Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
Spring Semester of 2009, Friday, 9:30am-11:30am

Yunping Jiang

Purpose of this course

The purpose of this course is designed to study geometric structures of invariant sets of discrete dynamical systems in real one-dimension, complex one-dimension, and real two-dimension. We would like to bring quasiconformal mapping theory into this study. We would also like to study some geometric structure of some space of dynamical systems with a same topological type. In this study, we would like to bring Teichmüller theory into the study. Some key words that we will constantly deal with are

- fixed point and periodic point
- invariant set and Julia set
- invariant measure and Gibbs measure
- transfer operator and Hilbert metric
- Markov partition and symbolic coding
- quasiconformal mapping and quasisymmetric mapping
- holomorphic motion and Teichmüller space
- quasiconformal rigidity and smooth rigidity and rigidity

Main goal in the study of dynamical systems: Make a prediction for the future from past known data under certain rules for a typical starting point.

Setting:

- A topological space X called "phase space", whose elements are called points;
- "time", which could be integer time n or continuous time t;
- a map $f : X \to X$ or a flow ϕ^t on X;

What we want to know is the limits of $\{f^n(x)\}_{n=0}^{\infty}$ for a typical x in X under iterations or the limits of the flow line $\phi^t(x)$ for a typical x when the time t goes to infinity for a typical point x.

We call the group $\{f^n\}_{n=-\infty}^{\infty}$ if f is invertible or the semi-group $\{f^n\}_{n=0}^{\infty}$ if f is non-invertible a discrete dynamical system. We call the flow ϕ^t a continuous dynamical system.

Definition 1. A continuous map $F(t,x) = \phi^t(x) : \mathbb{R} \times X \to X$ is called a flow if for every $t \in \mathbb{R}$, $\phi^t : X \to X$ is a homeomorphism and if it satisfies

$$\phi^{t+s}(x) = \phi^t(\phi^s(x))$$

for all $t, s \in \mathbb{R}$ and $x \in X$.

If X is a smooth manifold and $F(t,x)$ is a smooth map, then for any $x \in X$, $\{\phi^t(x)\}_{t\in\mathbb{R}}$ is a smooth curve in X. Let

$$V(x) = \frac{\partial F(t,x)}{\partial t}\Big|_{t=0} = \frac{d\phi^t(x)}{dt}\Big|_{t=0}.$$

Then V defines a continuous vector field on the tangent bundle on $TX = \cup_{x\in X} T_x X$. Thus the flow $\phi^t(x)$ is just the solution of the ordinary differential equation

$$\frac{dy}{dt} = V(x) \quad \text{with the initial condition } y(0) = x.$$

Actually solutions of an ordinary differential equation on a complete smooth manifold for a vector field with certain smooth regularity form a flow. In many situations, we can use a discrete dynamical system to study a continuous dynamical system. This is due to a smart observation from Poincaré.

Poincaré map: Suppose $\phi^t(x)$ is a continuous flow on an m-dimensional smooth manifold. Suppose there are a point $x_0 \in X$ and a time $t_0 > 0$ such that $\phi^{t_0}(x_0) = x_0$. Then $\{\phi^t(x_0)\}_{t\in\mathbb{R}}$ is called a closed orbit, where the smallest such $t_0 > 0$ is called the period. Suppose $V(x_0) \neq 0$. Since M is locally $\mathbb{R}^m$, we can find an $(m-1)$-dimensional submanifold N of X transversal to $V(x_0)$ at x_0, that is,

$$T_{x_0}X = T_{x_0}N \oplus \{tV(x_0)\}.$$

Since $\phi^{t_0}(x_0) = x_0$ and since $F(t,x) = \phi^t(x)$ is continuous, we have a neighborhood U about x_0 in N such that for every $x \in N$, there is the smallest $t(x) > 0$ close to t_0 such that $\phi^{t(x)}(x) \in N$. Define

$$f(x) = \phi^{t(x)}(x) : U \to N.$$

Then f defines a discrete dynamical system with $f(x_0) = x_0$. So the study of the closed orbit $\{\phi^t(x_0)\}_{t\in\mathbb{R}}$ for the continuous dynamical system ϕ^t is equivalent to the study of the fixed point of the discrete dynamical system $\{f^n\}_{n=0}^{\infty}$. Here f is called a section map, or Pincaré map, or first return map.

Suspension: For a diffeomorphism $f : X \to X$ of a smooth manifold X, we can construct a suspension flow on the suspension manifold X_f. Here the suspension manifold is obtained by gluing $(f(x), 0)$ and $(x, 1)$ on $X \times [0, 1]$. The suspension flow ϕ^t is integral curves of the "vertical" vector field $\partial/\partial t$ on X_f.

Example 1. *Let $X = [0, 1]$ and $f(x) = 1 - x : X \to X$. Then X_f is the Möbius strip. The suspension flow ϕ^t has a period one closed orbit which does not separate M_f and one period two closed orbit which is the boundary of M_f and infinitely many period two closed orbits, each of them cuts M_f into two parts: one is topologically equivalent to M_f and the other is topologically equivalent to $[0, 1] \times S^1$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle.*

Example 2. *Let $X = S^1$ and $f(x) = -z : X \to X$. Then X_f is topologically equivalent to the two torus $T^2 = S^1 \times S^1$.*

In this course, we will more concentrate on discrete dynamical systems.

Dynamics of a discrete dynamical system with contraction property:

Suppose (X, d) is a metric space where $d = d(\cdot, \cdot)$ is the metric. A map $f : X \to X$ is called contracting if there exists $0 < \lambda < 1$ such that

$$d(fx, fy) \le \lambda d(x, y), \quad \forall x, y \in X,$$

where λ is called a contracting constant. A point $x \in X$ is called a fixed point if $f(x) = x$.

Theorem 1 (Contacting Mapping Principle)**.** *If (X, d) is a complete metric space and if $f : X \to X$ is contracting with the contracting constant λ, then f has a unique fixed point x_0 in X and $f^n(x)$ tends to x_0 exponentially when n goes to infinity for every x in X.*

Proof. For $x, y \in X$ and $n > 0$, we have

$$d(f^n(x), f^n(y)) \le \lambda d(f^{n-1}(x), f^{n-1}(y)) \le \cdots \le \lambda^n d(x, y)$$

for some $0 < \lambda < 1$. This implies that $\{f^n(x)\}_{n=0}^{\infty}$ have the same asymptotic behavior for all $x \in X$ as $n \to \infty$.

Now consider a sequence $\{f^n(x)\}_{n=0}^{\infty}$. For any $m > n > 0$,

$$d(f^m(x), f^n(x)) \leq \sum_{k=0}^{n-m-1} d(f^{n+k+1}(x), f^{n+k}(x))$$

$$\leq \sum_{k=0}^{n-m-1} \lambda^{n+k} d(f(x), x) \leq \frac{\lambda^n}{1-\lambda} d(f(x), x) \to 0, \quad \text{as} \quad n \to \infty.$$

Thus $\{f^n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence and since X is complete, it has a limiting point $x_0 \in X$. Because

$$f(x_0) = f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = x_0,$$

x_0 is a fixed point of f.

If y_0 is another fixed point, since

$$d(y_0, x_0) = d(f(y_0), f(x_0)) \leq \lambda d(x_0, y_0),$$

we have that $d(y_0, x_0) = 0$. So $y_0 = x_0$. This is the uniqueness. We complete the proof. □

The following result says that the unique fixed point x_0 of a contracting map f is preserved by C^0-perturbation and the fixed point changes continuously under C^0-topology.

Corollary 1. *Take f as that in the above theorem. For any $\epsilon > 0$, there is a $\delta \in (0, 1-\lambda)$ such that for any map $g : X \to X$ with*

1) *$d(f(x), g(x)) < \delta$ for all $x \in X$ and*
2) *$d(g(x), g(y)) \leq (\lambda + \delta) d(x, y)$ for all $x, y \in X$.*

the fixed point y_0 of g satisfies $d(y_0, x_0) < \epsilon$.

Proof. Take $\delta = \epsilon(1-\lambda)/(1+\epsilon)$. Since $g^n(x_0) \to y_0$, we have

$$d(x_0, y_0) \leq \sum_{k=0}^{\infty} d(g^k(x_0), g^{k+1}(x_0)) \leq \sum_{k=0}^{\infty} (\lambda+\delta)^k d(x_0, g(x_0))$$

$$\leq \frac{\delta}{1-\lambda-\delta} = \epsilon.$$

□

Now we can formulate the above theorem and the corollary into a modern version. Suppose $C(X)$ be the space of all continuous maps from X into X. A map $f \in C(X)$ is called Lipschitz if there is a constant $L > 0$ such that

$$d(f(x), f(y)) \leq L d(x, y), \quad \forall\, x, y \in X.$$

The smallest number $L > 0$ is called the Lipschitz constant and denoted as L_f. Then

$$L_f = \sup_{x \neq y \in X} \frac{d(f(x), f(y))}{d(x, y)}.$$

Let $LC(X)$ be the space of all Lipschitz map in $C(X)$. We can define a Lipschitz metric d_L on $LC(X)$ as

$$d_L(f, g) = \sup_{x \in X} d(f(x), g(x)) + |L_f - L_g|.$$

It is clear that (1) $d_L(f, g) = d_L(g, f)$, (2) $d_L(f, g) = 0$ if and only if $f = g$, and (3)

$$\begin{aligned} d_L(f, g) &= \sup_{x \in X} d(f(x), g(x)) + |L_f - L_g| \\ &\leq \sup_{x \in X} d(f(x), h(x)) + \sup_{x \in X} d(h(x), g(x)) + |L_f - L_h + L_h - L_g| \\ &\leq [\sup_{x \in X} d(f(x), h(x)) + |L_f - L_h|] + [(\sup_{x \in X} d(h(x), g(x)) + |L_h - L_g|] \\ &= d_L(f, h) + d_L(h, g). \end{aligned}$$

Therefore, d_L is a metric on $LC(X)$. (Note that it may happen that $d_L(f, g) = \infty$ for some $f, g \in LC(X)$, but one can modify it by defining

$$\tilde{d}_L(f, g) = \max\{d_L(f, g), 1\}$$

in this case. Since we only care about the case when $d_L(f, g)$ small, so we still use the original form of d_L.) All contracting maps from X to X form a subspace

$$LC_1(X) = \{f \in LC(X) \mid L_f < 1\}.$$

The metric is d_L restricted on $LC_1(X)$.

Theorem 2 (Modern Version of Contacting Mapping Principle)**.** *If (X, d) is a complete metric space, then for every $f \in LC_1(X)$, there is a unique $x(f) \in X$ such that $f(x(f)) = x(f)$. Moreover,*

$$F : (LC_1(X), d_L) \to (X, d); \quad F(f) = x(f)$$

is a continuous operator from the metric space $(LC_1(X), d_L)$ to the metric space (X, d).

Periodic point:

For a dynamical system, $f : X \to X$, we call a point $x \in X$ a periodic point of period $m > 0$ if $f^m(x) = x$ but $f^i(x) \neq x$ for all $1 \leq i \leq m - 1$. We use $Per_m(f)$ to denote the set of all period points of period m. In particular, if $m = 1$, x is called a fixed point. We use $Fix(f)$ to denote the set of all fixed points.

Exerice 1. *Suppose* $f(z) = z^d : S^1 \to S^1$. *Find* $Fix(f^m)$, $Per_m(f)$.

Suppose X is an n-dimensional C^1 manifold and $f : X \to X$ is a C^1 map. Suppose p is a periodic point of period m. Then $f^m(p) = p$. The derivative $Df^m(p) : T_pX \to T_pX$ is a linear operator where T_pX is the tangent space of X at p. Since we only consider local dynamics for f near p, we can introduce local coordinates near p with p as the origin 0. In these coordinates $Df^m(0)$ becomes an $n \times n$ matrix. So suppose U is a neighborhood of 0 in $\mathbb{R}^n$ and f is a C^1 map defined on U ($f : U \to f(U)$ with $f(0) = 0$). Let $\|\cdot\|$ be a fixed norm on $\mathbb{R}^n$. Let

$$B(r) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$$

be the closed ball of radius $r > 0$ and centered 0. For r small, $B(r) \subset U$. Consider $X = B(r)$. Let $C^1(X)$ be the space of all C^1 maps; each is defined on some neighborhood of X. For any $g \in C^1$, we can define the C^0 norm for g as

$$\|g\|_0 = \sup_{x \in X} \|g(x)\|.$$

Let

$$Dg(x) : \mathbb{R}^n \to \mathbb{R}^n$$

be the derivative of g at x. As a linear operator, we can define its C^0 norm as

$$\|Dg(x)\|_0 = \sup_{\|v\|=1} \|Dg(x)v\|.$$

Then C^0 norm for Dg is defined as

$$\|Dg\| = \sup_{x \in X} \|Dg(x)\|_0.$$

The C^1 norm of g is

$$\|g\|_1 = \|g\|_0 + \|Dg\|_0.$$

The C^1 distance between two maps $g, h \in C^1(X)$ is

$$d_1(g, h) = \|g - h\|_1.$$

We say g is close to f in the C^1-topology if their C^1 distance $d_1(g, f)$ is small.

Theorem 3 (Preserving of Periodic Point by Small Perturbation)**.** *If* $Df^m(0)$ *does not have one as an eigenvalue then every map* g *sufficiently closed to* f *in the* C^1*-topology has a unique periodic point of period* m *close to* 0.

Proof. Consider $F = f^m - id$. Then its derivative $DF(0) = Df^m(0) - I$. Since 1 is not among the eigenvalues of $DF(0)$, the map $F = f^m - id$ is locally invertible by the Inverse Function Theorem. More precisely, by picking r small, $F : X \to F(X)$ is invertible and its inverse $F^{-1} : F(X) \to X$ is also a C^1 map.

Suppose g is a map close to f in the C^1-topology. We can write $g^m = f^m - \eta$ where η has a small C^1 norm

$$\|\eta\|_1 = ||\eta||_0 + ||D\eta||_0.$$

Now we need to solve the equation,

$$x = g^m(x) = f^m(x) - \eta(x) = F(x) + x - \eta(x).$$

Equivalently, solve the equation

$$F(x) - \eta(x) = 0$$

or the equation

$$x = F^{-1}(\eta(x)).$$

A key argument in the proof is that since the C^0 norm of the derivative DF^{-1} is bounded and the C^1 norm of η is small,

$$F^{-1} \circ \eta : X \to X$$

is a contracting map (note that X is a complete space with the distance $d(x, y) = \|x - y\|$). More precisely, let $L = ||DF^{-1}||_0$ be the C^0-norm of DF^{-1} and let

$$||\eta||_1 = \|\eta\|_0 + ||D\eta||_0 \leq \epsilon$$

be the C^1 norm of η. We get

$$||F^{-1}(\eta(x)) - F^{-1}(\eta(y))|| \leq \epsilon L||x - y||$$

for $x, y \in X$ and, since $F^{-1}(0) = 0$,

$$||F^{-1}(\eta(0))|| = ||F^{-1}(\eta(0)) - F^{-1}(0)|| \leq L||\eta(0)|| \leq \epsilon L,$$

and hence

$$||F^{-1}(\eta(x))|| \leq ||F^{-1}(\eta(x)) - F^{-1}(\eta(0))|| + ||F^{-1}(\eta(0))|| \leq \epsilon L||x|| + \epsilon L.$$

Thus if

$$\epsilon \leq \frac{r}{L(1+r)},$$

then

$$F^{-1} \circ \eta : X \to X$$

is a contracting map. From the Contracting Mapping Principle, $F^{-1} \circ \eta$ has a unique fixed point $x = x(g) \in B(r)$. Thus $g^m(x) = x$. Furthermore, $\|x(g)\| \to 0$ as $d_1(g, f) \to 0$.

By taking $r > 0$ small, we can assume that $f^i(B(r)) \cap B(r) = \emptyset$ for $1 \le i \le n-1$. When g is C^1 close to f, this property is still kept, that is, $g^i(B(r)) \cap B(r) = \emptyset$ for $1 \le i \le n-1$. So $g^i(x) \ne x$ for $1 \le i \le n-1$. This says that x is a periodic point of period m for g. □

Dynamics of Linear Maps and Local Dynamics:

Suppose X is an n-dimensional C^1-manifold and suppose $f : X \to X$ is a C^1 map with a periodic point p of period $m > 0$. Then the derivative $Df^m(p)$ is a linear map from $T_pX \to T_pX$ where T_pX is the tangent space of X at p. In a "good" situation, the local dynamics of f near p is determined by this linear map. Here a "good" situation, we will refer to when $Df^m(p)$ is "hyperbolic" as we will explain later. By considering local coordinates at p, we can think $Df^m(p)$ as an $n \times n$ matrix A. Thus the linear map $Df^m(p) : T_pX \to T_pX$ can be thought as

$$y = Ax : \mathbb{R}^n \to \mathbb{R}^n.$$

Let $sp(A)$ be the set of all eigenvalues of A and define

$$r(A) = \max\{|a| \mid a \in sp(A)\}$$

be the spectral radius of A. It is a quantity independent of choice of norms on $\mathbb{R}^n$.

Given any norm $||\cdot||$ on $\mathbb{R}^n$, we define the norm of A as

$$||A|| = \sup_{||v||=1} ||Av||.$$

Clearly, $||A|| \ge r(A)$.

Lemma 1. *For any $\delta > 0$ there exists a norm on $\mathbb{R}^n$ such that*

$$||A|| < r(A) + \delta.$$

Proof. First we can find a basis in $\mathbb{R}^n$ such that A has Jordan normal form. That is,

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix}$$

where each block is either a Jordan block

$$A = \begin{pmatrix} \lambda & 1 & & & 0 \\ 0 & \lambda & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & \lambda & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

for a real eigenvalue λ or

$$A = \begin{pmatrix} \rho R_\phi & Id & \cdots & 0 \\ & \rho R_\phi & Id & 0 \\ & & \ddots & \\ 0 & & \cdots & \rho R_\phi \end{pmatrix}$$

for a pair of complex eigenvalues $\lambda = \rho e^{i\phi}$ and $\overline{\lambda} = \rho e^{-i\phi}$, where

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$R_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

Now consider a diagonal change of coordinate

$$\begin{pmatrix} 1 & & & 0 \\ & \delta^{-1} & & \\ & & \ddots & \\ 0 & 0 & \cdots & \delta^{-m+1} \end{pmatrix}$$

for a Jordan block for a real eigenvalue and

$$\begin{pmatrix} Id & & & 0 \\ & \delta^{-1} Id & & \\ & & \ddots & \\ 0 & 0 & \cdots & \delta^{-m+1} Id \end{pmatrix}$$

for a pair of complex eigenvalues. Now for the standard Euclidean norm with respect to this new basis, we have that

$$||A|| = \sup_{||v||=1} ||Av|| \leq r(A) + \delta.$$

□

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ are said to be equivalent if there is a constant $C > 0$ such that

$$C^{-1}\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1 \quad \forall v \in \mathbb{R}^n.$$

Since all norms in $\mathbb{R}^n$ are equivalent, we have that

Corollary 2. *Given any norm $||\cdot||$ on $\mathbb{R}^n$, for any $\epsilon > 0$, there exists a constant $C = C(\epsilon) > 0$ such that*

$$||A^n v|| < C(r(A) + \epsilon)^n ||v||.$$

Definition 2. In general, we call a map $f : X \to X$ contracting if there are two constants $C > 0$ and $0 < \lambda < 1$ such that

$$d(f^n(x), f^n(y)) \leq C\lambda^n d(x, y)$$

for all $n > 0$ and $x, y \in X$.

Exerice 2. *Prove the Contracting Mapping Principle (Theorem 1) under this more general definition.*

Corollary 3. *If all eigenvalues of A have absolute values less than one, then $A : \mathbb{R}^n \to \mathbb{R}^n$ is contracting with respect to any norm on $\mathbb{R}^n$. And A has a unique fixed point 0 and $A^m v \to 0$ exponentially as $m \to \infty$ for every $v \in \mathbb{R}^n$.*

Exerice 3. *Suppose A is a 2×2 matrix whose all eigenvalues have absolute values less than one. Discuss the dynamics of $A : \mathbb{R}^2 \to \mathbb{R}$ in different cases.*

Definition 3. A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is called hyperbolic if all of its eigenvalues have absolute values different from one.

For every real eigenvalue λ, let

$$E_\lambda = \{v \in \mathbb{R}^n \mid (A - \lambda Id)^k v = 0 \quad \text{for some } k\}$$

be its root space. Similarly, for a pair of complex eigenvalues λ and $\overline{\lambda}$, we can consider $A : \mathbb{C}^n \to \mathbb{C}^n$ as a complex map. Then we can define its root space E_λ and $E_{\overline{\lambda}}$. Let

$$E_{\lambda,\overline{\lambda}} = \mathbb{R}^n \cap (E_\lambda \otimes E_{\overline{\lambda}}).$$

Let

$$E^s = E^s(A) = \bigoplus_{|\lambda|<1} E_\lambda \oplus \bigoplus_{|\lambda|<1} E_{\lambda,\overline{\lambda}}$$

and

$$E^u = E^u(A) = \bigoplus_{|\lambda|>1} E_\lambda \oplus \bigoplus_{|\lambda|>1} E_{\lambda,\overline{\lambda}}$$

and

$$E^0 = E^0(A) = \bigoplus_{\lambda=1,-1} E_\lambda \oplus \bigoplus_{|\lambda|=1} E_{\lambda,\overline{\lambda}}.$$

We call E^s, E^u, and E^0 the stable space, the unstable space, and the central space, for A, respectively. Then

$$\mathbb{R}^n = E^s \oplus E^0 \oplus E^u$$

and

$$A(E^s) \subseteq E^s, \quad A(E^0) \subseteq E^0, \quad \text{and} \quad A(E^u) \subseteq E^u.$$

Moreover, $A^m v \to 0$ for $v \in E^s$ and $A^{-m} v \to 0$ for $v \in E^u$ as $m \to +\infty$. We say that A is hyperbolic is equivalent to say that $E^0 = \{0\}$, that is,

$$\mathbb{R}^n = E^s \oplus E^u.$$

Definition 4. Suppose X is an n-dimensional C^1 manifold and $f : X \to X$ is a C^1 map. Suppose p is a periodic point of period $m > 0$ of f. We call p a hyperbolic periodic point if the derivative $Df^m(p) : T_pX \to T_pX$ is a hyperbolic linear map.

Theorem 4. *(Hartman-Grobman Theorem) Suppose $f : X \to X$ is a C^1 map from an n-dimensional C^1 manifold X to itself. Suppose p is a hyperbolic fixed point of f. Then there are neighborhoods U_1 and $U_2 = f^{-1}(U_1)$ of p in X and neighborhoods V_1 and $V_2 = (Df(p))^{-1}(V_1)$ of 0 in $T_pX = \mathbb{R}^n$ and a homeomorphism $h : U_1 \to V_1$ such that $h(U_1 \cap U_2) = V_1 \cap V_2$ and*

$$h \circ f = Df(p) \circ h$$

on $U_1 \cap U_2$. In other words, $f|(U_1 \cap U_2)$ and $Df(p)|(V_1 \cap V_2)$ are conjugate.

The above theorem says that for a hyperbolic fixed point p of f, the local dynamics of f near p is same as the local dynamics of the linear map $Df(p)$ topologically.

Exerice 4. *Prove the Hartman-Grobman Theorem when all eigenvalues of $Df(p) : T_pX \to T_pX$ have absolute values > 1 by using the Contracting Mapping Principle. Can you find another way to prove it under the same assumption?*

Problem 1. *Under what condition, can we promote h to be C^1. More general, under what condition, can we promote h to be C^2, $\cdots$, C^k, or C^ω if f is a C^2. $\cdots$, C^ω. This discussion should divide into cases: real one-dimension and real higher dimension. Similarly, for a complex map, we should discuss it into cases: complex one-dimension and complex higher dimension.*

In one real and complex dimensional cases, we have some kinds of complete solutions, which we will talk this later. However, in higher dimensional cases, it is a difficult problem.

In order to understand a non-hyperbolic linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, we need consider $A : E^0 \to E^0$. This is divided into three cases: (1) $A : E_1 \to E_1$; (2) $A : E_{-1} \to E_{-1}$; (3) $A : E_{\lambda,\overline{\lambda}} \to E_{\lambda,\overline{\lambda}}$. The case (1) is an identity map when A is restricted on the eigenspace and the case (2) is a reflection when A is restricted on the eigenspace. These two cases are trivial. The case (3) is rotation when it is restricted on the eigenspace, that is, the eigenspace $\mathbb{R}^2$ is foliated into circles and A restricted on each circle is a rotation. Thus we only need to discuss the dynamics of A restricted on the unit circle. Let

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

be the unit circle and, if $\lambda = e^{2\pi i\phi}$, then

$$A(z) = e^{2\pi i\phi} z : S^1 \to S^1.$$

If $\phi = p/q$ with $(p, q) = 1$, then every point in S^1 is a periodic point of period q. If we arrange the orbit $\{A^n z\}_{n=0}^{\infty}$ counter-clockwise on S^1 as $z_0 = z,\ z_1,\ \cdots,\ z_{q-1}$, then $Az_i = z_{i+p \pmod q}$ for all $0 \le i \le q-1$.

More interesting dynamics of a rotation is when ϕ is an irrational number.

Definition 5. Suppose X is a topological space and $f : X \to X$ is an invertible map. We call f topologically transitive if there exists a point $x \in X$ such that its orbit $O(x) = \{f^m(x)\}_{m=-\infty}^{\infty}$ is dense in X. It is called minimal if the orbit $O(x)$ is dense in X for every $x \in X$.

A subset $Y \subset X$ is called invariant if $f(Y) \subset Y$. Then the statement that f is minimal is equivalent to say that f has no invariant closed subset.

Theorem 5. *If ϕ is irrational, then $A : S^1 \to S^1$ is minimal.*

Proof. Let $\overline{O}(x) = \overline{\{f^m(x)\}}_{m=-\infty}^{\infty}$ be the closure of the orbit of any point $x \in S^1$. If it is not dense, the complement $S^1 \setminus \overline{O}(x)$ is an non-empty invariant open set, which is a union of open intervals. Let I be one of the longest intervals. Since the rotation preserves the length, we have that $|I| = |A^m I|$ for all m. But $A^k I \cap A^l = \emptyset$ for any $k \ne l$. This is because that if $A^k I \cap A^l I \ne \emptyset$, then $A^k I = A^l I$ (otherwise $A^k I \cup A^l I$ is a longer interval than I in $S^1 \setminus \overline{O}(x)$, contradiction). This implies that $A^{k-l} : I \to I$. With loss of generality, suppose $k > l$. Then there is a point $x \in S^1$ such that $A^{k-l}x = x$. Suppose $x = e^{2\pi i\theta}$. Then $e^{2\pi i[(k-l)\phi+\theta]} = e^{2\pi i\theta}$. Thus

$$[(k-l)\phi + \theta] = \theta + p$$

for some integer p. Therefore, $\phi = p/(k-l)$ is a rational number. This contradicts to our assumption that ϕ is irrational.

Since $\cup_{n=\infty}^{\infty} A^n I \subset S^1$, we have that

$$\infty = \sum_{n=-\infty}^{\infty} |A^n I| \le |S^1| < \infty.$$

This is a contradiction. The contradiction says that $S^1 \setminus \overline{O}(x)$ must be non-empty. We have proved the theorem. $\square$

Irrational circle rotations is a basic example for the study of dynamics of topological groups. A topological group G is a group with a topology such that each group multiplication by a fixed element in the group is a continuous map from the topological space into itself and the inverse is also a continuous map from the topological space into itself. For example, the unit circle S^1 is an Abelian topological group.

Let $L_{g_0} : G \to G$ be the group multiplication by g_0

$$L_{g_0} g = g_0 g : G \to G.$$

The orbit of the unit element $e \in G$ is cyclic group $\{g_0^n\}_{n=-\infty}^{\infty}$. From Theorem 5, S^1 has not proper infinite closed subgroup.

Theorem 6. *If L_{g_0} is topologically transitive, then it is minimal.*

Proof. Since L_{g_0} is topologically transitive, there is a g such that $O(g) = \{g_0^n g\}_{n=-\infty}^{\infty}$ is dense in G. Now for any $g' \in G$, $g_0^n g' = g_0^n g((g^{-1}g')$. Thus the orbit $O(g') = O(g)(g^{-1}g')$. Since $g^{-1}g' : G \to G$ is continuous, $O(g')$ is dense in G. $\square$

Another example of an Abelian topological groups is a torus

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n} = \{w = (z_1, \cdots, z_n) \mid z_i \in S^1\}$$

which plays a central role in the study of completely integrable Hamiltonian systems. The multiplication

$$w \cdot w' = (z_1 z_1', \cdots, z_n z_n)$$

if $w = (z_1, \cdots, z_n)$ and $w' = (z_1', \cdots, z_n')$. The group multiplication by w_0 is

$$L_{w_0} w = w_0 \cdot w.$$

Theorem 7. *Suppose $w_0 = (e^{2\pi i\phi_1}, \cdots, e^{2\pi i\phi_1})$. The group multiplication L_{w_0} is minimal if and only if the number $\phi_1, \cdots, \phi_n$, and 1 are rationally independent, that is, if $\sum_{i=1}^n k_i\phi_i = 0 \pmod 1$, then $k_1 = k_2 = \cdots = k_n = 0$.*

Exerice 5. *Prove this theorem. You can refer to Katok and Hassenlblatt's book "Introduction to the Modern Theory of Dynamical Systems", Cambridge University, pages 28-31.*

Maps with extremely complicate dynamics but we now understood quite well:

Suppose (X, d) is a metric space. By the definition, X is called compact if every cover $\{U_\alpha\}_{\alpha\in\Lambda}$ by open sets has a sub-cover $\{U_{\alpha_n}\}_{n=1}^{m}$ consisting of finite number of open sets in this cover. Equivalently, X is compact if every sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$. A closed subset of a compact space is compact. The space X is called totally disconnect if every connected component of X contains only one point. A point $x \in X$ is called a limiting point if for every neighborhood U of x, there is a $y \neq x$ in U. Let X' be the set of all limiting points. Then X is called perfect if $X = X'$. Topologically, a Cantor set is a compact, totally disconnected, and perfect metric space.

Suppose $I = [0, 1]$ is the unit interval and $I_0 = [0, a]$ and $I_1 = [b, 1]$ are two subintervals where $0 < a < b < 1$. A C^1 map f defined on $I_0 \cup I_1$ is said to be degree two if $f|I_i$ from I_i to I is bijective for $i = 0$, 1; f is said to be expanding if there are constants $C > 0$ and $\lambda > 1$ so that $|(f^{\circ n})'(x)| \geq C\lambda^n$ whenever $f^{\circ i}(x)$ are in $I_0 \cup I_1$ for all $i = 0$, 1, $\ldots$, $n-1$.

Suppose $f : I_0 \cup I_1 \to I$ is a degree two expanding map. Let $G = (a, b)$ be the complement of $I_0 \cup I_1$ in I. A number x in I is said to be escaping to G if $f^{\circ k}(x)$ is in G for some integer $k \geq 0$ (where $f^{\circ 0}$ is the identity). The set $\Omega \subseteq I$ of escaping points is an open subset of the real line. The complement Λ of Ω in I is called the non-escaping set under f. It is a compact (closed and bounded) subset of the real line $\mathbf{R}$. We also call Λ the maximal invariant set of f in $[0, 1]$.

A linear example

Example 3. *($\frac{1}{3}$-Cantor set) . Suppose $a = 1/3$ and $b = 2/3$. Define*

$$f(x) = \begin{cases} 3x, & if\ 0 \leq x \leq a; \\ 3x - 2, & if\ b \leq x \leq 1 \end{cases}$$

Then f is a degree two expanding map for which the non-escaping set Λ under f is the famous $\frac{1}{3}$-Cantor set.

Proof. Every point $x \in [0, 1]$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{i_n}{3^n}$$

where $i_n \in \{0, 1, 2\}$. Then

$$\Lambda = \{x = \sum_{n=1}^{\infty} \frac{i_n}{3^n} \in [0, 1] \mid i_n \in \{0, 2\}\}.$$

It is a closed subset. So it is compact.

For any

$$x = \sum_{n=1}^{\infty} \frac{i_n}{3^n}$$

and any $m > 0$, let

$$x_m = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i_m^*}{3^m} + \sum_{n=m+1}^{\infty} \frac{i_n}{3^n},$$

where $i_m^* = 2 - i_m$. Then $x_m \neq x$ and $x_m \to x$ as $m \to \infty$. This says $x \in \Lambda'$. So Λ is perfect.

Suppose $x \neq y \in \Lambda$. Then there is an integer $m > 0$ such that

$$x = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i_m}{3^m} + \sum_{n=m+1}^{\infty} \frac{i_n}{3^n}$$

and

$$y = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i'_m}{3^m} + \sum_{n=m+1}^{\infty} \frac{i'_n}{3^n}$$

where $i_m \neq i'_m$. Let us assume that $i_m = 0$ and $i'_m = 2$ and define

$$z = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{1}{3^m} + \sum_{n=m+1}^{\infty} \frac{1}{3^n}.$$

Then

$$U = (-\infty, z) \quad \text{and} \quad V = (z, +\infty)$$

are two open sets. It is clear that $U \cap V = \emptyset$ and $\Lambda \subset U \cup V$ and $x \in U$ and $y \in V$. Therefore, Λ is a Cantor set. □

Theorem 8. *Any two Cantor sets are topologically equivalent.*

Proof. Suppose (X, d) is a Cantor set. Let Λ be the 1/3-Cantor set. We only need to prove that X is homeomorphic to Λ.

Each non-empty open set U of X (respectively, Λ) can be divide into two non-empty open sets U_1 and U_2 such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Therefore, since X and Λ are compact, we can divide

$$X = X_1 \cup \cdots X_n \quad \text{and} \quad \Lambda = \Lambda_1 \cup \cdots \Lambda_n$$

into disjoint non-empty compact sets such that all of them have diameters < 1. Suppose we have divided

$$X = \cup_{i_0\cdots i_k} X_{i_0\cdots i_k} \quad \text{and} \quad \Lambda = \cup_{i_0\cdots i_k} \Lambda_{i_0\cdots i_k}$$

into disjoint non-empty compact sets such that all of them have diameters $< 1/k$. Then each pair $X_{i_0\cdots i_k}$ and $\Lambda_{i_0\cdots i_k}$ can be divided into same number disjoint non-empty compact sets such that all of them have diameters $< 1/(k+1)$, that is,

$$X_{i_0\cdots i_k} = \cup_{l=1}^{n_{k+1}} X_{i_0\cdots i_k l} \quad \text{and} \quad \Lambda_{i_0\cdots i_k} = \cup_{l=1}^{n_{k+1}} \Lambda_{i_0\cdots i_k l}$$

and $d(X_{i_0\cdots i_k l}), |\Lambda_{i_0\cdots i_k l}| < 1/(k+1)$. Thus we have a sequence of nested non-empty compact set

$$\cdots \subset X_{i_0\cdots i_k i_{k+1}} \subset X_{i_0\cdots i_k} \subset \cdots \subset X_{i_0} \subset X$$

and

$$\cdots \subset \Lambda_{i_0\cdots i_k i_{k+1}} \subset \Lambda_{i_0\cdots i_k} \subset \cdots \subset \Lambda_{i_0} \subset \Lambda.$$

Since $d(X_{i_0\cdots i_k}), |\Lambda_{i_0\cdots i_k}| < 1/k$,

$$\cap_{k=1}^{\infty} X_{i_0\cdots i_k} = \{x_{i_0\cdots i_k\cdots}\} \quad \text{and} \quad \cap_{k=1}^{\infty} \Lambda_{i_0\cdots i_k} = \{p_{i_0\cdots i_k\cdots}\}$$

both contain one point each. Set

$$h(p_{i_0\cdots i_k\cdots}) = x_{i_0\cdots i_k\cdots}.$$

Then it is a homeomorphism between X and Λ. □

The map f in Example 3 or in Theorem 5 is a linear map. In order to study the dynamics of a non-linear map, we need first to study some distortion property.

Naive distortion lemmas:

Let f be a function defined on a set U of the real line $\mathbb{R}$. It is said to be C^1 (or $C^{1+\alpha}$ for $0 < \alpha \leq 1$ or C^{1+bv}) if it can be extended to a differentiable function defined on an open set containing U and if the derivative of the extension is continuous (or is α-Hölder continuous or is of bounded variation).

Suppose f is a C^1 function on a set U of the real line $\mathbb{R}$ and $P_1 = \{x_i\}_{i=1}^n$ and $P_2 = \{y_i\}_{i=1}^n$ are two sequences of points in U. The number

$$\log \Big| \prod_{i=1}^{n} \frac{f'(x_i)}{f'(y_i)} \Big|$$

is called the distortion of f along P_1 and P_2.

Lemma 2. *Suppose* $\kappa = \inf_{x\in U} |f'(x)| > 0$*. Then the distortion of* f *along* P_1 *and* P_2 *can be estimated as*

$$\Big|\log\Big|\prod_{i=1}^{n}\frac{f'(x_i)}{f'(y_i)}\Big|\Big| \le \frac{1}{\kappa}\sum_{i=1}^{n}|f'(x_i)-f'(y_i)|$$

Proof. The proof of this lemma is easy because

$$\Big|\log\Big|\prod_{i=1}^{n}\frac{f'(x_i)}{f'(y_i)}\Big|\Big| \le \sum_{i=1}^{n}\Big|\log|f'(x_i)| - \log|f'(y_i)|\Big|$$

$$\le \frac{1}{\kappa}\sum_{i=1}^{n}|f'(x_i)-f'(y_i)|.$$

□

The next two lemmas are easily derived from Lemma 2

Lemma 3. *($C^{1+\alpha}$-Denjoy distortion lemma) . Suppose* f *is* $C^{1+\alpha}$ *for some* $0<\alpha\le 1$ *and* $\kappa = \inf_{x\in U}|f'(x)|>0$*. Let* $\iota = \sup_{x\neq y\in U}\big(|f'(x)-f'(y)|/|x-y|^{\alpha}\big) < \infty$*. Then the distortion of* f *along* P_1 *and* P_2 *is bounded by* $(\iota/\kappa)\sum_{i=1}^{n}|x_i-y_i|^{\alpha}$*, that is,*

$$\Big|\log\Big|\prod_{i=1}^{n}\frac{f'(x_i)}{f'(y_i)}\Big|\Big| \le \frac{\iota}{\kappa}\sum_{i=1}^{n}|x_i-y_i|^{\alpha}$$

Proof. Since $|f'(x_i)-f'(y_i)| \le \iota|x_i-y_i|^{\alpha}$, it follows directly from Lemma 2. □

Lemma 4. *(C^{1+bv}-Denjoy distortion lemma) . Suppose* f *is* C^{1+bv}*. Then there is a constant* $C>0$ *so that the distortion of* f *along* P_1 *and* P_2 *is bounded by* C*, that is,*

$$\Big|\log\Big|\prod_{i=1}^{n}\frac{f'(x_i)}{f'(y_i)}\Big|\Big| \le C,$$

whenever the open intervals I_i*, bounded by* x_i *and* y_i*, for* $i=1, \ldots, n$*, are pairwise disjoint.*

Proof. Let V be the total variation of f on U. Then $\sum_{i=1}^{n}|f'(x_i)-f'(y_i)|$ is bounded by V for $\{I_i\}_{i=1}^{n}$ are pairwise disjoint. We can take $C=V/\kappa$. □

Dynamics of non-linear expanding maps.

Theorem 9. *($C^{1+\alpha}$-hyperbolic Cantor set) If $f : I_0 \cup I_1 \to I$ is a $C^{1+\alpha}$ degree two expanding map for some $0 < \alpha \leq 1$. Then the non-escaping set Λ under f is a Cantor set whose Lebesgue measure is zero.*

Proof. Let f_i be the restriction of the function f to I_i, and $g_i = f_i^{-1} : I \to I_i$ be the inverse of f_i for $i = 0$ or 1. We can consider compositions $g_{w_n} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_n}$ for all strings $w_n = i_0 i_1 \ldots i_n$ of $0's$ and $1's$. These compositions are contracting; this means that there are constants $C > 0$ and $0 < \mu < 1$ so that $|g'_{w_n}(x)| < C\mu^n$ for all x in I.

Suppose $w_n = i_0 i_1 \ldots i_n$ is a string of $0's$ and $1's$. Let $I_{w_n} = g_{w_n}(I)$ be the image of I under g_{w_n}, and let $G_{w_n} = g_{w_n}(G)$ be the set of all the points escaping to G under $g_{w_n}^{-1}$. The union $\cup_{w_n} I_{w_n}$ is the set of all points not escaping to G under the iterates $f^{\circ k}$ for $k = 0, 1, \ldots, n$, where w_n runs over all the strings of $0's$ and $1's$ of length $n + 1$. The set $\{I_{w_n}\}$ is a collection of pairwise disjoint closed intervals and one to one correspondence with the set $\{w_n\}$ of all the strings of $0's$ and $1's$ of length $n + 1$. Hence $\Lambda = \cap_{n=0}^{\infty} \cup_{w_n} I_{w_n}$, where w_n runs over all the strings of $0's$ and $1's$ of length $n + 1$.

Let us first prove that Λ is uncountable. For a string $w_n = i_0 i_1 \ldots i_n$ of $0's$ and $1's$ and a digit $i_{n+1} = 0$ or 1, $I_{w_n i_{n+1}} \subseteq I_{w_n}$ since $I_{i_{n+1}} \subseteq I$. This implies that

$$\cdots \subseteq I_{i_0 i_1 \ldots i_n} \subseteq \cdots \subseteq I_{i_0 i_1} \subseteq I_{i_0}$$

and that $I_w = \cap_{n=0}^{\infty} I_{i_0 i_1 \ldots i_n}$ is a non-empty closed subset for any infinite string $w = i_0 i_1 \ldots$ of $0's$ and $1's$. Hence the set $\{I_w\}$ is a collection of pairwise disjoint non-empty closed subsets and is in one to one correspondence with the uncountable set $\{w = i_0 i_1 \ldots\}$ of all infinite strings of $0's$ and $1's$. Hence the set $\{I_w\}$ is uncountable. So too is the set Λ because $\Lambda = \cup_w I_w$ where $w = i_0 i_1 \ldots$ runs over all infinite strings of $0's$ and $1's$.

Since g_{w_n} is contracting, the length of I_{w_n} is less than $C\mu^n$ for any string $w_n = i_0 i_1 \ldots i_n$ of $0's$ and $1's$ of length $n + 1$. This implies that I_w contains a single number x_w, and the map $\pi(w) = x_w$ from $\{w\}$ to Λ is bijective. We use this to prove that Λ is totally disconnected, that is, every (connected) component Π of Λ contains only one number. Suppose there is a component Π of Λ which contains two different numbers x_w and $x_{w'}$ where $w = i_0 i_1 \ldots i_n i_{n+1} \ldots$ and $w' = i_0 i_1 \ldots i_n i'_{n+1} \ldots$ where $i_{n+1} \neq i'_{n+1}$. Both x_w and $x_{w'}$ are in I_{w_n} where $w_n = i_0 i_1 \ldots i_n$. The set I_{w_n} is the union of an open interval G_{w_n} and two closed intervals $I_{w_n i_{n+1}}$ and $I_{w_n i'_{n+1}}$ which are on different sides of G_{w_n}. The numbers x_w and $x_{w'}$ are in $I_{w_n i_{n+1}}$ and $I_{w_n i'_{n+1}}$, respectively. Take a point z in G_{w_n}. Then

$$\Pi = \Big(\Pi \cap (-\infty, z)\Big) \cup \Big(\Pi \cap (z, \infty)\Big).$$

This contradicts the statement that Π is a component of Λ and proves that Λ is totally disconnected.

Since Λ is closed, the set Λ' of limit points of Λ is contained in Λ. To prove that Λ is a perfect set, we only need to show that Λ is contained in Λ'. Let x_w be a number in Λ and $w = i_0 i_1 \ldots i_n i_{n+1} \ldots$. Let $r(i) = i + 1 \pmod{2}$; $r(i)$ is 1 for $i = 0$ and 0 for $i = 1$. Take $w^{(n)} = i_0 \ldots i_{n-1} r(i_n) i_{n+1} \ldots$; $w^{(n)}$ differs from w at $(n+1)^{th}$ position. Then $x_{w^{(n)}} \neq x_w$ and both of them are in $I_{i_0 \ldots i_{n-1}}$. Since the length of $I_{i_0 \ldots i_{n-1}}$ tends to zero as n goes to infinity, $x_{w^{(n)}}$ tends to x_w as n goes to infinity. This says that x_w is a limit point of Λ. So Λ is contained in Λ'. Hence Λ is a Cantor set.

Now let us prove that the Lebesgue measure of the Cantor set Λ is zero. Let $m(\cdot)$ mean the Lebesgue measure and let $|J|$ mean the length of an interval. An inequality which can be easily obtained is

$$m(\Lambda) \leq \sum_{w_n} |I_{w_n}| < C 2^{n+1} \mu^n,$$

where w_n runs over all the strings of $0's$ and $1's$ of length $n+1$. This inequality is true because $\{I_{w_n}\}$ is a cover of Λ and the total number of the strings of $0's$ and $1's$ of length $n+1$ is 2^{n+1}. If $\mu < 1/2$, it is much easier to see $m(\Lambda) = 0$. However, to prove that the Lebesgue measure of Λ is zero for any $0 < \mu < 1$, we need help from Lemma 3.

Suppose $w_n = i_0 \ldots i_n$ is a string of $0's$ and $1's$ of length $n+1$. The map f^{n+1} from I_{w_n} to I is a monotone function and its inverse is g_{w_n}. For any two numbers x and y in I_{w_n}, let $x_i = f^{\circ(n-i+1)}(x)$ and $y_i = f^{\circ(n-i+1)}(y)$ for $i = 0, 1, \ldots, n+1$. By the mean value theorem and the chain rule, $|x_i - y_i| < C\mu^i$ and $\sum_{i=0}^{n+1} |x_i - y_i|^\alpha < C/(1-\mu^\alpha)$. According to Lemma 1.2, the distortion of f along $X = \{x_i\}$ and $Y = \{y_i\}$ is bounded by the constant $C' = (\iota/\kappa)\big(C/(1-\mu^\alpha)\big)$, that is,

$$\Big|\log\Big|\frac{\Big(f^{\circ(n+1)}\Big)'(x)}{\Big(f^{\circ(n+1)}\Big)'(y)}\Big|\Big| \leq C',$$

where ι is the Hölder constant of f' on $I_0 \cup I_1$ and $\kappa = \inf_{x \in I_0 \cup I_1} |f'(x)|$. This implies that

$$\frac{|G_{w_n}|}{|I_{w_n}|} \geq c = e^{-C'} |G|,$$

since $G = f^{\circ(n+1)}(G_{w_n})$ and $I = f^{\circ(n+1)}(I_{w_n})$. Now we have that

$$|I_{w_n0}| + |I_{w_n1}| \leq (1-c)|I_{w_n}|$$

because $I_{w_n} = I_{w_n0} \cup G_{w_n} \cup I_{w_n1}$; moreover,

$$m(\Lambda) \leq \sum_{w_{n+1}} |I_{w_{n+1}}| = \sum_{w_n} (|I_{w_n0}| + |I_{w_n1}|)$$

$$\leq (1-c) \sum_{w_n} |I_{w_n}| \leq \cdots \leq (1-c)^{n+1}$$

for all positive integers n. Hence the Lebesgue measure of Λ is zero. □

Remark and Exercise 1. *From the proof, one can see that the non-escaping set Λ of a C^1 degree two expanding map is a Cantor set in the real line. There is a Cantor set with positive Lebesgue measure. An interesting problem is to construct a C^1 degree two expanding map whose non-escaping set is a Cantor set with positive Lebesgue measure. Try to study this counter-example. You can refer to Bowen's paper "A horseshoe with positive measure." Invent. Math.,* **29** *(1975), 203-204.*

Horseshoe Maps

Let R be a rectangle in $\mathbb{R}^2$ with two horizontal sides and two vertical sides. Suppose $f : R \to f(R) \subset \mathbb{R}^2$ is a diffeomorphism such that $R \cap f(R)$ are two separate rectangles R_0 and R_1 such that their horizontal sides parallel to the horizontal sides of R and and their vertical sides are parts of the vertical sides of R. Let R^0 and R^1 be two components of $f^{-1}(R) \cap R$. Suppose R^0 and R^1 are also two rectangles whose vertical sides are parallel to those vertical sides of R and whose horizontal sides are parts of the horizontal sides of R. Suppose $f : R^0 \to R_0$ and $f : R^1 \to R_1$ are hyperbolic affine maps, contracting in the vertical direction and expanding in the horizontal direction. This map is called a Smale horseshoe map. Let Λ be the maximal invariant set of f in R. Then

$$\Lambda = \cap_{n=-\infty}^{\infty} f^n(R).$$

The intersection

$$R \cap f(R) \cap f^2(R) = (R_0 \cap R_1) \cap f^2(R)$$

consists of four rectangles R_{ij} for $i, j \in \{0, 1\}$ whose horizontal sides parallel to the horizontal sides of R and and their vertical sides are parts of the vertical sides of R. Inductively, $\cap_{k=0}^{n} f^k(R)$ consists of 2^n thin rectangles whose horizontal sides parallel to the horizontal sides of R and and their vertical sides are parts of the vertical sides of R. Let us denote these rectangles as $R_{i_0 i_1 \cdots i_{n-1}}$ for $i_k \in 0, 1$, $k = 0, 1, \cdots n-1$. Here we label them as $f(R_{i_0 \cdots i_{n-1}}) = R_{i_1 \cdots i_{n-1}}$. Then we have that

$$R_{i_0 \cdots i_{n-1}} \subset R_{i_0 \cdots i_{n-2}} \subset \cdots \subset R_{i_0}.$$

This implies that

$$R_{i_0 \cdots i_{n-1} \cdots} = \cap_{k=0}^{\infty} R_{i_0 \cdots i_{n-1}}$$

is a horizontal line in R. Then

$$R_+ = \cap_{k=0}^{\infty} f^k(R)$$

consists of uncountably many horizontal lines.

Similarly, the intersection

$$R \cap f^{-1}(R) \cap f^{-2}(R) = (R^0 \cap R^1) \cap f^{-2}(R)$$

consists of four rectangles R^{ij} for $i, j \in \{0, 1\}$ whose vertical sides parallel to the vertical sides of R and and their horizontal sides are parts of the horizontal sides of R. Inductively, $\cap_{k=1}^{n} f^{-k}(R)$ consists of 2^n thin rectangles whose horizontal sides parallel to the horizontal sides of R and and their vertical sides are parts of the vertical sides of R. Let us denote these rectangles as $R^{j_{n-1} j_{n-2} \cdots j_1}$ for $j_k \in 0, 1$, $k = 1, \cdots n-1$. Here we label them as $f^{-1}(R_{j_{n-1} \cdots j_2 j_1}) = R_{j_{n-1} \cdots j_2}$. Then we have that

$$R_{j_{n-1} j_{n-2} \cdots j_1} \subset R_{j_{n-2}} \subset \cdots \subset R_{j_1}.$$

This implies that

$$R_{\cdots j_{n-1} \cdots j_1} = \cap_{k=1}^{\infty} R_{j_{n-1} \cdots j_1}$$

is a vertical line in R.

$$R_- = \cap_{k=0}^{\infty} f^k(R)$$

consists of uncountably many horizontal lines.

Give any $w_+ = i_0 \cdots i_{n-1} \cdots$, $R_{w_+} \cap R_-$ is a Cantor set on the line R_{w_+}. Give any $w_- = \cdots j_{n-1} \cdots j_1$, $R_{w_-} \cap R_+$ is a Cantor set on the line R_{w_-}. Then

$$\Lambda = R_+ \cap R_-$$

can be thought as a product of two Cantor set on a horizontal line and on a vertical line.

Theorem 10. *Consider the dynamical system $f : \Lambda \to \Lambda$. Then the set of periodic points of f are dense in Λ. And $Per_n(f)$ contains 2^n different points. Furthermore, $f : \Lambda \to \Lambda$ is topologically mixing, that is, for any two open sets U and $V \subset \Lambda$, there exists an integer $N = N(U, V) > 0$ such that $f^n(U) \cap V$ is nonempty for every $n > N$.*

One-side full shift on symbolic space.

In one-dimensional expanding maps, we used symbols to label intervals and points. Here we study a symbolic dynamical system. Through the study of symbolic dynamical systems, we can have a better understanding of dynamical systems.

Let $S = \{0, 1, \cdots, d-1\}$ be a set with the discrete topology. Consider the space

$$\Sigma = \Sigma_d = \prod_{k=0}^{\infty} S = \{w = i_0 i_1 \cdots i_{k-1} \cdots \mid i_k \in S,\ k = 0, 1, \cdots\}.$$

We give Σ the product topology as follows. Give any

$$w = i_0 i_1 \cdots i_{k-1} \cdots .$$

Define

$$[w]_n = \{w' = i_0 i_1 \cdots i_{n-1} j_n \cdots j_{k-1} \cdots \in \Sigma\}.$$

It is called an n-cylinder of Σ. We call a cylinder an open set. The set of all cylinders forms a topological basis. Then Σ with this topology is a topological space.

We can also introduce a metric on Σ:

$$d(w, w') = \sum_{k=1}^{\infty} \frac{|i_{k-1} - j_{k-1}|}{d^k}$$

if $w = i_0 \cdots i_k \cdots$ and $w' = j_0 \cdots j_{k-1} \cdots$. One can check that $d(\cdot, \cdot)$ is a metric on Σ and the topology induced from this metric on Σ is the same as the product topology.

One side full shift is defined as

$$\sigma : \Sigma \to \Sigma; \quad \sigma(i_0 i_1 \cdots i_{k-1} \cdots) = i_1 \cdots i_{k-1} \cdots .$$

Then it is a continuous map. The dynamical system σ is called a symbolic dynamical system.

Suppose $I = [0, 1]$ and $I_0 = [0, b_0]$, $I_1 = [a_1, b_1]$, $\cdots$, $I_{d-1} = [a_{d-1}, b_{d-1}]$ where

$$a_0 = 0 < b_0 < a_1 < b_1 < \cdots < a_{d-1} < b_{d-1}.$$

Let

$$f : U = \cup_{i=0}^{d-1} \to I$$

be a map such that $f : I_i \to I$ is a C^1-diffeomorphism for each $i = 0, \cdots, d-1$. We say f is expanding if there are two constants $C > 0$ and $\lambda > 1$ such that

$$|(f^n)'(x)| \geq C\lambda^n$$

for all $x \in I$ and $n > 0$ such that $f^{n-1}(x)$ is defined. Let

$$\Lambda = \cap_{k=0}^{\infty} f^{-k}(I)$$

be the maximal invariant set of f.

Theorem 11. *Suppose $f : U \to I$ is a C^1 expanding map. Then $f : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \Sigma \to \Sigma$. That is, there is a homeomorphism $h : \Sigma \to \Lambda$ such that*

$$f \circ h = h \circ \sigma$$

on Σ.

Proof. For each $0 \leq i \leq d - 1$, let $g_i = (f|I_i)^{-1} : I \to I_i$. For each $w_m = i_0 i_1 \cdots i_{m-1}$, define

$$g_{w_m} = g_{i_0} \circ \cdots \circ g_{i_{m-1}}$$

and

$$I_{w_m} = g_{w_m}(I).$$

Then for every $w = w_m \cdots \in \Sigma$,

$$\cdots \subseteq I_{w_m} \subset I_{w_{m-1}} \subset \cdots \subset I_{w_1}.$$

Since f is expanding, $|I_{w_m}| \to 0$ as $m \to \infty$. So

$$I_w = \cap_{m=1}^{\infty} I_{w_m} = \{x_w\}$$

contains only one point. Define $h : \Sigma \to \Lambda$ as $h(w) = x_w$, we have that h is a homeomorphism and

$$f \circ h = h \circ \sigma$$

on Σ. □

Lemma 5. *All periodic points of period $m > 0$ are*

$$Per_m(f) = \{x_w \mid w = (i_0 \cdots i_{m-1})^{\infty}\}.$$

Exerice 6. *Prove that σ (as well as f) is topologically transitive but not minimal.*

Exerice 7. *Find all proper closed subsets A contains only finitely many points in Σ which is forward invariant, that is, $\sigma(A) \subseteq A$. Find a proper closed subset A contains infinitely many points in Σ which is forward invariant.*

Gibbs Measures

If a physical system of n states with the energies of these states are $E_1, \cdots, E_n$. Suppose that this system is put in contact with a much larger "heat source" which is at temperature T. Energy is therefore allowed to pass between the original system and the heat source. Suppose the temperature T of the heat source remains constant. It is a physical fact derived in statistical mechanics that the probability p_j that state j occurs is given by the Gibbs distribution

$$p_j = \frac{e^{-\beta E_j}}{\sum_{i=1}^{n} e^{-\beta E_i}}$$

where $\beta = \frac{1}{kT}$ and k is a physical constant. This is the starting point for the thermodynamical formalism. However, thermodynamical formalism is a mathematical subject which studies Gibbs measures for more general systems.

Consider the symbolic dynamical system $\sigma : \Sigma \to \Sigma$ (or $f : \Lambda \to \Lambda$). A probability measure μ on Σ is called an invariant measure if

$$\mu(A) = \mu(\sigma^{-1}(A))$$

for any Borel set A. A function $\phi : \Sigma \to \mathbb{R}$ is called positive if $\phi > 0$ and Hölder continuous if there are two constant $C > 0$ and $0 < \tau < 1$ such that

$$|\phi(w) - \phi(w')| \leq C\tau^m$$

for any $w, w' \in \Sigma$ such that their first m digits are the same. For a positive and Hölder continuous function ϕ, the function $\psi = \log \phi$ is also Hölder continuous.

Theorem 12. *Consider the system* $(\sigma, \Sigma, \log \phi)$ *where* ϕ *is a positive Hölder continuous function. There is a unique probability measure* $\mu = \mu_{\log \phi}$ *such that*

$$C^{-1} \leq \frac{\mu([w_m])}{\exp[-mP + \sum_{k=0}^{m-1} \log \phi(\sigma^k(w))]} \leq C.$$

where $C > 0$ *is a constant and where* $[w_m]$ *is any cylinder of* Σ *and* $w \in [w_m]$*. Here* $P = P(\log \phi)$ *is a constant depending on* $\log \phi$ *and is called the pressure for the system.*

Remark 1. *If we consider* $f : \Lambda \to \Lambda$ *as a* $C^{1+\alpha}$ *expanding map, let* $h : \Sigma \to \Lambda$ *be the conjugacy from* $\Sigma \to \Lambda$*. Consider* $\phi(w) = 1/|f'(h(w))|$*. Then it is a positive Hölder continuous function on* Σ*. Consider*

$$t \log \phi(w) = -t \log |f'(h(w))|.$$

Then the pressure $P(-t \log |f'(h(w))|$ *is a function of* t*. There is a unique number* $0 < t_0 < 1$ *such that* $P(-t \log |f'(h(w)|) = 0$*. This number is just the Hausdorff dimension of* Λ*. For more details about Gibss theory, you can refer to my lecture notes (download site: http://qcpages.qc.cuny.edu/˜yjiang/HomePageYJ/Download/*

JiangNJLectureIFM.pdf) "Nanjing Lecture Notes In Dynamical Systems. Part One: Transfer Operators in Thermodynamical Formalism."

Dynamics of Circle endomorphisms and Teichmüller theory

Circle endomorphisms

Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$. Suppose

$$f : T \to T$$

is an orientation-preserving covering map of degree $d \geq 2$. We call it a circle endomorphism. Suppose

$$h : T \to T$$

is an orientation-preserving homeomorphism. We call it in this paper a circle homeomorphism.

For a circle endomorphism f, it has a fixed point. We will assume that $f(1) = 1$.

The universal cover of T is the real line $\mathbb{R}$ with a covering map

$$\pi(x) = e^{2\pi i x} : \mathbb{R} \to T.$$

Then every circle endomorphism f can be lifted to an orientation-preserving homeomorphism

$$F : \mathbb{R} \to \mathbb{R}, \quad F(x+1) = F(x) + d, \quad \forall x \in \mathbb{R}.$$

We will assume that $F(0) = 0$. Then there is a one-to-one correspondence between f and F. Therefore, we also call such an F a circle endomorphism.

Every orientation-preserving circle homeomorphism h can be lifted to an orientation-preserving homeomorphism

$$H : \mathbb{R} \to \mathbb{R}, \quad H(x+1) = H(x) + 1, \quad \forall x \in \mathbb{R}.$$

We will assume throughout this paper that $0 \leq H(0) < 1$. Then there is a one-to-one correspondence between h and H. Therefore, we also call such an H a circle homeomorphism.

A circle endomorphism f is C^k for $k \geq 1$ if the k^{th}-derivative $F^{(k)}$ exists and is continuous. And, furthermore, it is called $C^{k+\alpha}$ for some $0 < \alpha \leq 1$ if $F^{(k)}$ is α-Hölder continuous, that is,

$$\sup_{x \neq y \in \mathbb{R}} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x-y|^\alpha} = \sup_{x \neq y \in [0,1]} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x-y|^\alpha} < \infty.$$

A C^1 circle endomorphism f is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C\lambda^n, \quad n = 1, 2, \cdots .$$

Definition 6. A circle homeomorphism h is called quasisymmetric if there is a constant $M \geq 1$ such that

$$M^{-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq M, \quad \forall x \in \mathbb{R},\ \forall t > 0.$$

Furthermore, it is called symmetric if there is a bounded function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that

$$\frac{1}{1+\varepsilon(t)} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R},\ \forall t > 0.$$

Example 4. *A C^1-diffeomorphism of T is symmetric.*

However, the class of symmetric homeomorphisms is larger than the class of C^1-diffeomorphisms. For example, a symmetric homeomorphism may not necessarily be absolutely continuous.

Definition 7. A circle endomorphism f is called uniformly quasisymmetric if there is a constant $M > 0$ such that

$$M^{-1} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq M$$

for all $x \in \mathbb{R}$ and $t > 0$ and any $n > 0$.

Definition 8. A circle endomorphism f is called uniformly symmetric if there is a bounded function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that

$$\frac{1}{1+\varepsilon(t)} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R},\ \forall t > 0.$$

Example 5. *A $C^{1+\alpha}$, for some $0 < \alpha \leq 1$, circle expanding endomorphism f is uniformly symmetric. Furthermore, $\varepsilon(t) \leq Dt^\alpha$ for some constant $D > 0$ and $0 \leq t \leq 1$.*

Proof. Since $F(x+1) = F(x) + d$, then $F'(x+1) = F'(x)$ is a periodic function. Since F is $C^{1+\alpha}$, we have a constant $C_1 > 0$ such that

$$|F'(x) - F'(y)| \leq C_1 |x - y|^\alpha, \quad \forall x, y \in \mathbb{R}.$$

Since F is expanding, we have a constant $C_2 > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C_2 \lambda^n, \quad \forall x \in \mathbb{R}, \ n > 0.$$

For any $x, y \in \mathbb{R}$ and $n > 0$, let $x_k = F^{-k}(x)$ and $y_k = F^{-k}(y)$, $0 \leq k \leq n$. Then

$$\Big| \log \frac{(F^{-n})'(x)}{(F^{-n})'(y)} \Big| = \Big| \log \frac{(F^{n})'(y_n)}{(F^{n})'(x_n)} \Big| \leq \sum_{k=1}^{n} | \log F'(x_k) - \log F'(y_k)|$$

$$\leq \frac{1}{C_2 \lambda} \sum_{k=1}^{n} |F'(x_k) - F'(y_k)| \leq \frac{C_1}{C_2 \lambda} \sum_{k=1}^{n} |x_k - y_y|^{\alpha} \leq \frac{C_1}{C_2^{1+\alpha} \lambda} \sum_{k=1}^{n} \lambda^{-\alpha k} |x-y|^{\alpha}.$$

Let

$$C = \frac{C_1 \lambda^{\alpha}}{C_2^{1+\alpha} (\lambda^{\alpha} - 1) \lambda}.$$

Then we have the following Hölder distortion property:

$$(1) \qquad e^{-C|x-y|^{\alpha}} \leq \frac{(F^{-n})'(x)}{(F^{-n})'(y)} \leq e^{C|x-y|^{\alpha}}, \quad \forall x, y \in \mathbb{R}, \ \ \forall n > 0.$$

Furthermore, let

$$\varepsilon(t) = \begin{cases} e^{Ct^{\alpha}} - 1, & 0 < t \leq 1, \\ e^{C} - 1, & t > 1. \end{cases}$$

Then $\varepsilon(t) > 0$ is a bounded function such that $\varepsilon(t) \to 0$ as $t \to 0^+$ and such that

$$\frac{1}{1 + \varepsilon(t)} \leq \frac{(F^{-n})'(\xi)}{(F^{-n})'(\eta)} = \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1 + \varepsilon(t), \ \ \forall x \in \mathbb{R}, \ \forall t > 0,$$

where ξ and η are two numbers in $[0, 1]$. Thus F is uniformly symmetric. Furthermore, one can see that $\varepsilon(t) \leq Dt^{\alpha}$ for some constant $D > 0$ and $0 \leq t \leq 1$. We have proved the example. □

Remark 2. *The uniformly symmetric condition is a weaker condition than the $C^{1+\alpha}$ expanding condition for any $0 < \alpha \leq 1$. For example, a uniformly symmetric circle endomorphism could be totally singular, that is, it could map a set with positive Lebesgue measure to a set with zero Lebesgue measure.*

Another example of a uniformly symmetric circle endomorphism is a C^1 Dini expanding circle endomorphism as follows. Suppose f is a C^1 circle endomorphism. The function

$$\omega(t) = \sup_{|x-y| \leq t} |F'(x) - F'(y)|, \quad t > 0,$$

is called the modulus of continuity of F'. Then f is called C^1 Dini if

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

Suppose f is a C^1 Dini expanding circle endomorphism. Let $C > 0$ and $\lambda > 1$ be two constants such that

$$(F^n)'(x) \geq C\lambda^n, \quad x \in \mathbb{R}, \ n \geq 1.$$

Define

$$\tilde{\omega}(t) = \sum_{n=1}^{\infty} \omega(C^{-1}\lambda^{-n}t).$$

Then

$$\tilde{\omega}(t) \leq \int_0^{\infty} \omega(C^{-1}\lambda^{-x}t)dx = \frac{1}{\log\lambda}\int_0^{C^{-1}\lambda^{-1}t} \frac{\omega(y)}{y}dy < \infty$$

for all $0 \leq t \leq 1$ and $\tilde{\omega}(t) \to 0$ as $t \to 0$.

Example and Exercise 1. *A C^1 Dini circle expanding endomorphism f is uniformly symmetric. Furthermore, $\varepsilon(t) \leq D\tilde{\omega}(t)$ for some constant $D > 0$ and $0 \leq t \leq 1$. You can refer to my paper (download site:http://qcpages.qc.cuny.edu/ yjiang/HomePageYJ/Download/ 2008TeichAndGibbsTh.pdf) "Teichmller structures and dual geometric Gibbs type measure theory for continuous potentials".*

Symbolic space and topological representation

Suppose f is a circle endomorphism with $f(1) = 1$. Consider the preimage $f^{-1}(1)$. Then $f^{-1}(1)$ cuts T into d closed intervals J_0, J_1, $\cdots$, J_{d-1}, ordered by the counter-clockwise order of T. Suppose J_0 has an endpoint 1. Then J_{d-1} also has an endpoint 1. Let

$$\varpi_0 = \{J_0, J_1, \cdots, J_{d-1}\}.$$

Then it is a Markov partition, that is,

i. $T = \cup_{k=0}^{d-1} J_k$,
ii. the restriction of f to the interior of J_i is injective for every $0 \leq i \leq d-1$,
iii. $f(J_i) = T$ for every $0 \leq i \leq d-1$.

Let I_0, I_1, $\cdots$, I_{d-1} be the lifts of J_0, J_1, $\cdots$, J_{d-1} in $[0,1]$. Then we have that

i) $[0,1] = \cup_{k=0}^{d-1} I_k$,
ii) $F(I_i) = [i, i+1]$ for every $0 \leq i \leq d-1$.

Let

$$\eta_0 = \{I_0, I_1, \cdots, I_{d-1}\}.$$

Then it is a partition of $[0,1]$.

Consider the pull-back partition $\varpi_n = f^{-n}\varpi_0$ of ϖ_0 by f^n. It contains d^n intervals and is also a Markov partition of T. Intervals J in ϖ_n can be labeled as follows. Let $w_n = i_0 i_1 \cdots i_{n-1}$ be a word of length n of $0's$, $1's$, $\cdots$, and $(d-1)'s$. Then $J_{w_n} \in \varpi_n$ if $f^k(J_{w_n}) \subset J_{i_k}$ for $0 \le k \le n-1$. Then

$$\varpi_n = \{J_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0,1,\cdots,d-1\}, k = 0,1,\cdots,d-1\}.$$

Let η_n be the corresponding lift partition of ϖ_n in $[0,1]$ with the same labelings. Then

$$\eta_n = \{I_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0,1,\cdots,d-1\}, k = 0,1,\cdots,d-1\}.$$

Consider the space

$$\Sigma = \prod_{n=0}^{\infty}\{0,1,\cdots,d-1\}$$

$$= \{w = i_0 i_1 \cdots i_k \cdots i_{n-1} \cdots \mid i_k \in \{0,1,\cdots,d-1\},\ k = 0,1,\cdots\}$$

with the product topology. It is a compact topological space. A cylinder for a fixed word $w_n = i_0 i_1 \cdots i_{n-1}$ of length n is

$$[w_n] = \{w' = i_0 i_1 \cdots i_{n-1} i'_n i'_{n+1} \cdots \mid i'_{n+k} \in \{0,1,\cdots,d-1\}, k = 0,1,\cdots\}$$

All left cylinders form a topological basis of Σ. We call it the *left topology*. The space Σ with this left topology is called the *symbolic space*.

For any $w = i_0 i_1 \cdots i_{n-1} i_n \cdots$, let

$$\sigma(w) = i_1 \cdots i_{n-1} i_n \cdots$$

be the shift map. Then (Σ, σ) is called a symbolic dynamical system.

For a point $w = i_0 \cdots i_{n-1} i_n \cdots \in \Sigma$, let $w_n = i_0 \cdots i_{n-1}$. Then

$$\cdots \subset J_{w_n} \subset J_{w_{n-1}} \subset \cdots J_{w_1} \subset T.$$

Since each J_{w_n} is compact,

$$J_w = \cap_{n=1}^{\infty} J_{w_n} \neq \emptyset.$$

If every $J_w = \{x_w\}$ contains only one point, then we define the projection π_f from Σ onto T as

$$\pi_f(w) = x_w.$$

The projection π_f is 1-1 except for a countable set

$$B = \{w = i_0 i_1 \cdots i_{n-1} 1000 \cdots, i_0 i_1 \cdots i_{n-1} 0(d-1)(d-1)(d-1) \cdots\}.$$

From our construction, one can check that

$$\pi_f \circ \sigma(w) = f \circ \pi_f(w), \quad w \in \Sigma.$$

For any interval $I = [a, b]$ in $[0, 1]$, we use $|I| = b - a$ to mean its Lebesgue length. Let

$$\iota_{n,f} = \max_{w_n} |I_{w_n}|,$$

where w_n runs over all words of $\{0, 1, \cdots, d-1\}$ of length n.

Two circle endomorphisms f and g are topologically conjugate if there is an orientation-preserving circle homeomorphism h of T such that

$$f \circ h = h \circ g.$$

The following result is first proved by Shub for C^2 expanding circle endomorphisms 1960's by using the contracting mapping theorem.

Theorem 13. *Let f and g be two circle endomorphisms such that both $\iota_{n,f}$ and $\iota_{n,g}$ tend to zero as $n \to \infty$. Then f and g are topologically conjugate if and only if their topological degrees are the same.*

Proof. Topological conjugacy preserves the topological degree. Thus if f and g are topologically conjugate, then their topological degrees are the same.

Now suppose f and g have the same topological degree. Then they have the same symbolic space. Since both sets $J_{w,f} = \{x_w\}$ and $J_{w,g} = \{y_w\}$ contain only a single point for each w, we can define

$$h(x_w) = y_w.$$

One can check that h is an orientation-preserving homeomorphism with the inverse

$$h^{-1}(y_w) = x_w.$$

□

Therefore, for a fixed degree $d > 1$, there is only one topological model (Σ, σ) for dynamics of all circle endomorphisms of degree d with $\iota_n \to 0$ as $n \to \infty$.

Bounded geometry and uniformly quasisymmetry and quasisymmetric conjugacy

Definition 9. The sequence $\{\varpi_n\}_{n=0}^{\infty}$ of nested partitions of T is said to have bounded nearby geometry if there is a constant $C > 0$ such that for any $n \geq 0$ and any two intervals $I, I' \in \eta_n$ with a same endpoint or one has an endpoint 0 and the other has an endpoint 1 (in which case we say they have a common endpoint by modulo 1),

$$C^{-1} \leq \frac{|I'|}{|I|} \leq C.$$

The sequence $\{\varpi_n\}_{n=0}^{\infty}$ of nested partitions of T is said to have bounded geometry if there is a constant $C > 0$ such that

$$\frac{|L|}{|I|} \geq C, \quad \forall\, L \subset I, \;\; L \in \eta_{n+1}, \; I \in \eta_n, \quad \forall\, n \geq 0.$$

The bounded nearby geometry implies the bounded geometry since each interval $I \in \eta_n$ is divided into d subintervals in η_{n+1}. But it is not true for the other direction.

Theorem 14. *If f is a uniformly quasisymmetric circle endomorphisms, then the sequence $\{\varpi_n\}_{n=0}^{\infty}$ of nested partitions of T has bounded nearby geometry and thus bounded geometry.*

Proof. Let F with $F(0) = 0$ be the lift of f. Define

$$G_k(x) = F^{-1}(x+k) : [0,1] \to [0,1], \quad \text{for} \quad k = 0, 1, \cdots, n-1.$$

For any word $w_n = i_0 i_1 \cdots i_{n-1}$, define

$$G_{w_n} = G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_{n-1}}.$$

Then

$$I_{w_n} = G_{w_n}([0,1]) = F^{-n}([m, m+1]),$$

where $m = i_{n-1} + i_{n-2} d + \cdots + i_0 d^{n-1}$. Suppose I'_{w_n} is an interval in η_n having a common endpoint with I_{w_n} modulo 1. Then

$$I'_{w_n} = F^{-n}([m+1, m+2]) \quad \text{or} \quad F^{-n}([m-1, m]).$$

Thus

$$\frac{1}{1+\varepsilon(1)} \leq \frac{|I_{w_n}|}{|I'_{w_n}|} \leq 1 + \varepsilon(1).$$

Let $C = 1 + \varepsilon(1)$. Then we have that

$$C^{-1} \leq \frac{|I|}{|I'|} \leq C$$

for any intervals $I, I' \in \eta_n$ with a common endpoint modulo 1, $n = 0, 1, \cdots$. This means that $\{\varpi_n\}_{n=0}^{\infty}$ has the bounded nearby geometry. We proved the theorem. □

Remark 3. *The converse is also true in the above theorem. Refer to the proof of the following corollary.*

Example 6. *Consider* $q_d(z) = z^d$, $d \geq 2$. *Then it is a circle endomorphism of degree* d. *The lift of* q *is* $Q(x) = dx$. *Then we have*

$$\eta_0 = \Big\{[\frac{k}{d}, \frac{k+1}{d}]\Big\}_{k=0}^{d-1}$$

and

$$\eta_{n-1} = \Big\{[\frac{k}{d^n}, \frac{k+1}{d^n}]\Big\}_{i=0}^{d^n-1}.$$

Every $0 \leq k < d^{n+1}$, *can be expressed as*

$$k = i_{n-1} + i_{n-2}d + \cdots + i_0 d^{n-1}.$$

So for $w_n = i_0 i_1 \cdots i_{n-1}$, $I_{w_n} = [i/d^n, (i+1)/d^n]$.

Theorem 15. *Any two uniformly quasisymmetric circle endomorphisms* f *and* g *of the same degree* $d > 1$ *are topologically conjugate and the conjugacy is a quasisymmetric homeomorphism.*

Proof. From $f \circ h = h \circ g$ and $g(1) = 1$, $h(1)$ is a fixed point of f, that is, $f(h(1)) = h(1)$. Let $k(z) = z/h(1)$ and $\tilde{f} = k \circ f \circ k^{-1}$. Then $\tilde{f}(1) = 1$. Take $\tilde{h} = k \circ h$. We have that $\tilde{h}(1) = 1$ and $\tilde{f} \circ \tilde{h} = \tilde{h} \circ g$. So $\tilde{h}$ is quasisymmetric if and only if h is quasisymmetric. So, without loss of generality, we assume that $h(1) = 1$.

Suppose

$$\eta_{n,f} = \{I_{w_n,f}\} \quad \text{and} \quad \eta_{n,g} = \{I_{w_n,g}\}, \quad n = 1, 2, \cdots$$

are two sequences of Markov partitions for f and g, respectively.

From the bounded geometry property (Theorem 14), we have a constant $0 < \tau < 1$ such that

$$\iota_{n,f} = \max_{w_n} |I_{w_n,f}|, \quad \iota_{n,g} = \max_{w_n} |I_{w_n,g}| \leq \tau^n, \quad \forall\, n = 1, 2, \cdots.$$

Then Theorem 13 implies that f and g are topologically conjugate.

Suppose h is the topological conjugacy between f and g and H is its lift to $\mathbb{R}$. By adding all integers, the sequence of partitions $\eta_{n,f}$ and $\eta_{n,g}$ induce two sequences of partitions of $\mathbb{R}$, which we still denoted as $\eta_{n,f}$ and $\eta_{n,g}$. Both of these sequences of partitions have bounded nearby geometry.

Let Ω be the set of all endpoints of intervals $I \in \eta_n$, $n = 0, 1 \cdots, \infty$. Then it is dense in $\mathbb{R}$.

For $x \in \Omega$. Consider the interval $[x - t, x]$. There is a largest integer $n \geq 0$ such that there is an interval $I = [a, x] \in \eta_{n,f}$ satisfying

$[x-t,x] \subseteq I$. Suppose $J=[b,x] \in \eta_{n+1,f}$. Then $J \subseteq [x-t,x]$. Let $J'=[x,c] \in \eta_{n+1,f}$. From Theorem 14, there is a constant $C>0$ such that

$$C^{-1} \leq \frac{|J'|}{|J|} \leq C.$$

If $|J'|>t$, we have $|J'| \leq Ct$. Let $J'_k=[x,c_k] \in \eta_{n+k+1,f}$ for $k>0$. From the bounded geometry, there is a $0<\tau<1$ such that

$$|J'_k| \leq \tau^k Ct.$$

Let k be the smallest integer greater than $-\log C/\log r$. Then $|J'_k| \leq t$. This implies that $J'_k \subseteq [x,x+t]$. So we have

$$\frac{|H(J'_k)|}{|H(I)|} \leq \frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|} \leq \frac{|H(J')|}{|H(J)|},$$

where $H(I) \in \eta_{n,g}$, $H(J), H(J') \in \eta_{n+1,g}$, and $H(J'_k) \in \eta_{n+k+1,g}$. Now from the bounded geometry for g, we have a constant, still denote as $C>0$, such that

$$C^{-1} \leq \frac{|H(J'_k)|}{|H(I)|} \leq \frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|} \leq \frac{|H(J')|}{|H(J)|} \leq C.$$

If $|J'| \leq t$, we have $|J'| \geq C^{-1}t$. Let $J'_{-k}=[x,c_{-k}] \in \eta_{n-k+1,f}$ for $k \geq 0$. Then from the bounded geometry, there is a constant, which we still denote as $0<\tau<1$, such that $|J'_{-k}| \geq \tau^{-k}C^{-1}t$. Let k be the smallest integer greater than $-\log C/\log r$. Then $|J'_{-k}| \geq t$. This implies that $J'_{-k} \supseteq [x,x+t]$. So we have

$$\frac{|H(J')|}{|H(I)|} \leq \frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|} \leq \frac{|H(J'_{-k})|}{|H(J)|},$$

where $H(I) \in \eta_{n,g}$, $H(J), H(J') \in \eta_{n+1,g}$, and $H(J'_{-k}) \in \eta_{n-k+1,g}$. Now from the bounded geometry for g, we have a constant, which we still denote as $C>0$, such that

$$C^{-1} \leq \frac{|H(J')|}{|H(I)|} \leq \frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|} \leq \frac{|H(J'_{-k})|}{|H(J)|} \leq C.$$

For any $x \in \mathbb{R}$, since Ω is dense in $[0,1]$, we have a sequence $x_n \in \Omega$ such that $x_n \to x$ as $n \to \infty$. For any $t>0$, we have that

$$C^{-1} \leq \frac{|H(x_n+t)-H(x_n)|}{|H(x_n)-H(x_n-t)|} \leq C.$$

Since H is uniformly continuous on $\mathbb{R}$, we get that

$$C^{-1} \leq \frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|} \leq C.$$

We have proved the theorem. □

Corollary 4. *Suppose f is a circle endomorphism. If the sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$ has the bounded geometry. Then f is uniformly quasisymmetric.*

Proof. Suppose h is a conjugacy from f to q, that is, $f = h \circ q \circ h^{-1}$. Since both sequences of nested Markov partitions for f and for q have bounded nearby and bounded geometry. Then h is quasisymmetric. Since $f^n = h \circ q^n \circ h^{-1}$, so f is uniformly quasisymmetric. □

Note from Robert Suzzi Valli on Lecture of Y. Jiang

in April 3rd, 2009, 9:30 am – 11:30 am.

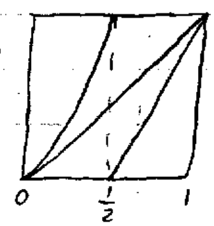
4/3/09

Thm: Spse f and g are uniformly quasisymmetric circle endomorphisms of deg $d>1$. Then they are topologically conjugate by h and h is quasisymmetric. Moreover, f and g both have bounded and bounded nearby geometry.

Thm: A circle endomorphism f of deg $d>1$ has bounded and bounded nearby geometry $\Leftrightarrow$ it is quasisymmetrically conjugate to $q(z)=z^d$.

Questions.

almost expanding circle endomorphism (Spse it's C^2) →



$\xrightarrow[\text{by } h]{\text{top conj}}$ $q(z)=z^2$

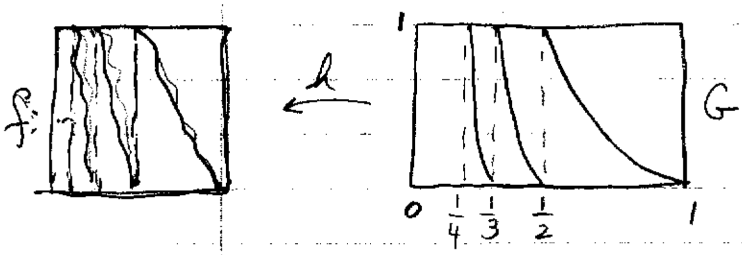
h is not q.s.

$f'(0)=1.\ f'(x)>1,\ \forall x\in(0,1]$

f has no bdd or bdd nearby geom. $\left(\frac{|I_{\overbrace{00\cdots0}^{n}}|}{|I_{\overbrace{00\cdots0}^{n-1}}|}\to 1 \ \&\ \frac{|I_{\overbrace{00\cdots01}^{n-1}}|}{|I_{\overbrace{00\cdots0}^{n-1}}|}\to 0\right)$

(open problem) **Q1:** Spse f and g are both almost expanding circle endomorphisms and C^2. Then f and g are top. conj by h. Is h q.s.?

$G(x)=\frac{1}{x}-\left[\frac{1}{x}\right]:(0,1]\to(0,1]$ (Gauss map)



$\frac{1}{2}<x<1$

$2>\frac{1}{x}>1$

$\left[\frac{1}{x}\right]=1$

$\frac{1}{x}-1$

$x=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}}=[a_1,a_2,\ldots$

$\frac{1}{x}=a_1+\frac{1}{a_2+\cdots}$

$\left[\frac{1}{x}\right]=a_1$

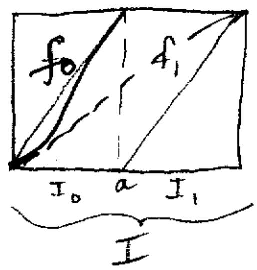
$|G'(x)|=\left|-\frac{1}{x^2}\right|>1,\ x\in\left(\frac{1}{n+1},\frac{1}{n}\right)$

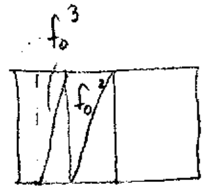
$n=1,\ldots$

$G(x)=\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}=[a_2a_3\ldots]$

Q2: Is h q.s.?

G(x) is just shift map in symbolic space of infinite letters.



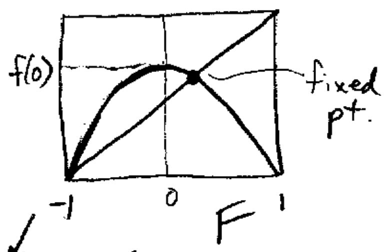


$I_{0_n} = f_0^{-1}(I_0)$

$0_n = \overbrace{00\cdots0}^{n}$

$\hat{f}(x) = f^n : I_{0_n} \to I$
$|\hat{f}'(x)| \geq \lambda' > 1, \ \forall x \in I_{0_n}$ } this relates Q1 to Q2 ($\hat{f}$ like G(x)-Gauss map)

(bigger problem) **Q3**:



"folding map" → unimodal map

$f(x) = \mu x(1-x)$

$f(x) = F(-|x|^\alpha)$, $\alpha > 1$

$F: [-1,1] \to [-1, f(0)]$ is a C^3-diffeomorphism + negative Schwarzian derivative

Schwarzian derivative

$$S(F)(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2 < 0$$

$$\Downarrow$$

$$S(F^{-1}) > 0$$

Q3: Spse f and g are both unimodal and are top. conj. by h, i.e. $f \circ h = h \circ g$. Is h q.s.?

specific case: $f(x) = F(-|x|^{2n})$ ← cx. analytic
$f(z) = F(-z^{2n})$ ← analytic — extend to complex map

(*) $x \mapsto |x|^2$ "x^2"
can extend this map to: $z \mapsto z^2$

· can't do: $x \mapsto |x|^3$
can't extend to: $z \mapsto z^3$.

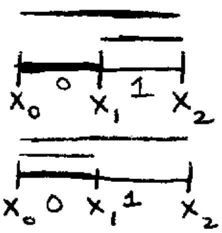
(★)

Test Ques: 2

<u>Thm</u>: $I=[0,1]$. Spse $f:[0,1]\xrightarrow{\text{cont.}}[0,1]$ has a periodic point of period 3. Then, $\forall m\geq 1$, f has a periodic point of period m.

(<u>Li-York</u>: period 3 ⇒ chaos)

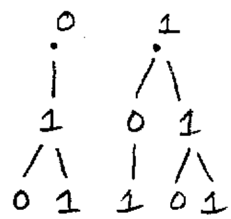
$x_0\neq x_1\neq x_2$
$f(x_0)=x_1$
$f(x_1)=x_2$
$f(x_2)=x_0$



$A=\begin{pmatrix}0&1\\1&1\end{pmatrix}$ $\Sigma_A=\{\omega=i_0i_1\ldots \mid a_{i_k i_{k+1}}=$

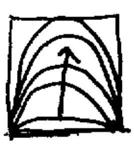
$A=\begin{pmatrix}1&1\\1&0\end{pmatrix}$

(★) use mean value thm.



$3\rhd 5\rhd 7\rhd\cdots\rhd 2\cdot 3\rhd 2\cdot 5\rhd\cdots\rhd 2^n\cdot 3\rhd 2^n\cdot 5\rhd\cdots\rhd 2^n\rhd 2^{n-1}\rhd\cdots\rhd 2\rhd 1$

<u>Thm</u> (Sarkowski): Spse $f:I\to I$ is a cont. map and has a periodic point of period m. Then f has a periodic point of period n for every $m\rhd n$.

$g_\mu(x)$: 

$a\leq\mu\leq b$

$\{\tilde{\mu}_n\}\to\tilde{\mu}_\infty$

Feigenbaum constant

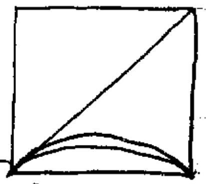
$\tilde{\delta}_n=\dfrac{\tilde{\mu}_n-\tilde{\mu}_{n-1}}{\tilde{\mu}_n-\tilde{\mu}_{n+1}}\longrightarrow\delta=$ ~~4.669...~~ (for $\alpha=2$)
4.669....

$\delta_n=\dfrac{\mu_n-\mu_{n-1}}{\mu_n+\mu_{n+1}}$

<u>Conj</u>: δ_α is universal and independent of any family as long as α is fixed.

$\alpha=2$ is OK

(3)



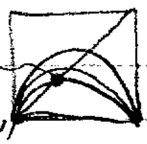
deriv 1 when $\mu = \frac{1}{2}$

For $0 < \mu \le \frac{1}{2}$, $f_\mu^n(x) \to -1$ as $n \to \infty$

$$f_\mu(x) = -\mu x^2 + (\mu - 1)$$
$$0 < \mu \le 2$$

$$f_\mu(-1) = -1$$
$$f_\mu(1) = -\mu$$
$$f_\mu'(0) = 0$$

for $\mu > \frac{1}{2}$



$\mu_n \nearrow \mu_\infty$

$\mu_{n-1} < \mu < \mu_n$

f_μ has a periodic pt. P_μ of period 2^n

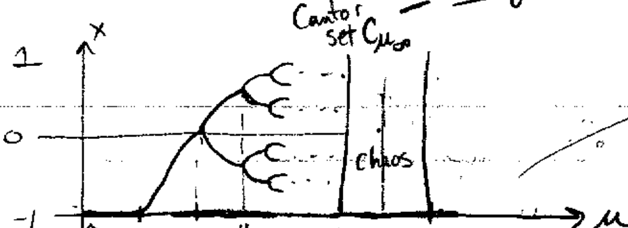
s.t. $f_\mu^n(x) \to \{P_\mu, \dots, f^{2^n-1}(P_\mu)\} = \mathcal{O}(P_\mu)$ → orbit of P_μ.

$(\mu > \frac{1}{2})$ as $n \to \infty$, for almost every x.

$f_{\mu_\infty}^n(x) \to ?$ as $n \to \infty$

Cantor set C_{μ_∞} — strange attractor

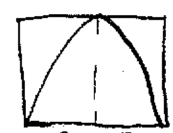
bifurcation



$\mu > 2$

horseshoe map

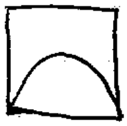
At $\mu = 2$



(works like circle expanding map)

Note: Spse $g \xrightarrow[\text{by } h]{\text{top.conj}} f_{\mu_\infty}^n(x) \Rightarrow h$ is q.s.

$\forall f$



+ some smoothing condition

$\exists \mu$ s.t. $f \overset{\text{top.conj}}{\sim} f_\mu$ on a dynamical interesting set

Let $\mathcal{QS}$ be the space of all (orientation-preserving) quasisymmetric circle homeomorphisms. Let $\mathcal{Q}$ be those in $\mathcal{QS}$ fixing 1. Let $\mathcal{F}$ be the space of all uniformly quasisymmetric circle endomorphisms of degree 2 fixing 1. (The following can be done for all degrees $d > 2$ with a little bit modification). Suppose $q(z) = z^2$. Let $\alpha : \mathcal{Q} \to \mathcal{F}$ be defined as

$$\alpha(h) = h \circ q \circ h^{-1}.$$

Theorem 16. *The map α is bijective.*

Proof. From Theorems 14 and 15 and Corollary 4, we have that $\alpha : \mathcal{Q} \to \mathcal{F}$ is a onto map. We only need to prove that α is injective. Suppose $\alpha(h_1) = \alpha(h_2)$ for $h_1, h_2 \in \mathcal{Q}$. Since 1 is the only fixed point of $\alpha(h_1)$, $\alpha(h_2)$, and q, $h_1(1) = h_2(1) = 1$. Since

$$h_1^{-1}(q^{-1}(1) = h_2^{-1}(q^{-1}(1)),$$

we have that $h_1(-1) = h_2(-1)$. Inductively, since

$$h_1^{-1}(q^{-n}(1) = h_2^{-1}(q^{-n}(1)),$$

we have that $h_1(e^{2\pi i(k/2^n)}) = h_2(e^{2\pi i(k/2^n)})$ for all $0 \leq k < 2^n$. Since the set of all numbers $\{k/2^n | 0 \leq k < 2^n, n = 1, 2, \cdots\}$ is dense in T, we have that $h_1 = h_2$. So α is injective. ☐

Remark 4. *The bounded nearby geometry and the quasisymmetric property for a conjugacy have been also studied for one-dimensional maps with critical points. Refer to (download site: http://qcpages.qc.cuny.edu/˜yjiang/HomePageYJ/Download/1993GeomFin.pdf) "Geometry of geometrically finite one-dimensional maps". Comm. in Math. Phys., 156 (1993), no. 3, 639-647. Or the book "Renormalization and Geometry in One-Dimensional and Complex Dynamics". Advanced Series in Nonlinear Dynamics, Vol. 10 (1996) World Scientific Publishing Co. Pte. Ltd., River Edge, NJ, xvi+309 pp. ISBN 981-02-2326-9.*

Dual symbolic space and scaling model for $\mathcal{Q}$ and $\mathcal{F}$.

Suppose f is a circle endomorphism of degree 2. As we have seen, we have a sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$. Here η_n contains 2^n intervals, each interval has a unique label $w_n = i_0 i_1 \cdots i_{n-1}$ which we denote as I_{w_n}, where $i_k \in \{0, 1\}$, such that

$$f(I_{w_n}) = I_{\sigma(w_n)}.$$

Now we would like to relabel these intervals. For $w_n = i_0 i_1 \cdots i_{n-1}$, let

$$w_n^* = j_{n-1} \cdots j_1 j_0,$$

where $j_{n-1} = i_0, \cdots, j_0 = i_{n-1}$. Define the dual shift map

$$\sigma^*(w_n^*) = j_{n-1} \cdots j_1.$$

Then we have the following

$$I_{w_n^*} \subset I_{w_n^*}.$$

We call

$$\Sigma^* = \{w^* = \cdots j_{n-1} \cdots j_0\} = \prod_{-\infty}^{0} \{0, 1\}$$

with the product topology the dual symbolic space. Then the dual shift map is

$$\sigma^* : w^* = \cdots j_{n-1} \cdots j_1 j_0 \to \sigma^*(w^*) = \cdots j_{n-1} \cdots j_1.$$

Then we call (Σ^*, σ^*) the dual symbolic dynamical system. The dual cylinder for a given $w_n^* = j_{n-1} \cdots j_0$ is

$$[w_n^*] = \{w^* = \cdots j'_{m-1} \cdots j'_n j_{n-1} \cdots j_0 \mid j'_k \in \{0, 1\}, k \geq n\}.$$

Then all these dual cylinders form a topological basis for Σ^*.

Now we define another operator in Σ^* called the adding machine $a(w^*)$ as follows: If $w^* = \cdots j_{n-1} \cdots j_1 j_0$ and $j_0 = 0$, then $a(w^*) = \cdots j_{n-1} \cdots j_1 (j_0 + 1)$ and if $j_0 = 1$, then $j_0 + 1 = 0$ and then consider $j_1 + 1$, so on. For each w_n^*, we can also define the adding machine a. We have that for any w_n^*, $I_{w_n^*}$ and $I_{a(w_n^*)}$ are two adjacent intervals. (Note that if $w_n^* = \underbrace{1 \cdots 1}_{n}$, then we define $a(w_n^*) = \underbrace{0 \cdots 0}_{n}$).

For each w_n^*, we define two scalings, one is for bounded nearby geometry and one is for bounded geometry:

$$bng(w_n^*) = \Big| \log \frac{|I_{w_n^*}|}{|I_{a(w_n^*)}|} \Big|$$

and

$$bg(w_n^*) = \frac{|I_{w_n^*}|}{|I_{\sigma^*(w_n^*)}|}.$$

Thus we have two sets of scalings

$$BNG = \{bng(w_n^*) \mid w_n^* = j_{n-1} \cdots j_0,\ j_k \in \{0, 1\}\}$$

and

$$BG = \{bg(w_n^*) \mid w_n^* = j_{n-1} \cdots j_0,\ j_k \in \{0, 1\}\}.$$

These two sets can be determined to each others.

Exerice 8. *Express scalings in BG in terms of scalings in BNG and verse versa.*

Therefore, the sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$ has bounded nearby geometry if and only if there is a constant $C > 0$ such that

$$bng(w_n^*) \leq C, \ \forall w_n^*.$$

And it has bounded geometry if and only if there is a constant $C > 0$ such that

$$bg(w_n^*) \geq C, \ \forall w_n^*.$$

It is also clear that

$$bg(w_n 0) + bg(w_n 1) = 1 \tag{2}$$

which we call it **the summation condition**. Now let us consider the space of scalings (means positive numbers)

$$\mathcal{S} = \{(S(w_n^*))\}$$

satisfies the summation condition (2) and the induced bounded nearby geometry $(bng(w_n^*))$ is in l^∞. Then for $f \in \mathcal{F}$, we have that

$$\gamma(f) = (bg(w_n^*)) \in \mathcal{S}.$$

This induces a map

$$\beta(h) = \gamma \circ \alpha \in \mathcal{S}$$

and

Theorem 17. *The maps $\beta : \mathcal{Q} \to \mathcal{S}$ and $\gamma : \mathcal{F} \to \mathcal{S}$ are both bijective.*

Universal Teichmüller space and the space of uniformly quasisymmetric circle endomorphisms and space of scalings

Remember that $\mathcal{QS}$ is the space of all quasisymmetric homeomorphisms of T. Let $\mathcal{M}$ be the space of all Möbius transformations preserving the unit disk Δ. Then

$$\mathcal{M} = \{M(z) = e^{2\pi\theta i}\frac{z-a}{1-\overline{a}z} \mid |a| < 1\}$$

The quotient space

$$\mathcal{UT} = \mathcal{QS}/\mathcal{M}$$

is called the universal Teichmüller space. It is a complex Banach manifold (we will discuss this later, see Remark 5).

Since each element in $\mathcal{M}$ has one complex parameter $a \in \Delta$ and one real parameter $0 \leq \theta < 1$, for any pair of triple points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ arranged in the counter-clockwise order on T, there is a one and only one element $M \in \mathcal{M}$ such that

$$M(z_1) = w_1, \quad M(z_2) = w_2, \quad M(z_3) = w_3.$$

Thus the universal Teichmüller space can be thought as the space of all quasisymmetric homeomorphisms fixing three points on T. Let us assume that this three points are 1, i, -1. Then

$$\mathcal{UT} = \{h \in \mathcal{QS} \mid h(1) = 1,\ h(i) = i,\ h(-1) = -1\} = \mathcal{Q}/\mathcal{M}_1$$

where

$$\mathcal{M}_1 = \{M(z) = \frac{1-\overline{a}}{1-a}\frac{z-a}{1-\overline{a}z} \mid |a| < 1\}$$

is the space of all Möbius transformations preserving the unit disk Δ fixing 1.

Now consider the space of all uniformly quasisymmetric circle endomorphisms of a fixed degree $d > 1$ conjugated by $\mathcal{M}$, which we denote as $\mathcal{UF}_d$. For the simplicity, we assume $d = 2$ and write $\mathcal{UF} = \mathcal{UF}_2$.

Theorem 18. *The map α induces a bijective map $\varrho = [\alpha] : \mathcal{UT} \to \mathcal{UF}$.*

Proof. Let $q(z) = z^2$. Then it has a unique fixed point 1 and $q^{-1} = \{1, -1\}$ and $q^{-2}(1) = \{1, i, -1, -i\}$. The space $\mathcal{UF}$ can be thought as the space of those uniformly quasisymmetric circle endomorphisms f fixing 1 such that $f^{-1}(1) = \{1, -1\}$ and $f^{-2}(1) = \{1, i, -1, s\}$. It is clear that of $h \in \mathcal{Q}$ fixing 1, -1, i if and only if $f = \alpha(h) \in \mathcal{UF}$. Thus α induces a onto map ϱ from $\mathcal{UT}$ to $\mathcal{UF}$. We need to prove ϱ is one-to-one. Suppose $\varrho(h_1) = \varrho(h_2)$. Similar to the proof of Theorem 16, $h_1 = h_2$. □

Let $\mathcal{US}$ be the space of scalings $(s(w_n^*))$ in $\mathcal{S}$ such that $s(0) = s(1) = s(00) = 1/2$.

Corollary 5. *The maps γ and β induce bijective maps $[\gamma] : \mathcal{UF} \to \mathcal{US}$ and $[\beta] : \mathcal{UT} \to \mathcal{US}$.*

Remark 5. *Note that*

$$\mathcal{Q} = \mathcal{UT} \times \Delta$$

where $\Delta = \{a \mid |a| < 1\}$ is the unit disk in the complex plane. For any $h \in \mathcal{Q}$, let $t = h(i)$ and $s = h(-1)$ and

$$a = \frac{1 - i\frac{s(t-1)}{(t-s)}}{1 - i\frac{t-1}{t-s}} \in \Delta.$$

Let

$$G_a(z) = \frac{1-\overline{a}}{1-a}\frac{z-a}{1-\overline{a}z}.$$

Then G_a *maps* $1, t, s$ *to* $1, i, -1$. *So* $G_a \circ h$ *fixes* $1, i, -1$ *and is in* $\mathcal{UT}$. *So if we define* $\chi(h) = (G_a \circ h, a)$. *Then it is a bijective map from* $\mathcal{Q}$ *to* $\mathcal{UT} \times \Delta$. *So* $\mathcal{Q}$ *as well as* $\mathcal{F}$ *and* $\mathcal{S}$ *has induced complex Banach manifold structure such that it is a complex Banach manifold. (We will discuss this later.)*

Lecture 12 5/1/09

10:00 am – 12:00 noon

by Fred Gardiner.

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} \qquad t^2 - x^2 - y^2 - z^2$$

Minkowski space in $\mathbb{R}^4$.

$t^2 - x^2$ in $\mathbb{R}^2$ $\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$, $A = \begin{pmatrix} \frac{1}{\sqrt{1-k^2}} & \frac{k}{\sqrt{1-k^2}} \\ \frac{k}{\sqrt{1-k^2}} & \frac{1}{\sqrt{1-k^2}} \end{pmatrix}$

Title:

Complex structure on Teichmüller space

$A\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix}$ $\quad t^2 - x^2 = (t')^2 - (x')^2$ · invariant under A.

In Euclidean: $t^2 + x^2$ $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ invariant for $t^2 + x^2$.

$PSL(2, \mathbb{C}) \cong$ Lorentz gp $= SO^+(1,3)$

$PSL(2, \mathbb{R})$.

$t^2 = x^2 + y^2 + z^2$ is a cone.

$\zeta = u + iv$

$$\begin{pmatrix} u^2+v^2+1 \\ 2u \\ -2v \\ u^2+v^2-1 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad t^2 = 4u^2+4v^2+u^4+v^4+1-2u^2-2v^2+2u^2v^2$$

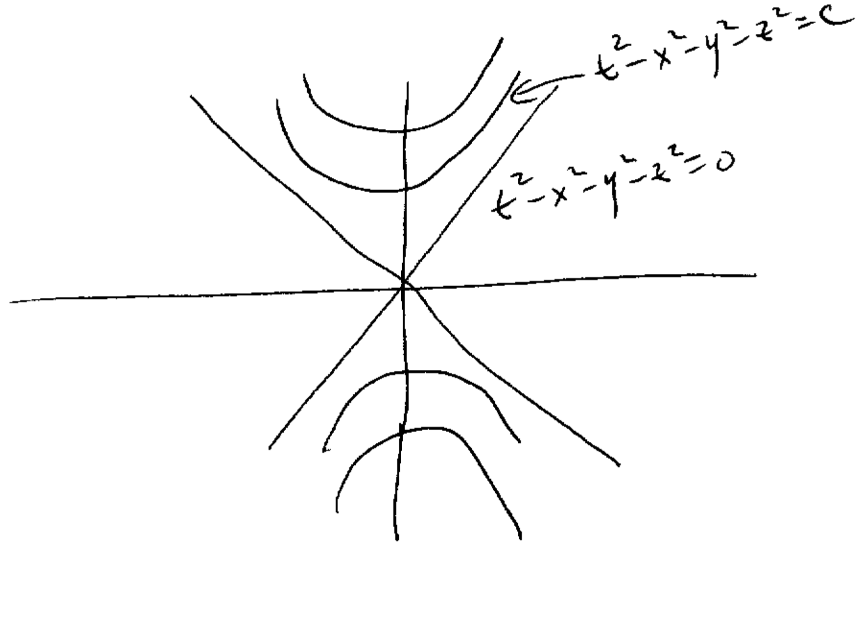
$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = \text{a Hermitian } 2\times 2 \text{ matrix}$$

(i.e. $X^t = \bar{X}$)

$\det = t^2 - x^2 - y^2 - z^2$. $\quad x, y, z, t$ are all real.

$P \in SL(2, \mathbb{C})$, $\det = 1$.

$X \longrightarrow PXP^*$, $\quad P^* =$ conjugate transpose. $= \bar{P}^t$



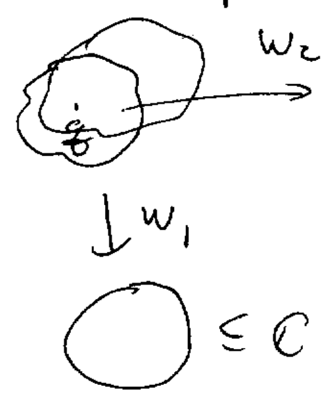
manifold structure

complex manifold structure

$R \Leftarrow$ Hausdorff space, Riemann surface.

$S = \{ f : \Delta \to R,\ f(0) = p.\ f \text{ is holomorphic} \}$

$f(z) = q \in R$



$w_1 \circ w_2^{-1}$: holo.

$\zeta \in \Delta \xrightarrow{f} R.$

$$(*)\quad \inf \left\{ \frac{1}{|f'(0)|} \;\middle|\; f \in S \right\}$$

; f' means fixed local coordinate $\left. \dfrac{d(w \circ f)}{d\zeta} \right|_{\zeta = 0} = f'(0).$

$\parallel$

$\rho(p)$. depends on local coordinate w.

$\rho(p)\,|dp|$ is invariant by local coordinate w.

i.e. $\rho(w_1)\,|dw_1| = \rho(w_2)\,|dw_2|.$

The f that realizes the infimum in $(*)$ is the universal covering of R by Δ.

z as coordinate

$\cdot$ 0 $\quad f(0)=p \quad$ 1 $\quad\quad \infty \quad\quad R = \mathbb{C} \setminus \{0,1\}$.

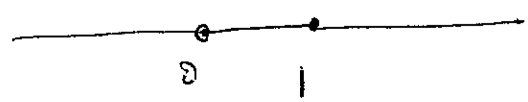
$\Delta \xrightarrow{f}$

one linear map $f_r : \Delta \to \Delta_r(p)$

$\Rightarrow f_r'(0) = r$.

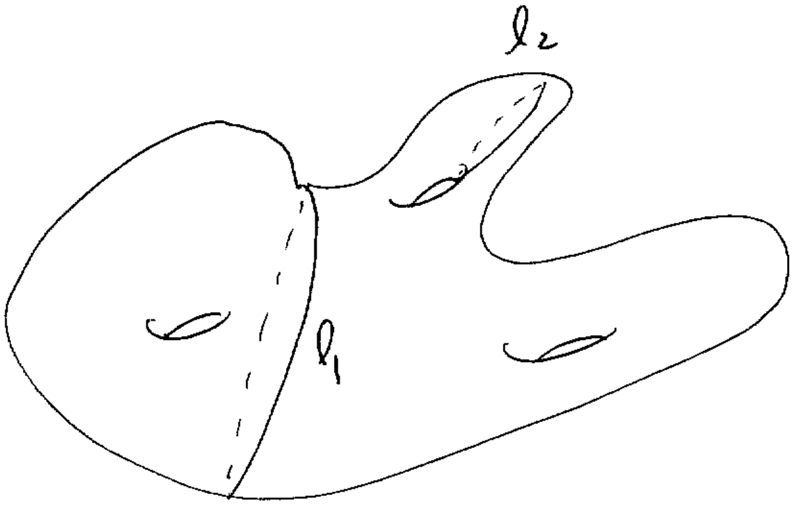
Riemann mapping maps $\Delta \to$ upper half plane with boundary segments $(0,1)$, $(1,\infty)$, $(-\infty,0)$.

$a, b, c \to 0, 1, \infty$.

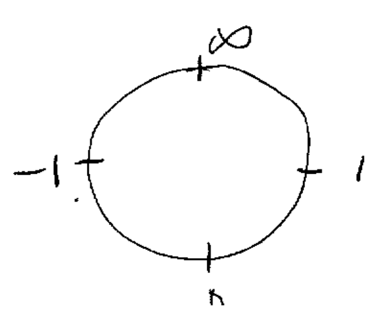


reflect in the triangle $\Rightarrow f : \Delta \to \mathbb{C}\setminus\{0,1\}$ is the universal covering of $\mathbb{C}\setminus\{0,1\}$ by Δ.

In general:

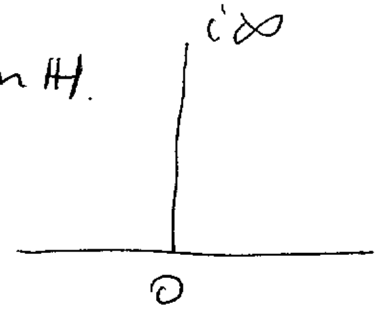


$\cong \Delta / \Gamma$ = covering group $\subset PSL(2,\mathbb{R})$



pick one closed curve and assume it

$A \in \Gamma$, $Az = \lambda z$, $\lambda > 1$ $(0, +i\infty)$ in $\mathbb{H}$.



$\ln \lambda$, hyperbolic ~~length~~ of l_1

$A'(0) = \lambda$

Tsuji, Fricke-Klein. If R is of finite type, then a finite number of lengths of homotopic classes of closed curve on R determines the covering group.

Γ_1. Γ_2 acting on $\mathbb{H}$.

$w : \mathbb{H} \to \mathbb{H}$ $\quad w\Gamma_1 w^{-1} = \Gamma_2$, i.e. $w \gamma w^{-1} = \chi(\gamma) \in \Gamma_2$ $\forall \gamma \in \Gamma_1$

χ is an isomorphism from Γ_1 on Γ_2.

L a Möbius trans. in determinated by its values at 3 pts $\quad \Lambda(\Gamma) =$ limit set of Γ.

Thinking about

W restricted to $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$

~~$W: R \to R$~~

$$\begin{array}{ccc} \Delta & \xrightarrow{\tilde{w}} & \Delta \\ \downarrow & & \downarrow \\ R_1 & \xrightarrow{w} & R_2 \end{array} \qquad M^{-1} \le \frac{|\tilde{w}(x+t) - \tilde{w}(x)|}{|\tilde{w}(x) - \tilde{w}(x-t)|} \le M \quad m > 0.$$

$T_2 = \omega T_1 \omega^{-1}$. $\quad \omega(0)=0, \quad \omega(1)=1. \quad \omega(\infty)=\infty.$

$\omega(z) = z + \varepsilon V(z) + O(\varepsilon) \quad z \in \mathbb{H}.$

$\omega_\varepsilon(x) = x + \varepsilon V(x) + O(\varepsilon).$

$$M \le \overset{\rho(x,t)}{\overset{\shortparallel}{\frac{|\omega(x+t) - \omega(x)|}{|\omega(x) - \omega(x-t)|}}} \le M$$

If $\varepsilon = 0$, $\rho(x,t)=1$. if ε small

$$\frac{1}{1+\delta(\varepsilon)} \le \frac{|\omega_\varepsilon(x+t) - \omega_\varepsilon(x)|}{|\omega_\varepsilon(x) - \omega_\varepsilon(x-t)|} \le 1+\delta(\varepsilon) \qquad \delta(\varepsilon)\to 0 \text{ as } \varepsilon \to 0.$$

$$\left| \frac{V(x+t) - V(x-t) - 2V(x)}{t} \right| \le C_\varepsilon \quad (**)$$

$T_\varepsilon = \omega_\varepsilon T \omega_\varepsilon^{-1}$

$$\frac{V(\gamma(x))}{\gamma'(x)} - V(x) = \text{quadratic polynomial in } x. \ (***)$$

$$\gamma_\delta(z) = \frac{a_\delta z + b_\delta}{c_\delta z + d_\delta} \qquad \gamma_0(z) = z. \quad a_\delta, b_\delta, c_\delta, d_\delta \text{ are differentiable functions of } \delta.$$

$$\lim_{\delta\to 0}\frac{1}{\delta}\left(\gamma_\delta(z)-z\right) = \frac{1}{\delta}\left(\frac{a_\delta z+b_\delta}{c_\delta z+d_\delta}-z\right)$$

$$=\lim_{\delta\to 0}\frac{1}{\delta}\,\frac{a_\delta z+b_\delta-c_\delta z^2-d_\delta z}{c_\delta z+d_\delta}.$$

$a_0=1$, $d_0=1$

$c_0=0$, $b_0=0$.

$$=-\dot c z^2+(\dot a-\dot d)z+\dot b.$$

$\Rightarrow$ The V's ~~that~~ satisfy (**) (***) ~~comprise~~ factored by quad polys comprise the tangent space to Teich(Γ).

Z = a real vector space

$I: Z\to Z$. $I^2=-$identity

$(a+ib)V \doteq aV+IbV$, $V\in Z$.

$c_1(c_2V)=(c_1c_2)V$.

$(a_1+ib_1)\left((a_2+ib_2)V\right)=(a_1+ib_1)(a_2+ib_2)\,V$

Bers embedding: Schwarzian derivative

Hilbert transform:

1) Harmonic conjugates

2) $PV - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{V(t)}{t-x} \, dt$

3) Same sort of integral to periodic functions

cotangent $(t-x)$. ~~by rep~~ t replace $(t-x)$.

4) Pompeiu formula.

– Cauchy

$$V(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \left(\frac{1}{\zeta - z} + \text{convergence term} \right) \mu(\zeta) \, d\xi \, d\eta.$$

$\mu(\bar\zeta) = \mu(\zeta)$

$\mu \in L_\infty$. Hilbert transf is just $I\mu$, ie

$$IV = -\frac{1}{\pi} \iint \left(\frac{1}{\zeta - z} + \text{con. term} \right) I\mu(\zeta) \, d\xi \, d\eta.$$

$$I\mu = \begin{cases} i\mu & \text{in } \mathbb{H}. \\ -i\mu & \text{in } \mathbb{H}^*. \end{cases}$$

$\Rightarrow$ Z under this Hilbert transform is a Complex Banach manifold.

Still in Draft version

Symmetric homeomorphisms

Let $\mathcal{Q}_0$ be the space of all symmetric homeomorphisms in $\mathcal{Q}$, that is, $h \in \mathcal{Q}_0$ if and only if $h(1) = 1$ and there is a bounded positive function $\epsilon(t) > 0$ such that $\epsilon(t) \to 0$ as $t \to 0$,

$$\frac{1}{1+\epsilon(t)} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq 1 + \epsilon(t)$$

for all $x \in \mathbb{R}$ and $t > 0$.

For any $h \in \mathcal{Q}_0$, consider $f = h \circ q \circ h^{-1}$ and the corresponding sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$ and corresponding scalings $(s(w_n^*))$.

Theorem 19. *The homeomorphism $h \in \mathcal{Q}_0$ if any only if for any $w^* \in \Sigma^*$, the limit $S(w^*) = \lim_{n\to\infty} s(w_n^*)$ exists and is $1/2$, that is, it defines a constant function*

$$S(w^*) = \frac{1}{2} : \Sigma^* \to \mathbb{R}.$$

Before to prove this theorem, we need to have the following concept about quasisymmetry on an interval. Suppose I and J are two intervals on the real line $\mathbb{R}$. Suppose $Q : I \to J$ is a homeomorphism. Let $M \geq 1$. We say that Q is M-quasisymmetric on I if

$$M^{-1} \leq \frac{|Q(x+t) - Q(x)|}{|Q(x) - Q(x-t)|} \leq M, \quad \forall\, x-t,\ x,\ x+t \in I, t > 0.$$

Lemma 6. *Suppose $I = J = [0,1]$. There is a function $\zeta(M) > 0$ satisfying $\zeta(M) \to 0$ as $M \to 1$ such that for any M-quasisymmetric homeomorphism $Q : [0,1] \to [0,1]$ such that $Q(0) = 0$ and $Q(1) = 1$,*

$$|Q(x) - x| \leq \zeta(M), \quad \forall\, x \in [0,1].$$

Proof. Consider points $x_n = 1/2^n$, $n = 0, 1, \cdots$. M-quasisymmetry and the normalization $Q(0) = 0, Q(1) = 1$ imply that

$$\frac{1}{1+M} H(\frac{1}{2^{n-1}}) \leq Q(\frac{1}{2^n}) \leq \frac{1}{1+M^{-1}} Q(\frac{1}{2^{n-1}}).$$

Similarly,

$$\Big(\frac{1}{1+M}\Big)^n \leq Q(\frac{1}{2^n}) \leq \Big(\frac{1}{1+M^{-1}}\Big)^n, \quad \forall\, n \geq 1.$$

Furthermore, by M-quasisymmetry and induction on $n = 1, 2, \cdots$, yield

$$\Big(\frac{1}{1+M}\Big)^n \leq Q(\frac{i}{2^n}) - Q(\frac{i-1}{2^n}) \leq \Big(\frac{1}{1+M^{-1}}\Big)^n, \quad \forall\, n \geq 1, \ \ 1 \leq i \leq 2^n.$$

Let

$$\tau_n = \max\left\{ \left(\frac{M}{M+1}\right)^n - \frac{1}{2^n}, \frac{1}{2^n} - \left(\frac{1}{M+1}\right)^n \right\}, \quad n = 1, 2, \cdots .$$

Then for $n = 1$,

$$|Q(\frac{1}{2}) - \frac{1}{2}| \leq \tau_1 = \frac{1}{2}\frac{M-1}{M+1},$$

and for any $n > 1$, we have

$$\max_{0 \leq i \leq 2^n} \left| Q(\frac{i}{2^n}) - \frac{i}{2^n} \right| \leq \max_{0 \leq i \leq 2^{n-1}} \left| Q(\frac{i}{2^{n-1}}) - \frac{i}{2^{n-1}} \right| + \tau_n$$

By summing over k for $1 \leq k \leq n$, we obtain

$$\max_{0 \leq i \leq 2^n} \left| Q(\frac{i}{2^n}) - \frac{i}{2^n} \right| \leq \delta_n = \sum_{k=1}^{n} \tau_k.$$

If we put $\zeta(M) = \sup_{1 \leq n < \infty} \{\delta_n\}$, by summing geometric series, we obtain

$$\zeta(M) = \max_{1 \leq n < \infty} \Big\{ M - 1 + \frac{1}{2^n} - M\Big(\frac{M}{1+M}\Big)^n, 1 - \frac{1}{M} + \frac{1}{M}\Big(\frac{1}{M}\Big)^n - \frac{1}{2^n} \Big\}.$$

Clearly, $\zeta(M) \to 0$ as $M \to 1$, and since the dyadic points

$$\{i/2^n \ \mid \ n = 1, 2, \cdots; 0 \leq i \leq 2^n\}$$

are dense in $[0, 1]$, we conclude

$$|Q(x) - x| \leq \zeta(M) \quad \forall\, x \in [0, 1],$$

which proves the lemma. □

Proof of Theorem 19. Suppose $\epsilon(t) > 0$ is the bounded positive function such that $\epsilon(t) \to 0$ as $t \to 0$,

$$\frac{1}{1+\epsilon(t)} \leq \frac{|H(x+t)-H(x)|}{|H(x)-H(x-t)|} \leq 1+\epsilon(t)$$

for all $x \in \mathbb{R}$ and $t > 0$.

For any $\epsilon > 0$, there is a $\delta > 0$ such that $\zeta(\delta) < \epsilon$. Then we have $\tau > 0$ such that $\epsilon(t) < \delta$ for any $0 < t < \tau$. Let $n_0 > 0$ be an integer such that $1/2^{n_0}) < \tau$. Then H on any interval I with $|I| < 1/2^{n_0}$ is a $(1+\delta)$-quasisymmetric homeomorphism.

Suppose $w^* = \cdots j_{n-1}\cdots j_1 j_0 \in \Sigma^* \in \Sigma^*$ is any point. Let

$$w_n^* = j_{n-1}\cdots j_1 j_0 \quad \text{and} \quad v_{n-1}^* = j_{n-1}\cdots j_1.$$

By definition,

$$S(w_n^*) = \frac{|I_{w_n}|}{|I_{\sigma^*(v_{n-1}^*)}|},$$

where $I_{w_n} \subset I_{v_{n-1}^*}$. Consider the sequence $\{S(w_n^*)\}_{n=1}^\infty$. Then we have that, for any $n > n_0$,

$$H : I_0 = [\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}}] \to I_{v_{n-1}^*}$$

is a $(1+\delta)$-quasisymmetric homeomorphism for some integer $0 \leq i \leq 2^{n-1}$. Let j be another integer such that $H([j/2^n, (j+1)/2^n] = I_{w_n^*}$. Let $J_0 = [j/2^n, (j+1)/2^n]$.

From Lemma 6 (by normalizing $I_{v_{n-1}}$ and I_0 to $[0,1]$ and J_0 to $[0,1/2]$ or $[1/2,1]$ and I_{w_m} to $[0,x]$ or $[x,1]$ by linear transformations), we get

$$|S(w_n^*) - \frac{1}{2}| \leq \zeta(1+\delta) < \epsilon.$$

This implies that $\{S(w_n^*)\}_{n=1}^\infty$ has a limit and the limit is $1/2$.

On the other hand if all $\{S(w_n^*)\}_{n=1}^\infty$ have limits and the limits are $1/2$. Then following a similar proof to Theorem 15, we can show that for any $\epsilon > 0$, there is a $\delta > 0$ such that H restricted on any interval I with $|I| < \delta$ is a $(1+\epsilon)$-quasisymmetric. □

The universal asymptotically affine Teichmüller space is defined as

$$\mathcal{T}_0 = \mathcal{Q}_0/\mathcal{M}_1 = \{h \in \mathcal{Q}_0 \mid h(0)=0, h(1)=1, h(i)=i\}.$$

Corollary 6. *The universal asymptotically affine Teichmüller space is one-to-one corresponding to the space of scalings* $(S(w_n^*)) \in \mathcal{US}$ *such that* $\lim_{n\to\infty} S(w_n^*) = 1/2$ *for all* $w^* = \cdots w_n^* \in \Sigma^*$.

Thus if we denote that

$$\mathcal{S}_0 = \{(S(w_n^*)) \in \mathcal{S} \mid \lim_{n\to\infty} S(w_n^*) = \frac{1}{2} \ \forall\, w^* = \cdots w_n^* \in \Sigma^*\}.$$

and

$$\mathcal{US}_0 = \{(S(w_n^*)) \in \mathcal{US} \mid \lim_{n\to\infty} S(w_n^*) = \frac{1}{2} \ \forall\, w^* = \cdots w_n^* \in \Sigma^*\}.$$

Then $\beta : \mathcal{Q}_0 \to \mathcal{S}_0$ and $\beta : \mathcal{T}_0 \to \mathcal{US}_0$ are bijective..

Remark 6. *Since $\mathcal{T}_0$ is a complex Banach manifold (we will discuss this later), so $\mathcal{US}_0$ is also a complex Banach manifold. Since*

$$\mathcal{Q}_0 = \mathcal{T}_0 \times \Delta \quad \textit{and} \quad \mathcal{S}_0 = \mathcal{US}_0 \times \Delta,$$

they are also complex Banach manifolds.

Limiting quasisymmetric homeomorphisms and uniformly symmetric circle endomorphisms

The above section gives us an idea to define the following space

$$\mathcal{LQ} = \{h \in \mathcal{Q} \mid S(w^*) = \lim_{n\to\infty} S(w_n^*) \text{ exists for all } w^* \in \Sigma^*\}.$$

Then for each $h \in \mathcal{LQ}$, we define a function

$$S(w^*) : \Sigma^* \to \mathbb{R}.$$

The scaling function satisfies the summation condition

$$S(w^*0) + S(w^*1) = 1 \quad \forall w^* \in \Sigma^*.$$

We call this function the scaling function for h as well as for $f = h \circ q \circ h^{-1}$. Following a similar proof to Theorem 19 we have that

Theorem 20. *The scaling functions for h and for $h_0 \circ h$ (or $h \circ h_0$) are the same for every $h_0 \in \mathcal{Q}_0$.*

Thus the space of all scaling functions is a representation of

$$\mathcal{LAT} = \mathcal{LQ}/\mathcal{Q}_0$$

which we call the universal limiting asymptotically conformal Teichmüller space. The universal asymptotically conformal Teichmüller space is defined as

$$\mathcal{AT} = \mathcal{Q}/\mathcal{Q}_0.$$

Then we have that

$$\mathcal{LAT} \subset \mathcal{AT}.$$

Remark and Problem 1. *The space $\mathcal{AT}$ can not be represented by a space of functions defined on ever points in Σ^*. We would like to use limiting Martingales to represent this space. Note that $\mathcal{AT}$ is a complex Banach manifold. (We will talk about this later.) However, we are still interested in to find the exact complex manifold structure of $\mathcal{LAT}$. It is an interesting question.*

Theorem 21. *Suppose $f = h \circ q \circ h^{-1}$ is a uniformly symmetric circle endomorphism. Then its scaling function*

$$S(w^*) : \Sigma^* \to \mathbb{R}^+$$

exists and is a continuous function. Furthermore, if f is $C^{1+\alpha}$, then $S(w^*)$ is Hölder continuous. Actually when f is C^1 Dini expanding, the modulus of continuity of $S(f)(w^*)$ is controlled by $\tilde{\omega}(t)$.

Proof. Suppose $w^* = \cdots j_{n-1} \cdots j_1 j_0 \in \Sigma^*$. Let

$$w_n^* = j_{n-1} \cdots j_1 j_0 \quad \text{and} \quad v_{n-1}^* = j_{n-1} \cdots j_1.$$

By definition,

$$S(w_n^*) = \frac{|I_{w_n}|}{|I_{v_{n-1}}|},$$

where $I_{w_n} \subset I_{v_{n-1}}$. Consider the sequence $\{S(w_n^*)\}_{n=1}^{\infty}$.

Let $0 < \tau < 1$ be a constant such that

$$\tilde{\iota}_n = \max_{w_n} |I_{w_n}| \leq \tau^n, \quad \forall n \geq 1.$$

For any $\epsilon > 0$, let $n_0 > 0$ be an integer such that $\zeta(1 + \varepsilon(\tau^{n-1})) \leq \epsilon$ for all $n > n_0$. Then for any $m > n > n_0$, we have that

$$F^{m-n}(I_{v_{m-1}}) = I_{v_{n-1}} \quad \text{and} \quad F^{m-n}(I_{w_m}) = I_{w_n}$$

Since $F^{-(m-n)}|I_{v_{n-1}}$ is a $(1+\varepsilon(\tau^{n-1}))$-quasisymmetric homeomorphism, from Lemma 6 (by normalizing $I_{v_{n-1}}$ to $[0,1]$ and I_{w_m} to $[0,x]$ by a linear transformation),

$$|S(w_m^*) - S(w_n^*)| = \Big| \frac{|F^{-(m-n)}(I_{w_n})|}{|F^{-(m-n)}(I_{v_{n-1}})|} - \frac{|I_{w_n}|}{|I_{v_{n-1}}|} \Big| \leq \zeta(1 + \varepsilon(\tau^{n-1})) \leq \epsilon.$$

This implies that $S(w_n^*)\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus the limit

$$S(w^*) = \lim_{n\to\infty} S(w_n^*)$$

exists.

Now consider two points

$$w^* = \cdots j_{m-1} \cdots j_n j_{n-1} \cdots j_0 \quad \text{and} \quad \tilde{w}^* = \cdots j_{m-1} \cdots j_n' j_{n-1} \cdots j_0.$$

Let $w_m^* = j_{m-1}\cdots j_n j_{n-1}\cdots j_0$ and $\tilde{w}_m^* = j_{m-1}\cdots j_n' j_{n-1}\cdots j_0$. Then $w_n^* = \tilde{w}_n^*$. For any $m > n$,

$$|S(w_m^*) - S(\tilde{w}_m^*)|$$
$$\leq |S(w_m^*) - S(w_n^*)| + |S(\tilde{w}_m^*) - S(w_n^*)| \leq 2\zeta(1+\varepsilon(\tau^{n-1})).$$

So by taking a limit,

$$|S(w^*) - S(\tilde{w}^*)| \leq 2\zeta(1+\varepsilon(\tau^{n-1})).$$

Thus we have that

$$S(w^*) : \Sigma^* \to \mathbb{R}^+$$

is a continuous function whose modulus of continuity is bounded by $2\zeta(1+\varepsilon(\tau^{n-1}))$.

Moreover, if f is a $C^{1+\alpha}$ expanding circle endomorphism for some $0 < \alpha \leq 1$, from the Hölder distortion property (1), there is a constant $C > 0$ such that

$$|S(w^*) - S(\tilde{w}^*)| \leq C\tau^{\alpha(n-1)}.$$

This implies that the scaling function $S(w^*)$ is Hölder continuous.

When f is C^1 Dini, then there is a constant $C > 0$ such that

$$|S(w^*) - S(\tilde{w}^*)| \leq C\tilde{\omega}(\tau^{n-1}).$$

Thus the scaling function $S(w^*)$ is continuous and its modulus of continuity is controlled by $\tilde{\omega}(\tau^{n-1})$. We have proved the theorem. $\square$

Define

$$\mathcal{UA}(q) = \{h \in \mathcal{Q} \mid f = h \circ q \circ h^{-1} \text{ is uniformly symmetric } \}$$

Then it can be represented by all scalings $(S(w_n^*))$ such that the limiting scaling functions $S(w^*)$ exists and is continuous. Since the limiting scaling functions are invariant under $\mathcal{Q}_0$, we have that the Teichmüller space of uniformly symmetric circle endomorphisms

$$\mathcal{AT}(q) = \mathcal{UA}(q)/\mathcal{Q}_0$$

is represented exactly by the space of all continuous scaling functions

$$\mathcal{CS} = \{S(w^*) \mid S : \Sigma^* \to \mathbb{R} \text{ is a continuous scaling function}\}.$$

Define

$$\mathcal{HA}(q) = \{h \in \mathcal{Q} \mid f = h \circ q \circ h^{-1} \text{ is } C^{1+\alpha} \text{ expanding for some } 0 < \alpha \leq 1\}$$

Then it can be represented by all scalings $(S(w_n^*))$ such that the limiting scaling functions $S(w^*)$ exists and is Hölder continuous. Since the limiting scaling functions are invariant under $\mathcal{Q}_0$, we have that the Teichmüller space of uniformly symmetric circle endomorphisms

$$\mathcal{HAT}(q) = \mathcal{HA}(q)/\mathcal{Q}_0 = \mathcal{HA}(q)/\mathcal{D}$$

is represented exactly by the space of all Hölder continuous scaling functions

$$\mathcal{HS} = \{S(w^*) \mid S : \Sigma^* \to \mathbb{R} \text{ is a Hölder continuous scaling function}\},$$

where $\mathcal{D}$ is the space of all circle diffeomorphisms.

Then we have that

$$\mathcal{HA}(q) \subset \mathcal{UA}(q) \subset \mathcal{LQ} \subset \mathcal{Q}$$

$$\mathcal{HAT}(q) \subset \mathcal{AT}(q) \subset \mathcal{LAT} \subset \mathcal{AT}$$

Remark 7. *Under the Teichmüller metric (we will talk later), the completion of $\mathcal{HAT}(q)$ is $\mathcal{AT}(q)$.*

Quasiconformal maps and quasisymmetric homeomorphisms

Theorem 22 (Riemann Mapping Theorem)**.** *Suppose D is a simply connected domain in the complex plan $\mathbb{C}$ such that its complement contains at least one point. Then there is a conformal map f from D onto the open unit disk Δ. Furthermore, if we specific a point z_0 in D such that $f(z_0) = 0$ and $f'(z_0) > 0$, then f is unique.*

In the theorem $f : D \to \Delta$ is called a Riemann mapping.

Theorem 23 (Caratheodory's Extension Theorem)**.** *If D is a Jordan domain, then the Riemann map $f : D \to \Delta$ can be extended to the boundary of D such that $f : \partial D \to S^1 = \partial D$ is a homeomorphism.*

Now we specific four points z_1, z_2, z_3, z_4 on the boundary of D and other four points w_1, w_2, w_3, w_4 on the boundary of Δ. Could you find a conformal $f : D \to \Delta$ such that its extension to the boundary maps z_i to w_i for $i = 1, 2, 3, 4$? This is Grötzsch's problem. Answer to this question is no in general.

Let $w = f(z)$ be a C^1 map. let $z = x + iy$ and $w = u + iv$. Then

$$du = u_x dx + u_y dy$$

$$dv = v_x dx + v_y dy.$$

If we write it into complex version, we have

$$dw = f_z dz + f_{\overline{z}} d\overline{z}$$

where

$$f_z = \frac{1}{2}(f_x - i f_y) \quad \text{and} \quad f_{\overline{z}} = \frac{1}{2}(f_x + i f_y).$$

Suppose f is orientation-preserving. Then the Jacobian

$$J(f)(z) = |f_z|^2 - |f_{\overline{z}}|^2 = u_x v_y - u_y v_x > 0.$$

So $|f_{\overline{z}}| < |f_z|$. This implies that

$$(|f_z| - |f_{\overline{z}}|)|dz| \leq |dw| \leq (|f_z| + |f_{\overline{z}}|)|dz|.$$

Thus dw maps the unit circle to an ellipse whose longest axis is $|f_z|+|f_{\overline{z}}|$ and whose shortest axis is $|f_z|-|f_{\overline{z}}|$. So the ratio of the major to minor axis is

$$D_f = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \geq 1.$$

This is called the dilatation of f at z. let

$$d_f = \frac{|f_{\overline{z}}|}{|f_z|} \leq 1.$$

Then

$$D_f = \frac{1+d_f}{1-d_f} \quad \text{and} \quad d_f = \frac{D_f - 1}{D_f + 1}.$$

The mapping f is conformal iff $D_f = 1$, $d_f = 0$.

Suppose $R = [0,a] \times [0,b]$ and $R' = [0,a'] \times [0,b']$ are two rectangle. Suppose $f : R \to R'$ is a C^1 map such that it maps the four corner points of R to the four corner points of R' in order. Assume that $m = a/b \leq m' = a'/b'$. A curve is called a a-side if its two endpoints are in $[0,a] \times \{0\}$ and $[0,a] \times \{1\}$, respectively. So f must take any a-side to a a'-side. This implies that

$$a' \leq \int_0^a |df(x+iy)| \leq \int_0^a (|f_z| + |f_{\overline{z}}|)dx.$$

So

$$a'b \leq \int_0 a \int_{0,b} (|f_z| + |f_{\overline{z}}|)dxdy$$

and

$$(a'b)^2 \leq \int_0 a \int_{0,b} \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dxdy \int_0 a \int_{0,b} (\frac{|f_z|^2 - |f_{\overline{z}}|^2}{d} xdy = a'b' \int_0 a \int_{0,b} D_f dxdy.$$

This implies that

$$\frac{m'}{m} \leq \frac{ab}{\int_0 a \int_{0,b} D_f dxdy}$$

Therefore,

$$\frac{m'}{m} \leq \sup D_f.$$

This says that if $m' \neq m$, then f can not be conformal. Let $K = supD_f \geq 1$. We call f a K-quasiconformal map from R to R'. This says that $K \geq m'/m$. The minimum is attained for the affine mapping

$$f(z) = \frac{1}{2}\Big(\frac{a'}{a} = \frac{b'}{b}\Big)z + \frac{1}{2}\Big(\frac{a'}{a} - \frac{b'}{b}\Big)\overline{z}.$$

A quadrilateral Q is a Jordan domain D, together with a pair of disjoint closed arcs on the boundary ∂D (called the b arcs). Then there is a conformal map f maps Q to a rectangle $R = [0, a] \times [0, b]$ such that the b-arcs are mapped to the b-sides of R. The number $m(Q) = a/b$ is unique. We call $m(Q)$ the module of Q.

Suppose Ω is a region in the complex plane. $f : \Omega \to f(\Omega)$ is a orientation-preserving homeomorphism. Then f maps any quadrilateral Q to a quadrilateral $f(Q)$ in $f(\Omega)$.

Definition 10 (Geometric Definition)**.** The map f is K-quasiconformal if

$$\frac{1}{K} \leq \frac{m(f(Q))}{m(Q)} \leq K$$

for any quadrilateral Q in Ω.

We say a real function $u(x, y)$ is ACL (absolutely continuous on lines) in the region Ω if for any closed rectangle $R \subset \Omega$ with sides parallel to the x-axis and to the y-axis, $u(x, y)$ is absolutely continuous on a.e. horizontal lines and a.e. on vertical lines. Such a function has partial derivatives a.e. in Ω.

Definition 11 (Analytic Definition)**.** The map f is K-quasiconformal if

1) f is ACL in Ω;
2) $f_{\overline{z}} \leq k|f_z|$, a.e. in Ω (where $k = (K-1)/(K+1)$.

Theorem 24. *The geometric definition and the analytic definition are equivalent.*