrm{grad} \ln \rho(x), X >) d\omega_2 \\
= & \int_M g(\mathrm{div}_{\omega_1} X + < \mathrm{grad} \ln \rho(x), X >) \rho(x) d\omega_1 \\
= & \int_M g\left[(\mathrm{div}_{\omega_1} X) \cdot \rho(x) + < \mathrm{grad}\rho(x), X >\right] d\omega_1 \\
= & \int_M g(\mathrm{div}_{\omega_1}(\rho(x) X)) d\omega_1 = -\int_M < \mathrm{grad} g, \rho(x) X > d\omega_1 \\
= & -\int_M < \mathrm{grad} g, X > \rho(x) d\omega_1 = -\int_M < \mathrm{grad} g, X > d\omega_2 \\
= & \int_M g(\mathrm{div}_{\omega_2} X) d\omega_2.
\end{aligned}$$

Remark 7. The same perturbation argument can be carried out along a periodic orbit of a volume preserving Anosov diffeomorphism f. Let $\{p_1, p_2, \cdots, p_k\}$ be any periodic orbit of f. Let g be the original Riemannian metric that defines the volume form preserved by f. Let g_l be a smooth (Lyapunov) metric under which the exponential splitting of the tangent bundle $E^s \oplus E^u$ satisfies the conditions $\|Dfv^s\| \leq \lambda_l \|v^s\|$ and $\|Dfv^u\| \geq \mu_l \|v^u\|$ for some $\lambda_l < 1$ and $\mu_l > 1$ and any nonzero vectors $v^s \in E^s, v^u \in E^u$. We first perturb f using Moser's theorem so that it preserves the volume form induced by g_l. We then choose an open neighborhood $O(p_i)$ sufficiently small around each point of the periodic orbit. To obtain the perturbation, we first identify the consecutive points $p_i \equiv p_{i+1}$, $i = 1, 2, \cdots, k; k+1 \equiv 1$ of the orbits, and treat the neighborhoods $O(p_i)$ and $O(p_{i+1})$ as the neighborhoods of the origin in a same Euclidean space. We further require that the identifications are chosen in a way that the eigenspaces of Df^n at p_i are invariant spaces of Df at p_i. With the identification, for $f : O(p_i) \to O(p_{i+1})$ we still have $Df_{p_i} = EA$, where E and A are matrices with the same properties as that in the proof of Theorem 1. Hence we can apply the process of perturbation

used in the proof of Theorem 1 to each of these k maps, we obtain map $\tilde{f}_w|_{O(p_i)} : O(p_i) \to O(p_{i+1})$. Now we go back to treat p_i and p_{i+1} as different points in the manifold, and the map $\tilde{f}_w|_{O(p_i)} : O(p_i) \to O(p_{i+1})$ can be extended to $\tilde{f}_w : M \to M$ by requiring $\tilde{f}_w = f$ outside $O(p_i)$ for $i = 1, \cdots, k$. Use this construction we can get that the entropy with respect to its SRB measure approaches zero as $w \to 0$.

Appendix B. **An example on the circle with a different approach.** In this appendix we adapt the example appeared in [12, Theorem 22] to construct a path of smooth circle expanding endomorphisms showing that at the end of the path, the hyperbolicity could disappear completely in the limiting map (that is, totally degenerate). This example serves as a one-dimensional model for our main results in this paper for higher-dimensional theorem (Theorem 1 and Theorem 2).

Let $\mathbb{S}^1$ be the unit circle and m denote the Lebesgue measure on $\mathbb{S}^1$. We will construct a C^1-path $t \in (0, \frac{1}{6}) \to f_t \in \mathcal{E}^r_m(\mathbb{S}^1)$ of orientation-preserving circle endomorphisms such that each map on the path is a C^r, $2 \le r \le \infty$, expanding map and preserves the Lebesgue measure. So the Lebesgue measure m is f_t's SRB-measure. Furthermore, we prove that $h_m(f_t) \to 0$ as $t \to 0^+$. The construction can be regarded as the smooth version of the following piecewise smooth map from $\big(-\dfrac{1}{2}, \dfrac{1}{2}\big]$ to $\big(-\dfrac{1}{2}, \dfrac{1}{2}\big]$,

$$L_t(x) = \begin{cases} \frac{1}{2t}\big(x + \frac{1}{2}\big), & x \in \big(-\frac{1}{2}, -\frac{1}{2} + t\big]; \\ \frac{x}{1-2t}, & x \in \big(-\frac{1}{2} + t, \frac{1}{2} - t\big]; \\ \frac{1}{2t}\big(x - \frac{1}{2}\big), & x \in \big(\frac{1}{2} - t, \frac{1}{2}\big] \end{cases}$$

It preserves the Lebesgue measure m and the metric entropy

$$h_m(L_t) = -2t\log(2t) - (1-2t)\log(1-2t).$$

Thus $h_m(L_t) \to 0$ as $t \to 0^+$.

In the following theorem and its proof we regard the unit circle as $\big[-\frac{1}{2}, \frac{1}{2}\big]$ and $\big[-\frac{1}{2} + t, \frac{1}{2} + t\big]$, $t < \frac{1}{6}$, alternatively.

Theorem 3 (One-Dimensional Model)**.** *There is a C^1-path $\{f_t\}_{0<t<\frac{1}{6}}$ of C^r, $2 \le r \le \infty$, orientation-preserving circle expanding endomorphisms of degree $d \ge 2$ such that each map f_t preserves m and the infimum of the metric entropy $h_m(f_t) : (0, \frac{1}{6}) \to \mathbb{R}^+$ is zero. Moreover, $f_t(x) \to x$ as $t \to 0^+$ for all $x \in \mathbb{S}^1 \setminus \{\frac{1}{2}\}$.*

Proof. Without loss of generality, we construct $t \in (0, \frac{1}{6}) \to f_t$ for $d = 2$.

Next we will smooth L_t such that the resulting map is a C^r circle endomorphism of degree 2 and preserves the Lebesgue measure m.

Construct a family of C^{r-1} functions $\{\phi_t(x)\}$ on $\big[\frac{1}{2} - t, \frac{1}{2} + t\big]$ satisfies the following conditions:

i) $\phi_t\big(\frac{1}{2} \pm t\big) = 2$; $\phi_t^{(k)}\big(\big(\frac{1}{2} - t\big)_+\big) = \phi_t^{(k)}\big(\big(\frac{1}{2} + t\big)_-\big) = 0$, $k = 1, 2, \cdots r-1$;

ii) $2 \le \phi_t(x) \le \frac{2}{t}$ for $|x - \frac{1}{2}| \le t$; $\phi_t(x) \ge \frac{1}{2t}$ for $|x - \frac{1}{2}| \le t - t^2$;

iii) $\int_{1/2-t}^{1/2} \phi_t(\xi)d\xi = \int_{1/2}^{1/2+t} \phi_t(\xi)d\xi = \frac{1}{2}$;

iv) ϕ_t depends on t smoothly.

We then define a map $f_1 = f_{t,1}$ from $\big[\frac{1}{2} - t, \frac{1}{2} + t\big]$ to $\big[-\frac{1}{2}, \frac{1}{2}\big]$ by

$$f_1(x) = -\frac{1}{2} + \int_{\frac{1}{2}-t}^{x} \phi(\xi)d\xi.$$

We have that

$$\begin{aligned} &f_1\Big(\frac{1}{2}\Big)=0;\quad f_1\Big(\frac{1}{2}\pm t\Big)=\pm\frac{1}{2};\quad f_1'\Big(\big(\frac{1}{2}-t\big)_+\Big)=2=f_1'\Big(\big(\frac{1}{2}+t\big)_-\Big);\\ &f_1^{(k)}\Big(\big(\frac{1}{2}-t\big)_+\Big)=0=f_1^{(k)}\Big(\big(\frac{1}{2}+t\big)_-\Big)\qquad \text{for } k=2,3,\cdots,r. \end{aligned} \tag{B.1}$$

Moreover,

$$2\le {f_1}'(x)\le\frac{2}{t}\quad\text{for}\quad \Big|x-\frac{1}{2}\Big|\le t;\quad\text{and}\quad {f_1}'(x)\ge\frac{1}{2t}\quad\text{for}\quad\Big|x-\frac{1}{2}\Big|\le t-t^2. \tag{B.2}$$

Let $g_1(x)$ be the inverse function of f_1. $g_1(x)$ is a C^r diffeomorphism from $\big[-\frac{1}{2},\frac{1}{2}\big]$ to $\big[\frac{1}{2}-t,\frac{1}{2}+t\big]$. Since $g_1'(x)=\dfrac{1}{\phi(g_1(x))}$, the properties of f_1 give that

$$\begin{aligned} &g_1(0)=\frac{1}{2};\quad g_1\Big(\pm\frac{1}{2}\Big)=\frac{1}{2}\pm t;\quad g_1'\Big(\big(-\frac{1}{2}\big)_+\Big)=\frac{1}{2}=g_1'\Big(\big(\frac{1}{2}\big)_-\Big);\\ &g_1^{(k)}\Big(\big(-\frac{1}{2}\big)_+\Big)=0=g_1^{(k)}\Big(\big(\frac{1}{2}\big)_-\Big)\qquad\text{for } k=2,3,\cdots,r. \end{aligned}$$

We note that the fact $f_1'(x)\le\frac{2}{t}$ on $\big[\frac{1}{2}-t,\frac{1}{2}+t\big]$ implies that the lengths of the intervals $f_1\big(\big[\frac{1}{2}-t,\frac{1}{2}-t+t^2\big]\big)$ and $f_1\big(\big[\frac{1}{2}+t-t^2,\frac{1}{2}+t\big]\big)$ are less than $2t$. Hence the inequalities in (B.2) imply

$$\frac{t}{2}\le {g_1}'(x)\le\frac{1}{2}\quad\text{for}\quad |x|\le\frac{1}{2}\quad\text{and}\quad {g_1}'(x)\le 2t\quad\text{for}\quad |x|\le\frac{1}{2}-2t. \tag{B.3}$$

We now define

$$g_0(x)=-\frac{1}{2}+t+\int_{-\frac{1}{2}}^{x}\big(1-g_1'(\xi)\big)d\xi=x+\frac{1}{2}-g_1(x),\quad x\in[0,1].$$

It is clear that

$$g_0(0)=0\quad\text{and}\quad g_0\Big(\pm\frac{1}{2}\Big)=\pm\Big(\frac{1}{2}-t\Big).$$

The definition also gives $g_0'(x)=1-g_1'(x)$, that is

$$g_0'(x)+g_1'(x)=1. \tag{B.4}$$

Hence we have

$$g_0'\Big(\big(-\frac{1}{2}\big)_+\Big)=\frac{1}{2}=g_0'\Big(\big(\frac{1}{2}\big)_-\Big);\quad\text{and}\quad g_0^{(k)}\Big(\big(-\frac{1}{2}\big)_+\Big)=0=g_0^{(k)}\Big(\big(\frac{1}{2}\big)_-\Big)$$

for $k=2,3,\cdots,r$. So $g_0:\big[-\frac{1}{2},\frac{1}{2}\big]\to\big[-\frac{1}{2}+t,\frac{1}{2}-t\big]$ is a C^r-diffeomorphism. Moreover, by (B.3)

$$\frac{1}{2}\le {g_0}'(x)\le 1-\frac{t}{2}\text{ for }|x|\le\frac{1}{2};\quad\text{and}\quad 1-2t\le {g_0}'(x)<1\text{ for }|x|\le\frac{1}{2}-2t. \tag{B.5}$$

Let $f_0(x):\big[-\frac{1}{2}+t,\frac{1}{2}-t\big]\to\big[-\frac{1}{2},\frac{1}{2}\big]$ be the inverse of $g_0(x)$. Then we have

$$\begin{aligned} &f_0(0)=0;\quad f_0\Big(\pm\big(\frac{1}{2}-t\big)\Big)=\pm\frac{1}{2};\quad f_0'\Big(\big(-\frac{1}{2}+t\big)_+\Big)=2=f_0'\Big(\big(\frac{1}{2}-t\big)_-\Big);\\ &f_0^{(k)}\Big(\big(-\frac{1}{2}+t\big)_+\Big)=0=f_0^{(k)}\Big(\big(\frac{1}{2}-t\big)_-\Big)\qquad\text{for } k=2,3,\cdots,r. \end{aligned} \tag{B.6}$$

Moreover, by the first inequalities in (B.5)

$$\frac{2}{2-t}\le {f_0}'(x)\le 2\qquad\text{for}\quad |x|\le\frac{1}{2}-t. \tag{B.7}$$

Note that the lengths of $g_0\big(\big[-\frac{1}{2},-\frac{1}{2}+2t\big]\big)$ and $g_0\big(\big[\frac{1}{2}-2t,\frac{1}{2}\big]\big)$ are less than $2t$. We have $g_0\big(\big[-\frac{1}{2}+2t,\frac{1}{2}-2t\big]\big)\supset\big[-\frac{1}{2}+3t,\frac{1}{2}-3t\big]$. By the second inequalities in (B.5) we get

$$1<{f_0}'(x)\le\frac{1}{1-2t}\quad\text{for}\quad\big|x\big|\le\frac{1}{2}-3t. \tag{B.8}$$

Now we define

$$f_t(x)=\begin{cases} f_0(x), & x\in\big[-\frac{1}{2}+t,\frac{1}{2}-t\big];\\ f_1(x), & x\in\big[\frac{1}{2}-t,\frac{1}{2}+t\big].\end{cases}$$

Note that on $\mathbb{S}^1$, $\frac{1}{2}+t$ and $-\frac{1}{2}+t$ represent the same point. By (B.1) and (B.6) we know that f_t is well defined and is a C^r expanding circle endomorphism.

For any $x\in\mathbb{S}^1$, let $\{x_0,x_1\}=f_t^{-1}(x)$ such that $x_0\in\big(-\dfrac{1}{2}+t,\dfrac{1}{2}-t\big]$ and $x_1\in\big(\dfrac{1}{2}-t,\dfrac{1}{2}+t\big]$. Then by (B.4) we have that

$$\frac{1}{f_t'(x_0)}+\frac{1}{f_t'(x_1)}=g_0'(x)+g_1'(x)=1.$$

So f_t preserves the Lebesgue measure m.

Since $f_0(0)=0$, by (B.8) we can obtain that

$$\big|f(x)-x\big|\le\int_0^x\big|f'(\xi)-1\big|d\xi\le\Big|\frac{1}{1-2t}-1\Big|=\frac{2t}{1-2t}\quad\text{for}\quad\big|x\big|\le\frac{1}{2}-3t.$$

This implies $\lim\limits_{t\to0^+}f_t(x)=x$ as $t\to0^+$ for all $x\in\mathbb{S}^1\setminus\big\{\frac{1}{2}\big\}$.

We are left to show that $\lim\limits_{t\to0^+}h_m(f_t)(x)=0$. This comes directly from the construction. Indeed, if we denote $I_0=\big(-\frac{1}{2}+3t,\frac{1}{2}-3t\big]$ and $I_1=\big(\frac{1}{2}-3t,\frac{1}{2}+3t\big]$, then $\{I_0,I_1\}$ is a partition of $\mathbb{S}^1$. By (B.2) and (B.7), $f_t'(x)\le\frac{2}{t}$ for $x\in I_1$. Hence, by (B.8) we can get

$$\begin{aligned}h_m(f_t)=&\int_{\mathbb{S}^1}\log f_t'(x)dx=\int_{I_0}\log f_t'(x)dx+\int_{I_1}\log f_t'(x)dx\\ \le&(1-6t)\cdot\log\frac{1}{1-2t}+6t\cdot\log\frac{2}{t},\end{aligned}$$

which converges to 0 as $t\to0^+$. This is what we need. □

Remark 8. In the above construction f_t sends the interval $\big[\frac{1}{2}-t,\frac{1}{2}+t\big]$ to the whole circle $\mathbb{S}^1$ and $f_t\big(\frac{1}{2}\big)=0$ for all t. Hence $\{f_t\}$ converges pointwise to a discontinuous map, and the convergence cannot be in C^0 topology. In particular, it holds that $\lim\limits_{t\to0^+}f_t(x)=x$ for $x\neq\dfrac{1}{2}$ and $\lim\limits_{t\to0^+}f_t\big(\dfrac{1}{2}\big)=0$.

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REFERENCES

[1] A. Arbieto and C. Matheus, A pasting lemma and some applications for conservative systems. With an appendix by David Diica and Yakov Simpson-Weller, *Ergodic Theory Dynam. Systems*, **27** (2007), 1399–1417.

[2] M. Benedicks and L.-S. Young, Sinai-Bowen-Ruelle measures for certain Hénon maps, *Invent. Math.*, **112** (1993), 541–576.

[3] J. Bochi, B. R. Fayad and E. Pujals, A remark on conservative diffeomorphisms, *C. R. Math. Acad. Sci. Paris,* **342** (2006), 763–766.

[4] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Mathematics, **470**, Springer-Verlag, New York, 1975.

[5] B. Dacorogna and J. Moser, On a partial differential equation involving the Jacobian determinant, (French summary) *Ann. Inst. H. Poincarè Anal. Non Linéaire*, **7** (1990), 1–26.

[6] M. Hirsch, C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics, **583**, Springer-Verlag, Berlin-New York, 1977.

[7] H. Hu, Conditions for the existence of SBR measures of "almost Anosov" diffeomorphisms, *Trans. Amer. Math. Soc.*, **352** (2000), 2331–2367.

[8] H. Hu, Statistical properties of some almost hyperbolic systems, in *Smooth Ergodic Theory and Its Applications*(Seattle, 1999), Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, **69** (2001), 367–384.

[9] H. Hu, M. Jiang and Y. Jiang, Infimum of the metric entropy of hyperbolic attractors with respect to the SRB measure, *Discrete Contin. Dyn. Syst.*, **22** (2008), 215–234.

[10] H. Hu and L.-S. Young, Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov", *Ergod. Th. & Dynam. Sys.*, **15** (1995), 67–76.

[11] M. Jiang, Differentiating potential functions of SRB measures on hyperbolic attractors, *Ergod. Th. & Dynam. Sys.*, **32** (2012), 1350–1369.

[12] Y. Jiang, Teichmüller structures and dual geometric Gibbs type measure theory for continuous potentials, `arXiv:0804.3104v3`

[13] A. Katok, Bernoulli diffeomorphisms on surfaces, *Ann. of Math.*, **110** (1979), 529–547.

[14] A. Katok and B. Hasselbratt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.

[15] F. Ledrappier, Propriétés ergodiques des measure de Sinai, *Publ. Math. I.H.E.S.*, **59** (1984), 163–188.

[16] F. Ledrappier and J.-M. Strelcyn, A proof of the estimation from below in Pesin's entropy formula, *Ergod. Th. & Dynam. Sys.*, **2** (1982), 203–219.

[17] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms I, Characterization of measures satisfying Pesin's entropy formula, *Ann. of Math.*, **122** (1985), 509–539.

[18] W. Li, J. Llibre and X. Zhang, Extension of Floquet's theory to nonlinear periodic differential systems and embedding diffeomorphisms in differential flows, *Amer. J. Math.*, **124** (2002), 107–127.

[19] J. Moser, On the volume elements on a manifold, *Trans. Amer. Math. Soc.*, **120** (1965), 286–294.

[20] V. A. Rokhlin, Lectures on the theory of entropy of transformations with invariant measures, (Russian)*Uspehi Mat. Nauk*, **22** (1967), 3–56.

[21] D. Ruelle, Differentiation of SRB states, *Commun. Math. Phys.*, **187** (1997), 227–241.

[22] Ya. Sinai, Gibbs measure in ergodic theory, *Russ. Math. Surveys*, **27** (1972), 21–64.

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