Dynamical Systems and Quasiconformal Mappings:
A Course Given in Department of Mathematics at CUNY Graduate Center
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Purpose of this course

The purpose of this course is designed to study geometric structures of invariant sets of discrete dynamical systems in real one-dimension, complex one-dimension, and real two-dimension. We would like to bring quasiconformal mapping theory into this study. We would also like to study some geometric structure of some space of dynamical systems with a same topological type. In this study, we would like to bring Teichmüller theory into the study. Some key words that we will constantly deal with are

- fixed point and periodic point
- invariant set and Julia set
- invariant measure and Gibbs measure
- transfer operator and Hilbert metric
- Markov partition and symbolic coding
- quasiconformal mapping and quasisymmetric mapping
- holomorphic motion and Teichmüller space
- quasiconformal rigidity and smooth rigidity and rigidity

Main goal in the study of dynamical systems: Make a prediction for the future from past known data under certain rules for a typical starting point.

Setting:

- A topological space \( X \) called ”phase space”, whose elements are called points;
- ”time”, which could be integer time \( n \) or continuous time \( t \);
- a map \( f : X \to X \) or a flow \( \phi^t \) on \( X \);

What we want to know is the limits of \( \{f^n(x)\}_{n=0}^\infty \) for a typical \( x \) in \( X \) under iterations or the limits of the flow line \( \phi^t(x) \) for a typical \( x \) when the time \( t \) goes to infinity for a typical point \( x \).
We call the group \( \{f^n\}_{n=-\infty}^{\infty} \) if \( f \) is invertible or the semi-group \( \{f^n\}_{n=0}^{\infty} \) if \( f \) is non-invertible a discrete dynamical system. We call the flow \( \phi^t \) a continuous dynamical system.

**Definition 1.** A continuous map \( F(t, x) = \phi^t(x) : \mathbb{R} \times X \to X \) is called a flow if for every \( t \in \mathbb{R} \), \( \phi^t : X \to X \) is a homeomorphism and if it satisfies

\[
\phi^{t+s}(x) = \phi^t(\phi^s(x))
\]

for all \( t, s \in \mathbb{R} \) and \( x \in X \).

If \( X \) is a smooth manifold and \( F(t, x) \) is a smooth map, then for any \( x \in X \), \( \{\phi^t(x)\}_{t \in \mathbb{R}} \) is a smooth curve in \( X \). Let

\[
V(x) = \left. \frac{\partial F(t, x)}{\partial t} \right|_{t=0} = \left. \frac{d\phi^t(x)}{dt} \right|_{t=0}.
\]

Then \( V \) defines a continuous vector field on the tangent bundle on \( TX = \bigcup_{x \in X} T_xX \). Thus the flow \( \phi^t(x) \) is just the solution of the ordinary differential equation

\[
\frac{dy}{dt} = V(x) \quad \text{with the initial condition } y(0) = x.
\]

Actually solutions of an ordinary differential equation on a complete smooth manifold for a vector field with certain smooth regularity form a flow. In many situations, we can use a discrete dynamical system to study a continuous dynamical system. This is due to a smart observation from Poincaré.

**Poincaré map:** Suppose \( \phi^t(x) \) is a continuous flow on an \( m \)-dimensional smooth manifold. Suppose there are a point \( x_0 \in X \) and a time \( t_0 > 0 \) such that \( \phi^{t_0}(x_0) = x_0 \). Then \( \{\phi^t(x_0)\}_{t \in \mathbb{R}} \) is called a closed orbit, where the smallest such \( t_0 > 0 \) is called the period. Suppose \( V(x_0) \neq 0 \). Since \( M \) is locally \( \mathbb{R}^m \), we can find an \( (m - 1) \)-dimensional submanifold \( N \) of \( X \) transversal to \( V(x_0) \) at \( x_0 \), that is,

\[
T_{x_0}X = T_{x_0}N \oplus \{tV(x_0)\}.
\]

Since \( \phi^{t_0}(x_0) = x_0 \) and since \( F(t, x) = \phi^t(x) \) is continuous, we have a neighborhood \( U \) about \( x_0 \) in \( N \) such that for every \( x \in N \), there is the smallest \( t(x) > 0 \) close to \( t_0 \) such that \( \phi^{t(x)}(x) \in N \). Define

\[
f(x) = \phi^{t(x)}(x) : U \to N.
\]

Then \( f \) defines a discrete dynamical system with \( f(x_0) = x_0 \). So the study of the closed orbit \( \{\phi^t(x_0)\}_{t \in \mathbb{R}} \) for the continuous dynamical system \( \phi^t \) is equivalent to the study of the fixed point of the discrete
dynamical system \( \{f^n\}_{n=0}^{\infty} \). Here \( f \) is called a section map, or Pincaré map, or first return map.

**Suspension:** For a diffeomorphism \( f : X \to X \) of a smooth manifold \( X \), we can construct a suspension flow on the suspension manifold \( X_f \). Here the suspension manifold is obtained by gluing \((f(x),0)\) and \((x,1)\) on \( X \times [0,1] \). The suspension flow \( \phi^t \) is integral curves of the "vertical" vector field \( \partial / \partial t \) on \( X_f \).

**Example 1.** Let \( X = [0,1] \) and \( f(x) = 1 - x : X \to X \). Then \( X_f \) is the Möbius strip. The suspension flow \( \phi^t \) has a period one closed orbit which does not separate \( M_f \) and one period two closed orbit which is the boundary of \( M_f \) and infinitely many period two closed orbits, each of them cuts \( M_f \) into two parts: one is topologically equivalent to \( M_f \) and the other is topologically equivalent to \([0,1] \times S^1\), where \( S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \) is the unit circle.

**Example 2.** Let \( X = S^1 \) and \( f(x) = -z : X \to X \). Then \( X_f \) is topologically equivalent to the two torus \( T^2 = S^1 \times S^1 \).

In this course, we will more concentrate on discrete dynamical systems.

**Dynamics of a discrete dynamical system with contraction property:**

Suppose \((X,d)\) is a metric space where \( d = d(\cdot,\cdot) \) is the metric. A map \( f : X \to X \) is called contracting if there exists \( 0 < \lambda < 1 \) such that

\[
d(f(x), f(y)) \leq \lambda d(x, y), \quad \forall x, y \in X,
\]

where \( \lambda \) is called a contracting constant. A point \( x \in X \) is called a fixed point if \( f(x) = x \).

**Theorem 1** ([Contacting Mapping Principle]). If \((X,d)\) is a complete metric space and if \( f : X \to X \) is contracting with the contracting constant \( \lambda \), then \( f \) has a unique fixed point \( x_0 \) in \( X \) and \( f^n(x) \) tends to \( x_0 \) exponentially when \( n \) goes to infinity for every \( x \) in \( X \).

**Proof.** For \( x, y \in X \) and \( n > 0 \), we have

\[
d(f^n(x), f^n(y)) \leq \lambda d(f^{n-1}(x), f^{n-1}(y)) \leq \cdots \leq \lambda^n d(x, y)
\]

for some \( 0 < \lambda < 1 \). This implies that \( \{f^n(x)\}_{n=0}^{\infty} \) have the same asymptotic behavior for all \( x \in X \) as \( n \to \infty \).
Now consider a sequence \( \{f^n(x)\}_{n=0}^{\infty} \). For any \( m > n > 0 \),
\[
d(f^m(x), f^n(x)) \leq \sum_{k=0}^{n-m-1} d(f^{n+k+1}(x), f^{n+k}(x))
\leq \sum_{k=0}^{n-m-1} \lambda^{n+k} d(f(x), x) \leq \frac{\lambda^n}{1-\lambda} d(f(x), x) \to 0, \quad \text{as} \quad n \to \infty.
\]

Thus \( \{f^n(x)\}_{n=0}^{\infty} \) is a Cauchy sequence and since \( X \) is complete, it has a limiting point \( x_0 \in X \). Because
\[
f(x_0) = f(\lim_{n \to \infty} f^n(x)) = \lim_{n \to \infty} f^{n+1}(x) = x_0,
\]
\( x_0 \) is a fixed point of \( f \).

If \( y_0 \) is another fixed point, since
\[
d(y_0, x_0) = d(f(y_0), f(x_0)) \leq \lambda d(x_0, y_0),
\]
we have that \( d(y_0, x_0) = 0 \). So \( y_0 = x_0 \). This is the uniqueness. We complete the proof. \( \square \)

The following result says that the unique fixed point \( x_0 \) of a contracting map \( f \) is preserved by \( C^0 \)-perturbation and the fixed point changes continuously under \( C^0 \)-topology.

**Corollary 1.** Take \( f \) as that in the above theorem. For any \( \epsilon > 0 \), there is a \( \delta \in (0, 1-\lambda) \) such that for any map \( g : X \to X \) with
1) \( d(f(x), g(x)) < \delta \) for all \( x \in X \) and
2) \( d(g(x), g(y)) \leq (\lambda + \delta)d(x, y) \) for all \( x, y \in X \).
the fixed point \( y_0 \) of \( g \) satisfies \( d(y_0, x_0) < \epsilon \).

**Proof.** Take \( \delta = \epsilon(1-\lambda)/(1+\epsilon) \). Since \( g^n(x_0) \to y_0 \), we have
\[
d(x_0, y_0) \leq \sum_{k=0}^{\infty} d(g^k(x_0), g^{k+1}(x_0)) \leq \sum_{k=0}^{\infty} (\lambda + \delta)^k d(x_0, g(x_0))
\leq \frac{\delta}{1-\lambda-\delta} = \epsilon.
\]

Now we can formulate the above theorem and the corollary into a modern version. Suppose \( C(X) \) be the space of all continuous maps from \( X \) into \( X \). A map \( f \in C(X) \) is called Lipschitz if there is a constant \( L > 0 \) such that
\[
d(f(x), f(y)) \leq Ld(x, y), \quad \forall x, y \in X.
\]
The smallest number $L > 0$ is called the Lipschitz constant and denoted as $L_f$. Then

$$L_f = \sup_{x \neq y \in X} \frac{d(f(x), f(y))}{d(x, y)}.$$ 

Let $LC(X)$ be the space of all Lipschitz map in $C(X)$. We can define a Lipschitz metric $d_L$ on $LC(X)$ as

$$d_L(f, g) = \sup_{x \in X} d(f(x), g(x)) + |L_f - L_g|.$$ 

It is clear that (1) $d_L(f, g) = d_L(g, f)$, (2) $d_L(f, g) = 0$ if and only if $f = g$, and (3)

$$d_L(f, g) = \sup_{x \in X} d(f(x), g(x)) + |L_f - L_g|$$

$$\leq \sup_{x \in X} d(f(x), h(x)) + \sup_{x \in X} d(h(x), g(x)) + |L_f - L_h + L_h - L_g|$$

$$\leq \sup_{x \in X} d(f(x), h(x)) + |L_f - L_h| + \sup_{x \in X} d(h(x), g(x)) + |L_h - L_g|$$

$$= d_L(f, h) + d_L(h, g).$$

Therefore, $d_L$ is a metric on $LC(X)$. (Note that it may happen that $d_L(f, g) = \infty$ for some $f, g \in LC(X)$, but one can modify it by defining

$$\tilde{d}_L(f, g) = \max\{d_L(f, g), 1\}$$

in this case. Since we only care about the case when $d_L(f, g)$ small, so we still use the original form of $d_L$. All contracting maps from $X$ to $X$ form a subspace

$$LC_1(X) = \{f \in LC(X) \mid L_f < 1\}.$$ 

The metric is $d_L$ restricted on $LC_1(X)$.

**Theorem 2** (Modern Version of Contacting Mapping Principle). If $(X, d)$ is a complete metric space, then for every $f \in LC_1(X)$, there is a unique $x(f) \in X$ such that $f(x(f)) = x(f)$. Moreover,

$$F : (LC_1(X), d_L) \rightarrow (X, d); \quad F(f) = x(f)$$

is a continuous operator from the metric space $(LC_1(X), d_L)$ to the metric space $(X, d)$.

**Periodic point:**

For a dynamical system, $f : X \rightarrow X$, we call a point $x \in X$ a periodic point of period $m > 0$ if $f^m(x) = x$ but $f^i(x) \neq x$ for all $1 \leq i \leq m - 1$. We use $Per_m(f)$ to denote the set of all period points.
of period $m$. In particular, if $m = 1$, $x$ is called a fixed point. We use $\text{Fix}(f)$ to denote the set of all fixed points.

**Exercise 1.** Suppose $f(z) = z^d : S^1 \to S^1$. Find $\text{Fix}(f^m)$, $\text{Per}_m(f)$.

Suppose $X$ is an $n$-dimensional $C^1$ manifold and $f : X \to X$ is a $C^1$ map. Suppose $p$ is a periodic point of period $m$. Then $f^m(p) = p$. The derivative $Df^m(p) : T_pX \to T_pX$ is a linear operator where $T_pX$ is the tangent space of $X$ at $p$. Since we only consider local dynamics for $f$ near $p$, we can introduce local coordinates near $p$ with $p$ as the origin 0. In these coordinates $Df^m(0)$ becomes an $n \times n$ matrix. So suppose $U$ is a neighborhood of 0 in $\mathbb{R}^n$ and $f$ is a $C^1$ map defined on $U$ ($f : U \to f(U)$ with $f(0) = 0$). Let $\| \cdot \|$ be a fixed norm on $\mathbb{R}^n$. Let

$$B(r) = \{ x \in \mathbb{R}^n \mid \| x \| \leq r \}$$

be the closed ball of radius $r > 0$ and centered 0. For $r$ small, $B(r) \subset U$. Consider $X = B(r)$. Let $C^1(X)$ be the space of all $C^1$ maps; each is defined on some neighborhood of $X$. For any $g \in C^1$, we can define the $C^0$ norm for $g$ as

$$\| g \|_0 = \sup_{x \in X} \| g(x) \|.$$

Let

$$Dg(x) : \mathbb{R}^n \to \mathbb{R}^n$$

be the derivative of $g$ at $x$. As a linear operator, we can define its $C^0$ norm as

$$\| Dg(x) \|_0 = \sup_{\| v \| = 1} \| Dg(x)v \|.$$

Then $C^0$ norm for $Dg$ is defined as

$$\| Dg \| = \sup_{x \in X} \| Dg(x) \|_0.$$

The $C^1$ norm of $g$ is

$$\| g \|_1 = \| g \|_0 + \| Dg \|_0.$$

The $C^1$ distance between two maps $g, h \in C^1(X)$ is

$$d_1(g, h) = \| g - h \|_1.$$

We say $g$ is close to $f$ in the $C^1$-topology if their $C^1$ distance $d_1(g, f)$ is small.

**Theorem 3** (Preserving of Periodic Point by Small Perturbation). *If $Df^m(0)$ does not have one as an eigenvalue then every map $g$ sufficiently closed to $f$ in the $C^1$-topology has a unique periodic point of period $m$ close to $0$.***
Proof. Consider \( F = f^m - id \). Then its derivative \( DF(0) = Df^m(0) - I \). Since 1 is not among the eigenvalues of \( DF(0) \), the map \( F = f^m - id \) is locally invertible by the Inverse Function Theorem. More precisely, by picking \( r \) small, \( F : X \to F(X) \) is invertible and its inverse \( F^{-1} : F(X) \to X \) is also a \( C^1 \) map.

Suppose \( g \) is a map close to \( f \) in the \( C^1 \)-topology. We can write \( g^m = f^m - \eta \) where \( \eta \) has a small \( C^1 \) norm
\[
\|\eta\|_1 = \|\eta\|_0 + \|D\eta\|_0.
\]

Now we need to solve the equation,
\[
x = g^m(x) = f^m(x) - \eta(x) = F(x) + x - \eta(x).
\]

Equivalently, solve the equation
\[
F(x) - \eta(x) = 0
\]
or the equation
\[
x = F^{-1}(\eta(x)).
\]

A key argument in the proof is that since the \( C^0 \) norm of the derivative \( DF^{-1} \) is bounded and the \( C^1 \) norm of \( \eta \) is small,
\[
F^{-1} \circ \eta : X \to X
\]
is a contracting map (note that \( X \) is a complete space with the distance \( d(x, y) = \|x - y\| \)). More precisely, let \( L = \|DF^{-1}\|_0 \) be the \( C^0 \)-norm of \( DF^{-1} \) and let
\[
\|\eta\|_1 = \|\eta\|_0 + \|D\eta\|_0 \leq \epsilon
\]
be the \( C^1 \) norm of \( \eta \). We get
\[
\|F^{-1}(\eta(x)) - F^{-1}(\eta(y))\| \leq \epsilon L \|x - y\|
\]
for \( x, y \in X \) and, since \( F^{-1}(0) = 0 \),
\[
\|F^{-1}(\eta(0))\| = \|F^{-1}(\eta(0)) - F^{-1}(0)\| \leq L \|\eta(0)\| \leq \epsilon L,
\]
and hence
\[
\|F^{-1}(\eta(x))\| \leq \|F^{-1}(\eta(x)) - F^{-1}(\eta(0))\| + \|F^{-1}(\eta(0))\| \leq \epsilon L \|x\| + \epsilon L.
\]
Thus if
\[
\epsilon \leq \frac{r}{L(1 + r)},
\]
then
\[
F^{-1} \circ \eta : X \to X
\]
is a contracting map. From the Contracting Mapping Principle, \( F^{-1} \circ \eta \) has a unique fixed point \( x = x(g) \in B(r) \). Thus \( g^m(x) = x \). Furthermore, \( \|x(g)\| \to 0 \) as \( d_1(g, f) \to 0 \).
By taking \( r > 0 \) small, we can assume that \( f^i(B(r)) \cap B(r) = \emptyset \) for \( 1 \leq i \leq n - 1 \). When \( g \) is \( C^1 \) close to \( f \), this property is still kept, that is, \( g^i(B(r)) \cap B(r) = \emptyset \) for \( 1 \leq i \leq n - 1 \). So \( g^i(x) \neq x \) for \( 1 \leq i \leq n - 1 \). This says that \( x \) is a periodic point of period \( m \) for \( g \). □

**Dynamics of Linear Maps and Local Dynamics:**

Suppose \( X \) is an \( n \)-dimensional \( C^1 \)-manifold and suppose \( f : X \to X \) is a \( C^1 \) map with a periodic point \( p \) of period \( m > 0 \). Then the derivative \( Df^m(p) \) is a linear map from \( T_pX \to T_pX \) where \( T_pX \) is the tangent space of \( X \) at \( p \). In a ”good” situation, the local dynamics of \( f \) near \( p \) is determined by this linear map. Here a ”good” situation, we will refer to when \( Df^m(p) \) is ”hyperbolic” as we will explain later. By considering local coordinates at \( p \), we can think \( Df^m(p) \) as an \( n \times n \) matrix \( A \). Thus the linear map \( Df^m(p) : T_pX \to T_pX \) can be thought as

\[
y = Ax : \mathbb{R}^n \to \mathbb{R}^n.
\]

Let \( sp(A) \) be the set of all eigenvalues of \( A \) and define

\[
r(A) = \max \{|a| \mid a \in sp(A)\}
\]

be the spectral radius of \( A \). It is a quantity independent of choice of norms on \( \mathbb{R}^n \).

Given any norm \( \| \cdot \| \) on \( \mathbb{R}^n \), we define the norm of \( A \) as

\[
\|A\| = \sup_{\|v\|=1} \|Av\|.
\]

Clearly, \( \|A\| \geq r(A) \).

**Lemma 1.** For any \( \delta > 0 \) there exists a norm on \( \mathbb{R}^n \) such that

\[
\|A\| < r(A) + \delta.
\]

**Proof.** First we can find a basis in \( \mathbb{R}^n \) such that \( A \) has Jordan normal form. That is,

\[
A = \begin{pmatrix}
A_1 & 0 \\
& \ddots & 0 \\
& 0 & A_k
\end{pmatrix}
\]

where each block is either a Jordan block

\[
A = \begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \lambda & 1 \\
0 & 0 & & \lambda
\end{pmatrix}
\]
for a real eigenvalue $\lambda$ or

$$A = \begin{pmatrix} \rho R \phi & Id & \cdots & 0 \\ \rho R \phi & Id & 0 \\ \vdots \\ 0 & \cdots & \rho R \phi \end{pmatrix}$$

for a pair of complex eigenvalues $\lambda = \rho e^{i\phi}$ and $\bar{\lambda} = \rho e^{-i\phi}$, where

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$R \phi = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$

Now consider a diagonal change of coordinate

$$\begin{pmatrix} 1 \\ \delta^{-1} \\ \vdots \\ 0 & 0 & \cdots & \delta^{-m+1} \end{pmatrix}$$

for a Jordan block for a real eigenvalue and

$$\begin{pmatrix} Id \\ \delta^{-1} Id \\ \vdots \\ 0 & 0 & \cdots & \delta^{-m+1} Id \end{pmatrix}$$

for a pair of complex eigenvalues. Now for the standard Euclidean norm with respect to this new basis, we have that

$$\|A\| = \sup_{\|v\|=1} \|Av\| \leq r(A) + \delta.$$ 

□

Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on $\mathbb{R}^n$ are said to be equivalent if there is a constant $C > 0$ such that

$$C^{-1}\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1 \quad \forall v \in \mathbb{R}^n.$$ 

Since all norms in $\mathbb{R}^n$ are equivalent, we have that

**Corollary 2.** Given any norm $\| \cdot \|$ on $\mathbb{R}^n$, for any $\epsilon > 0$, there exists a constant $C = C(\epsilon) > 0$ such that

$$\|A^n v\| < C(r(A) + \epsilon)^n\|v\|.$$
Definition 2. In general, we call a map \( f : X \to X \) contracting if there are two constants \( C > 0 \) and \( 0 < \lambda < 1 \) such that
\[
d(f^n(x), f^n(y)) \leq C\lambda^n d(x, y)
\]
for all \( n > 0 \) and \( x, y \in X \).

Exercise 2. Prove the Contracting Mapping Principle (Theorem 1) under this more general definition.

Corollary 3. If all eigenvalues of \( A \) have absolute values less than one, then \( A : \mathbb{R}^n \to \mathbb{R}^n \) is contracting with respect to any norm on \( \mathbb{R}^n \). And \( A \) has a unique fixed point \( 0 \) and \( A^m v \to 0 \) exponentially as \( m \to \infty \) for every \( v \in \mathbb{R}^n \).

Exercise 3. Suppose \( A \) is a \( 2 \times 2 \) matrix whose all eigenvalues have absolute values less than one. Discuss the dynamics of \( A : \mathbb{R}^2 \to \mathbb{R} \) in different cases.

Definition 3. A linear map \( A : \mathbb{R}^n \to \mathbb{R}^n \) is called hyperbolic if all of its eigenvalues have absolute values different from one.

For every real eigenvalue \( \lambda \), let
\[
E_{\lambda} = \{ v \in \mathbb{R}^n \mid (A - \lambda I)d)v = 0 \text{ for some } k \}
\]
be its root space. Similarly, for a pair of complex eigenvalues \( \lambda \) and \( \overline{\lambda} \), we can consider \( A : \mathbb{C}^n \to \mathbb{C}^n \) as a complex map. Then we can define its root space \( E_{\lambda} \) and \( E_{\overline{\lambda}} \). Let
\[
E_{\lambda, \overline{\lambda}} = \mathbb{R}^n \cap (E_{\lambda} \otimes E_{\overline{\lambda}}).
\]
Let
\[
E^s = E^s(A) = \bigoplus_{|\lambda|<1} E_{\lambda} \oplus \bigoplus_{|\lambda|<1} E_{\lambda, \overline{\lambda}}
\]
and
\[
E^u = E^u(A) = \bigoplus_{|\lambda|>1} E_{\lambda} \oplus \bigoplus_{|\lambda|>1} E_{\lambda, \overline{\lambda}}
\]
and
\[
E^0 = E^0(A) = \bigoplus_{\lambda=1,-1} E_{\lambda} \oplus \bigoplus_{|\lambda|=1} E_{\lambda, \overline{\lambda}}.
\]
We call \( E^s \), \( E^u \), and \( E^0 \) the stable space, the unstable space, and the central space, for \( A \), respectively. Then
\[
\mathbb{R}^n = E^s \oplus E^0 \oplus E^u
\]
and
\[
A(E^s) \subseteq E^s, \quad A(E^0) \subseteq E^0, \quad \text{and} \quad A(E^u) \subseteq E^u.
\]
Moreover, $A^m v \to 0$ for $v \in E^s$ and $A^{-m} v \to 0$ for $v \in E^u$ as $m \to +\infty$. We say that $A$ is hyperbolic is equivalent to say that $E^0 = \{0\}$, that is,

$$\mathbb{R}^n = E^s \oplus E^u.$$  

**Definition 4.** Suppose $X$ is an $n$-dimensional $C^1$ manifold and $f : X \to X$ is a $C^1$ map. Suppose $p$ is a periodic point of period $m > 0$ of $f$. We call $p$ a hyperbolic periodic point if the derivative $Df^m(p) : T_pX \to T_pX$ is a hyperbolic linear map.

**Theorem 4.** (Hartman-Grobman Theorem) Suppose $f : X \to X$ is a $C^1$ map from an $n$-dimensional $C^1$ manifold $X$ to itself. Suppose $p$ is a hyperbolic fixed point of $f$. Then there are neighborhoods $U_1$ and $U_2 = f^{-1}(U_1)$ of $p$ in $X$ and neighborhoods $V_1$ and $V_2 = (Df(p))^{-1}(V_1)$ of $0$ in $T_pX = \mathbb{R}^n$ and a homeomorphism $h : U_1 \to V_1$ such that $h(U_1 \cap U_2) = V_1 \cap V_2$ and

$$h \circ f = Df(p) \circ h$$
on $U_1 \cap U_2$. In other words, $f|(U_1 \cap U_2)$ and $Df(p)|(V_1 \cap V_2)$ are conjugate.

The above theorem says that for a hyperbolic fixed point $p$ of $f$, the local dynamics of $f$ near $p$ is same as the local dynamics of the linear map $Df(p)$ topologically.

**Exercice 4.** Prove the Hartman-Grobman Theorem when all eigenvalues of $Df(p) : T_pX \to T_pX$ have absolute values $> 1$ by using the Contracting Mapping Principle. Can you find another way to prove it under the same assumption?

**Problem 1.** Under what condition, can we promote $h$ to be $C^1$. More general, under what condition, can we promote $h$ to be $C^2$, $\ldots$, $C^k$, or $C^\omega$ if $f$ is a $C^2$, $\ldots$, $C^\omega$. This discussion should divide into cases: real one-dimension and real higher dimension. Similarly, for a complex map, we should discuss it into cases: complex one-dimension and complex higher dimension.

In one real and complex dimensional cases, we have some kinds of complete solutions, which we will talk this later. However, in higher dimensional cases, it is a difficult problem.

In order to understand a non-hyperbolic linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, we need consider $A : E^0 \to E^0$. This is divided into three cases: (1) $A : E_1 \to E_1$; (2) $A : E_{-1} \to E_{-1}$; (3) $A : E_\lambda \to E_\lambda$. The case (1) is an identity map when $A$ is restricted on the eigenspace and the case (2) is a reflection when $A$ is restricted on the eigenspace. These
two cases are trivial. The case (3) is rotation when it is restricted on the eigenspace, that is, the eigenspace $\mathbb{R}^2$ is foliated into circles and $A$ restricted on each circle is a rotation. Thus we only need to discuss the dynamics of $A$ restricted on the unit circle. Let

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

be the unit circle and, if $\lambda = e^{2\pi i \phi}$, then

$$A(z) = e^{2\pi i \phi} z : S^1 \to S^1.$$

If $\phi = p/q$ with $(p, q) = 1$, then every point in $S^1$ is a periodic point of period $q$. If we arrange the orbit $\{A^n z\}_{n=0}^\infty$ counter-clockwise on $S^1$ as $z_0 = z, z_1, \ldots, z_{q-1}$, then $A z_i = z_{i+p \ (\text{mod} \ q)}$ for all $0 \leq i \leq q - 1$.

More interesting dynamics of a rotation is when $\phi$ is an irrational number.

**Definition 5.** Suppose $X$ is a topological space and $f : X \to X$ is an invertible map. We call $f$ topologically transitive if there exists a point $x \in X$ such that its orbit $O(x) = \{f^m(x)\}_{m=-\infty}^\infty$ is dense in $X$. It is called minimal if the orbit $O(x)$ is dense in $X$ for every $x \in X$.

A subset $Y \subset X$ is called invariant if $f(Y) \subset Y$. Then the statement that $f$ is minimal is equivalent to say that $f$ has no invariant closed subset.

**Theorem 5.** If $\phi$ is irrational, then $A : S^1 \to S^1$ is minimal.

**Proof.** Let $\overline{O}(x) = \{f^m(x)\}_{m=-\infty}^\infty$ be the closure of the orbit of any point $x \in S^1$. If it is not dense, the complement $S^1 \setminus \overline{O}(x)$ is an non-empty invariant open set, which is a union of open intervals. Let $I$ be one of the longest intervals. Since the rotation preserves the length, we have that $|I| = |A^m I|$ for all $m$. But $A^k I \cap A^l I = \emptyset$ for any $k \neq l$. This is because that if $A^k I \cap A^l I \neq \emptyset$, then $A^k I = A^l I$ (otherwise $A^k I \cup A^l I$ is a longer interval than $I$ in $S^1 \setminus \overline{O}(x)$, contradiction). This implies that $A^{k-l} : I \to I$. With loss of generality, suppose $k > l$. Then there is a point $x \in S^1$ such that $A^{k-l} x = x$. Suppose $x = e^{2\pi i \theta}$. Then

$$e^{2\pi i ((k-l) \phi + \theta)} = e^{2\pi i \theta}.$$ 

Thus

$$[(k-l) \phi + \theta] = \theta + p$$

for some integer $p$. Therefore, $\phi = p/(k-l)$ is a rational number. This contradicts to our assumption that $\phi$ is irrational.

Since $\bigcup_{n=\infty}^{\infty} A^n I \subset S^1$, we have that

$$\infty = \sum_{n=-\infty}^{\infty} |A^n I| \leq |S^1| < \infty.$$
This is a contradiction. The contradiction says that $S^1 \setminus \overline{O}(x)$ must be non-empty. We have proved the theorem.

Irrational circle rotations is a basic example for the study of dynamics of topological groups. A topological group $G$ is a group with a topology such that each group multiplication by a fixed element in the group is a continuous map from the topological space into itself and the inverse is also a continuous map from the topological space into itself. For example, the unit circle $S^1$ is an Abelian topological group.

Let $L_{g_0} : G \to G$ be the group multiplication by $g_0$

$$L_{g_0}g = g_0g : G \to G.$$ 

The orbit of the unit element $e \in G$ is cyclic group $\{g_0^n\}_{n=-\infty}^{\infty}$. From Theorem 5, $S^1$ has not proper infinite closed subgroup.

**Theorem 6.** If $L_{g_0}$ is topologically transitive, then it is minimal.

**Proof.** Since $L_{g_0}$ is topologically transitive, there is a $g$ such that $O(g) = \{g_0^n g\}_{n=-\infty}^{\infty}$ is dense in $G$. Now for any $g' \in G$, $g_0^n g' = g_0^n g((g^{-1}g')$. Thus the orbit $O(g') = O(g)(g^{-1}g')$. Since $g^{-1}g' : G \to G$ is continuous, $O(g')$ is dense in $G$. □

Another example of an Abelian topological groups is a torus $T^n = \prod_{i=1}^{n} S^1 = \{w = (z_1, \cdots, z_n) | z_i \in S^1\}$

which plays a central role in the study of completely integrable Hamiltonian systems. The multiplication

$$w \cdot w' = (z_1 z'_1, \cdots, z_n z'_n)$$

if $w = (z_1, \cdots, z_n)$ and $w' = (z'_1, \cdots, z'_n)$. The group multiplication by $w_0$ is

$$L_{w_0}w = w_0 \cdot w.$$ 

**Theorem 7.** Suppose $w_0 = (e^{2\pi i \phi_1}, \cdots, e^{2\pi i \phi_n})$. The group multiplication $L_{w_0}$ is minimal if and only if the number $\phi_1, \cdots, \phi_n$, and 1 are rationally independent, that is, if $\sum_{i=1}^{n} k_i \phi_i = 0 \pmod{1}$, then $k_1 = k_2 = \cdots = k_n = 0$.

**Exercise 5.** Prove this theorem. You can refer to Katok and Hasselblatt's book "Introduction to the Modern Theory of Dynamical Systems", Cambridge University, pages 28-31.
Maps with extremely complicated dynamics but we now understand quite well:

Suppose \((X,d)\) is a metric space. By the definition, \(X\) is called compact if every cover \(\{U_\alpha\}_{\alpha \in \Lambda}\) by open sets has a sub-cover \(\{U_{\alpha_n}\}_{n=1}^m\) consisting of finite number of open sets in this cover. Equivalently, \(X\) is compact if every sequence \(\{x_n\}_{n=1}^\infty\) has a convergent subsequence \(\{x_{n_m}\}_{m=1}^\infty\). A closed subset of a compact space is compact. The space \(X\) is called totally disconnect if every connected component of \(X\) contains only one point. A point \(x \in X\) is called a limiting point if for every neighborhood \(U\) of \(x\), there is a \(y \neq x\) in \(U\). Let \(X'\) be the set of all limiting points. Then \(X\) is called perfect if \(X = X'\). Topologically, a Cantor set is a compact, totally disconnected, and perfect metric space.

Suppose \(I = [0,1]\) is the unit interval and \(I_0 = [0,a]\) and \(I_1 = [b,1]\) are two subintervals where \(0 < a < b < 1\). A \(C^1\) map \(f\) defined on \(I_0 \cup I_1\) is said to be degree two if \(f|I_i\) from \(I_i\) to \(I\) is bijective for \(i = 0, 1\); \(f\) is said to be expanding if there are constants \(C > 0\) and \(\lambda > 1\) so that \(|(f^m)'(x)| \geq C\lambda^m\) whenever \(f^m(x)\) are in \(I_0 \cup I_1\) for all \(i = 0, 1, \ldots, n - 1\).

Suppose \(f : I_0 \cup I_1 \to I\) is a degree two expanding map. Let \(G = (a,b)\) be the complement of \(I_0 \cup I_1\) in \(I\). A number \(x\) in \(I\) is said to be escaping to \(G\) if \(f^k(x)\) is in \(G\) for some integer \(k \geq 0\) (where \(f^0\) is the identity). The set \(\Omega \subseteq I\) of escaping points is an open subset of the real line. The complement \(\Lambda\) of \(\Omega\) in \(I\) is called the non-escaping set under \(f\). It is a compact (closed and bounded) subset of the real line \(\mathbb{R}\). We also call \(\Lambda\) the maximal invariant set of \(f\) in \([0,1]\).

A linear example

**Example 3.** (\(\frac{1}{3}\)-Cantor set). Suppose \(a = 1/3\) and \(b = 2/3\). Define

\[
f(x) = \begin{cases} 
3x, & \text{if } 0 \leq x \leq a; \\
3x - 2, & \text{if } b \leq x \leq 1 
\end{cases}
\]

Then \(f\) is a degree two expanding map for which the non-escaping set \(\Lambda\) under \(f\) is the famous \(\frac{1}{3}\)-Cantor set.

**Proof.** Every point \(x \in [0,1]\) can be written as

\[
x = \sum_{n=1}^{\infty} \frac{i_n}{3^n}
\]
where \( i_n \in \{0, 1, 2\} \). Then
\[
\Lambda = \{ x = \sum_{n=1}^{\infty} \frac{i_n}{3^n} \in [0, 1] \mid i_n \in \{0, 2\} \}.
\]

It is a closed subset. So it is compact.

For any
\[
x = \sum_{n=1}^{\infty} \frac{i_n}{3^n}
\]
and any \( m > 0 \), let
\[
x_m = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i_m^*}{3^m} + \sum_{n=m+1}^{\infty} \frac{i_n}{3^n},
\]
where \( i_m^* = 2 - i_m \). Then \( x_m \neq x \) and \( x_m \to x \) as \( m \to \infty \). This says \( x \in \Lambda' \). So \( \Lambda \) is perfect.

Suppose \( x \neq y \in \Lambda \). Then there is an integer \( m > 0 \) such that
\[
x = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i_m}{3^m} + \sum_{n=m+1}^{\infty} \frac{i_n}{3^n}
\]
and
\[
y = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i_m'}{3^m} + \sum_{n=m+1}^{\infty} \frac{i_n'}{3^n}
\]
where \( i_m \neq i_m' \). Let us assume that \( i_m = 0 \) and \( i_m' = 2 \) and define
\[
z = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{1}{3^m} + \sum_{n=m+1}^{\infty} \frac{1}{3^n}.
\]

Then
\[
U = (-\infty, z) \quad \text{and} \quad V = (z, +\infty)
\]
are two open sets. It is clear that \( U \cap V = \emptyset \) and \( \Lambda \subset U \cup V \) and \( x \in U \) and \( y \in V \). Therefore, \( \Lambda \) is a Cantor set.

**Theorem 8.** Any two Cantor sets are topologically equivalent.

**Proof.** Suppose \((X, d)\) is a Cantor set. Let \( \Lambda \) be the 1/3-Cantor set. We only need to prove that \( X \) is homeomorphic to \( \Lambda \).

Each non-empty open set \( U \) of \( X \) (respectively, \( \Lambda \)) can be divide into two non-empty open sets \( U_1 \) and \( U_2 \) such that \( U = U_1 \cup U_2 \) and \( U_1 \cap U_2 = \emptyset \). Therefore, since \( X \) and \( \Lambda \) are compact, we can divide
\[
X = X_1 \cup \cdots X_n \quad \text{and} \quad \Lambda = \Lambda_1 \cup \cdots \Lambda_n
\]

into disjoint non-empty compact sets such that all of them have diameters < 1. Suppose we have divided
\[ X = \bigcup_{i_{0}}^{i_{k}} X_{i_{0} \cdots i_{k}} \quad \text{and} \quad \Lambda = \bigcup_{i_{0}}^{i_{k}} \Lambda_{i_{0} \cdots i_{k}} \]
into disjoint non-empty compact sets such that all of them have diameters < 1/k. Then each pair \( X_{i_{0} \cdots i_{k}} \) and \( \Lambda_{i_{0} \cdots i_{k}} \) can be divided into same number disjoint non-empty compact sets such that all of them have diameters < \( 1/(k+1) \), that is,
\[ X_{i_{0} \cdots i_{k}} = \bigcup_{l=1}^{n_{k+1}} X_{i_{0} \cdots i_{k}l} \quad \text{and} \quad \Lambda_{i_{0} \cdots i_{k}} = \bigcup_{l=1}^{n_{k+1}} \Lambda_{i_{0} \cdots i_{k}l} \]
and \( d(X_{i_{0} \cdots i_{k}}), |\Lambda_{i_{0} \cdots i_{k}}| < 1/(k+1) \). Thus we have a sequence of nested non-empty compact set
\[ \cdots \subset X_{i_{0} \cdots i_{k+1}} \subset X_{i_{0} \cdots i_{k}} \subset \cdots \subset X_{i_{0}} \subset X \]
and
\[ \cdots \subset \Lambda_{i_{0} \cdots i_{k+1}} \subset \Lambda_{i_{0} \cdots i_{k}} \subset \cdots \subset \Lambda_{i_{0}} \subset \Lambda. \]
Since \( d(X_{i_{0} \cdots i_{k}}), |\Lambda_{i_{0} \cdots i_{k}}| < 1/k \),
\[ \cap_{k=1}^{\infty} X_{i_{0} \cdots i_{k}} = \{ x_{i_{0} \cdots i_{k}} \} \quad \text{and} \quad \cap_{k=1}^{\infty} \Lambda_{i_{0} \cdots i_{k}} = \{ p_{i_{0} \cdots i_{k}} \} \]
both contain one point each. Set
\[ h(p_{i_{0} \cdots i_{k} \cdots}) = x_{i_{0} \cdots i_{k} \cdots}. \]
Then it is a homeomorphism between \( X \) and \( \Lambda \). \( \square \)

The map \( f \) in Example 3 or in Theorem 5 is a linear map. In order to study the dynamics of a non-linear map, we need first to study some distortion property.

**Naive distortion lemmas:**

Let \( f \) be a function defined on a set \( U \) of the real line \( \mathbb{R} \). It is said to be \( C^1 \) (or \( C^{1+\alpha} \) for \( 0 < \alpha \leq 1 \) or \( C^{1+\text{bv}} \)) if it can be extended to a differentiable function defined on an open set containing \( U \) and if the derivative of the extension is continuous (or is \( \alpha \)-Hölder continuous or is of bounded variation).

Suppose \( f \) is a \( C^1 \) function on a set \( U \) of the real line \( \mathbb{R} \) and \( P_1 = \{ x_i \}_{i=1}^{n} \) and \( P_2 = \{ y_i \}_{i=1}^{n} \) are two sequences of points in \( U \). The number
\[ \log \left| \prod_{i=1}^{n} \frac{f'(x_i)}{f'(y_i)} \right| \]
is called the distortion of \( f \) along \( P_1 \) and \( P_2 \).
Lemma 2. Suppose \( \kappa = \inf_{x \in U} |f'(x)| > 0 \). Then the distortion of \( f \) along \( P_1 \) and \( P_2 \) can be estimated as

\[
\log \left| \prod_{i=1}^{n} \frac{f'(x_i)}{f'(y_i)} \right| \leq \frac{1}{\kappa} \sum_{i=1}^{n} |f'(x_i) - f'(y_i)|
\]

Proof. The proof of this lemma is easy because

\[
\log \left| \prod_{i=1}^{n} \frac{f'(x_i)}{f'(y_i)} \right| \leq \sum_{i=1}^{n} \log |f'(x_i)| - \log |f'(y_i)|
\]

\[
\leq \frac{1}{\kappa} \sum_{i=1}^{n} |f'(x_i) - f'(y_i)|.
\]

\[\square\]

The next two lemmas are easily derived from Lemma 2.

Lemma 3. (\( C^{1+\alpha} \)-Denjoy distortion lemma). Suppose \( f \) is \( C^{1+\alpha} \) for some \( 0 < \alpha \leq 1 \) and \( \kappa = \inf_{x \in U} |f'(x)| > 0 \). Let \( \iota = \sup_{x \neq y \in U} \left( |f'(x) - f'(y)| / |x - y|^{\alpha} \right) < \infty \). Then the distortion of \( f \) along \( P_1 \) and \( P_2 \) is bounded by \( (\iota/\kappa) \sum_{i=1}^{n} |x_i - y_i|^{\alpha} \), that is,

\[
\log \left| \prod_{i=1}^{n} \frac{f'(x_i)}{f'(y_i)} \right| \leq \frac{\iota}{\kappa} \sum_{i=1}^{n} |x_i - y_i|^{\alpha}
\]

Proof. Since \( |f'(x_i) - f'(y_i)| \leq \iota|x_i - y_i|^{\alpha} \), it follows directly from Lemma 2. \[\square\]

Lemma 4. (\( C^{1+bv} \)-Denjoy distortion lemma). Suppose \( f \) is \( C^{1+bv} \). Then there is a constant \( C > 0 \) so that the distortion of \( f \) along \( P_1 \) and \( P_2 \) is bounded by \( C \), that is,

\[
\log \left| \prod_{i=1}^{n} \frac{f'(x_i)}{f'(y_i)} \right| \leq C,
\]

whenever the open intervals \( I_i \), bounded by \( x_i \) and \( y_i \), for \( i = 1, \ldots, n \), are pairwise disjoint.

Proof. Let \( V \) be the total variation of \( f \) on \( U \). Then \( \sum_{i=1}^{n} |f'(x_i) - f'(y_i)| \) is bounded by \( V \) for \( \{I_i\}_{i=1}^{n} \) are pairwise disjoint. We can take \( C = V/\kappa \). \[\square\]

Dynamics of non-linear expanding maps.
Theorem 9. (C^{1+\alpha}-hyperbolic Cantor set) If $f : I_0 \cup I_1 \to I$ is a $C^{1+\alpha}$ degree two expanding map for some $0 < \alpha \leq 1$. Then the non-escaping set $\Lambda$ under $f$ is a Cantor set whose Lebesgue measure is zero.

Proof. Let $f_i$ be the restriction of the function $f$ to $I_i$, and $g_i = f_i^{-1} : I \to I_i$ be the inverse of $f_i$ for $i = 0$ or 1. We can consider compositions $g_{w_n} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_n}$ for all strings $w_n = i_0 i_1 \ldots i_n$ of 0’s and 1’s. These compositions are contracting; this means that there are constants $C > 0$ and $0 < \mu < 1$ so that $|g'_{w_n}(x)| < C\mu^n$ for all $x$ in $I$.

Suppose $w_n = i_0 i_1 \ldots i_n$ is a string of 0’s and 1’s. Let $I_{w_n} = g_{w_n}(I)$ be the image of $I$ under $g_{w_n}$, and let $G_{w_n} = g_{w_n}^{-1}(G)$ be the set of all the points escaping to $G$ under $g_{w_n}^{-1}$. The union $\bigcup_{w_n} I_{w_n}$ is the set of all points not escaping to $G$ under the iterates $f^k$ for $k = 0, 1, \ldots, n$, where $w_n$ runs over all the strings of 0’s and 1’s of length $n+1$. The set $\{I_{w_n}\}$ is a collection of pairwise disjoint closed intervals and one to one correspondence with the set $\{w_n\}$ of all the strings of 0’s and 1’s of length $n + 1$. Hence $\Lambda = \bigcap_{n=0}^{\infty} \bigcup_{w_n} I_{w_n}$, where $w_n$ runs over all the strings of 0’s and 1’s of length $n + 1$.

Let us first prove that $\Lambda$ is uncountable. For a string $w_n = i_0 i_1 \ldots i_n$ of 0’s and 1’s and a digit $i_{n+1} = 0$ or 1, $I_{w_n i_{n+1}} \subseteq I_{w_n}$ since $I_{i_{n+1}} \subseteq I$. This implies that

$$\cdots \subseteq I_{i_0 i_1 \ldots i_n} \subseteq \cdots \subseteq I_{i_0} \subseteq I$$

and that $I_w = \bigcap_{n=0}^{\infty} I_{i_0 i_1 \ldots i_n}$ is a non-empty closed subset for any infinite string $w = i_0 i_1 \ldots$ of 0’s and 1’s. Hence the set $\{I_w\}$ is a collection of pairwise disjoint non-empty closed subsets and is in one to one correspondence with the uncountable set $\{w = i_0 i_1 \ldots\}$ of all infinite strings of 0’s and 1’s. Hence the set $\{I_w\}$ is uncountable. So too is the set $\Lambda$ because $\Lambda = \bigcup_w I_w$ where $w = i_0 i_1 \ldots$ runs over all infinite strings of 0’s and 1’s.

Since $g_{w_n}$ is contracting, the length of $I_{w_n}$ is less than $C\mu^n$ for any string $w_n = i_0 i_1 \ldots i_n$ of 0’s and 1’s of length $n + 1$. This implies that $I_w$ contains a single number $x_w$, and the map $\pi(w) = x_w$ from $\{w\}$ to $\Lambda$ is bijective. We use this to prove that $\Lambda$ is totally disconnected, that is, every (connected) component $\Pi$ of $\Lambda$ contains only one number. Suppose there is a component $\Pi$ of $\Lambda$ which contains two different numbers $x_w$ and $x_{w'}$ where $w = i_0 i_1 \ldots i_n i_{n+1} \ldots$ and $w' = i_0 i_1 \ldots i_n' i_{n+1} \ldots$ where $i_{n+1} \neq i_{n+1}'$. Both $x_w$ and $x_{w'}$ are in $I_{w_n}$ where $w_n = i_0 i_1 \ldots i_n$. The set $I_{w_n}$ is the union of an open interval $G_{w_n}$ and two closed intervals $I_{w_n i_{n+1}}$ and $I_{w_n i'_{n+1}}$ which are on different sides of $G_{w_n}$. The numbers $x_w$ and $x_{w'}$ are in $I_{w_n i_{n+1}}$ and $I_{w_n i'_{n+1}}$, respectively. Take a
point \( z \) in \( G_{w_n} \). Then

\[
\Pi = \left( \Pi \cap (-\infty, z) \right) \cup \left( \Pi \cap (z, \infty) \right).
\]

This contradicts the statement that \( \Pi \) is a component of \( \Lambda \) and proves that \( \Lambda \) is totally disconnected.

Since \( \Lambda \) is closed, the set \( \Lambda' \) of limit points of \( \Lambda \) is contained in \( \Lambda \). To prove that \( \Lambda \) is a perfect set, we only need to show that \( \Lambda \) is contained in \( \Lambda' \). Let \( x_w \) be a number in \( \Lambda \) and \( w = i_0 i_1 \ldots i_n i_{n+1} \ldots \). Let \( r(i) = i + 1 \pmod{2} \); \( r(i) \) is 1 for \( i = 0 \) and 0 for \( i = 1 \). Take \( w(n) = i_0 \ldots i_{n-1} r(i_n) i_{n+1} \ldots \); \( w(n) \) differs from \( w \) at \( (n + 1)\)th position. Then \( x_{w(n)} \neq x_w \) and both of them are in \( I_{i_0 \ldots i_{n-1}} \). Since the length of \( I_{i_0 \ldots i_{n-1}} \) tends to zero as \( n \) goes to infinity, \( x_{w(n)} \) tends to \( x_w \) as \( n \) goes to infinity. This says that \( x_w \) is a limit point of \( \Lambda \). So \( \Lambda \) is contained in \( \Lambda' \). Hence \( \Lambda \) is a Cantor set.

Now let us prove that the Lebesgue measure of the Cantor set \( \Lambda \) is zero. Let \( m(\cdot) \) mean the Lebesgue measure and let \( |J| \) mean the length of an interval. An inequality which can be easily obtained is

\[
m(\Lambda) \leq \sum_{w_n} |I_{w_n}| < C2^{n+1} \mu^n,
\]

where \( w_n \) runs over all the strings of 0’s and 1’s of length \( n + 1 \). This inequality is true because \( \{I_{w_n}\} \) is a cover of \( \Lambda \) and the total number of the strings of 0’s and 1’s of length \( n + 1 \) is \( 2^{n+1} \). If \( \mu < 1/2 \), it is much easier to see \( m(\Lambda) = 0 \). However, to prove that the Lebesgue measure of \( \Lambda \) is zero for any \( 0 < \mu < 1 \), we need help from Lemma 3.

Suppose \( w_n = i_0 \ldots i_n \) is a string of 0’s and 1’s of length \( n + 1 \). The map \( f_{n+1} \) from \( I_{w_n} \) to \( I \) is a monotone function and its inverse is \( g_{w_n} \). For any two numbers \( x \) and \( y \) in \( I_{w_n} \), let \( x_i = f_{0}^{(n-i+1)}(x) \) and \( y_i = f_{0}^{(n-i+1)}(y) \) for \( i = 0, 1, \ldots, n \). By the mean value theorem and the chain rule, \( |x_i - y_i| < C \mu^i \) and \( \sum_{i=0}^{n+1} |x_i - y_i|^\alpha < C/(1 - \mu^\alpha) \).

According to Lemma 1.2, the distortion of \( f \) along \( X = \{x_i\} \) and \( Y = \{y_i\} \) is bounded by the constant \( C' = (i/\kappa)(C/(1 - \mu^\alpha)) \), that is,

\[
\left| \log \frac{f_{0}^{(n+1)}(x)'}{f_{0}^{(n+1)}(y)'} \right| \leq C',
\]

where \( \nu \) is the Hölder constant of \( f' \) on \( I_0 \cup I_1 \) and \( \kappa = \inf_{x \in I_0 \cup I_1} |f'(x)| \).

This implies that

\[
\frac{|G_{w_n}|}{|I_{w_n}|} \geq e^{-C'} |G|,
\]
since \( G = f^{o(n+1)}(G_{w_n}) \) and \( I = f^{o(n+1)}(I_{w_n}) \). Now we have that
\[
|I_{w_n,0}| + |I_{w_n,1}| \leq (1 - c)|I_{w_n}|
\]
because \( I_{w_n} = I_{w_n,0} \cup G_{w_n} \cup I_{w_n,1} \); moreover,
\[
m(\Lambda) \leq \sum_{w_{n+1}} |I_{w_{n+1}}| = \sum_{w_n} (|I_{w_n,0}| + |I_{w_n,1}|)
\]
\[
\leq (1 - c) \sum_{w_n} |I_{w_n}| \leq \cdots \leq (1 - c)^{n+1}
\]
for all positive integers \( n \). Hence the Lebesgue measure of \( \Lambda \) is zero. \( \square \)

**Remark and Exercise 1.** From the proof, one can see that the non-escaping set \( \Lambda \) of a \( C^1 \) degree two expanding map is a Cantor set in the real line. There is a Cantor set with positive Lebesgue measure. An interesting problem is to construct a \( C^1 \) degree two expanding map whose non-escaping set is a Cantor set with positive Lebesgue measure. Try to study this counter-example. You can refer to Bowen’s paper ”A horseshoe with positive measure.” Invent. Math., 29 (1975), 203-204.

**Horseshoe Maps**

Let \( R \) be a rectangle in \( \mathbb{R}^2 \) with two horizontal sides and two vertical sides. Suppose \( f : R \to f(R) \subset \mathbb{R}^2 \) is a diffeomorphism such that \( R \cap f(R) \) are two separate rectangles \( R_0 \) and \( R_1 \) such that their horizontal sides parallel to the horizontal sides of \( R \) and and their vertical sides are parts of the vertical sides of \( R \). Let \( R^0 \) and \( R^1 \) be two components of \( f^{-1}(R) \cap R \). Suppose \( R^0 \) and \( R^1 \) are also two rectangles whose vertical sides are parallel to those vertical sides of \( R \) and whose horizontal sides are parts of the horizontal sides of \( R \). Suppose \( f : R^0 \to R_0 \) and \( f : R^1 \to R_1 \) are hyperbolic affine maps, contracting in the vertical direction and expanding in the horizontal direction. This map is called a Smale horseshoe map. Let \( \Lambda \) be the maximal invariant set of \( f \) in \( R \).
Then
\[
\Lambda = \cap_{n=-\infty}^{\infty} f^n(R).
\]
The intersection \( R \cap f(R) \cap f^2(R) = (R_0 \cap R_1) \cap f^2(R) \) consists of four rectangles \( R_{ij} \) for \( i, j \in \{0, 1\} \) whose horizontal sides parallel to the horizontal sides of \( R \) and and their vertical sides are parts of the vertical sides of \( R \). Inductively, \( \cap_{k=0}^{n} f^k(R) \) consists of \( 2^n \) thin rectangles whose horizontal sides parallel to the horizontal sides of \( R \) and and their vertical sides are parts of the vertical sides of \( R \). Let
us denote these rectangles as $R_{i_0i_1\cdots i_{n-1}}$ for $i_k \in 0,1$; $k = 0,1,\cdots n-1$. Here we label them as $f(R_{i_0i_1\cdots i_{n-1}}) = R_{i_1\cdots i_{n-1}}$. Then we have that

$$R_{i_0i_1\cdots i_{n-1}} \subset R_{i_0i_1\cdots i_{n-2}} \subset \cdots \subset R_{i_0}.$$  

This implies that

$$R_{i_0i_1\cdots i_{n-1}} = \cap_{k=0}^{\infty} R_{i_0i_1\cdots i_{n-1}}$$

is a horizontal line in $R$. Then

$$R_+ = \cap_{k=0}^{\infty} f^k(R)$$

consists of uncountably many horizontal lines.

Similarly, the intersection

$$R \cap f^{-1}(R) \cap f^{-2}(R) = (R^0 \cap R^1) \cap f^{-2}(R)$$

consists of four rectangles $R^{ij}$ for $i,j \in \{0,1\}$ whose vertical sides parallel to the vertical sides of $R$ and and their horizontal sides are parts of the horizontal sides of $R$. Inductively, $\cap_{k=1}^{\infty} f^{-k}(R)$ consists of $2^n$ thin rectangles whose horizontal sides parallel to the horizontal sides of $R$ and and their vertical sides are parts of the vertical sides of $R$. Let us denote these rectangles as $R_{jn-1j_{n-2}\cdots j_1}$ for $j_k \in 0,1$; $k = 1,\cdots n-1$. Here we label them as $f^{-1}(R_{jn-1j_{n-2}\cdots j_1}) = R_{jn-1\cdots j_2}$. Then we have that

$$R_{jn-1j_{n-2}\cdots j_1} \subset R_{jn-2} \subset \cdots \subset R_{j_1}.$$  

This implies that

$$R_{\cdots j_{n-1}\cdots j_1} = \cap_{k=1}^{\infty} R_{jn-1\cdots j_1}$$

is a vertical line in $R$.

$$R_- = \cap_{k=0}^{\infty} f^k(R)$$

consists of uncountably many horizontal lines.

Give any $w_+ = i_0\cdots i_{n-1}\cdots$, $R_{w_+} \cap R_-$ is a Cantor set on the line $R_{w_+}$. Give any $w_- = \cdots j_{n-1}\cdots j_1$, $R_{w_-} \cap R_+$ is a Cantor set on the line $R_{w_-}$. Then

$$\Lambda = R_+ \cap R_-$$

can be thought as a product of two Cantor set on a horizontal line and on a vertical line.

**Theorem 10.** Consider the dynamical system $f : \Lambda \to \Lambda$. Then the set of periodic points of $f$ are dense in $\Lambda$. And $\text{Per}_n(f)$ contains $2^n$ different points. Furthermore, $f : \Lambda \to \Lambda$ is topologically mixing, that is, for any two open sets $U$ and $V \subset \Lambda$, there exists an integer $N = N(U,V) > 0$ such that $f^n(U) \cap V$ is nonempty for every $n > N$. 
One-side full shift on symbolic space.

In one-dimensional expanding maps, we used symbols to label intervals and points. Here we study a symbolic dynamical system. Through the study of symbolic dynamical systems, we can have a better understanding of dynamical systems.

Let $S = \{0, 1, \cdots , d-1\}$ be a set with the discrete topology. Consider the space

$$\Sigma = \Sigma_d = \prod_{k=0}^{\infty} S = \{w = i_0i_1\cdots i_{k-1}\cdots \mid i_k \in S, \ k = 0, 1, \cdots \}.$$  

We give $\Sigma$ the product topology as follows. Give any $w = i_0i_1\cdots i_{k-1}\cdots$.

Define $[w]_n = \{w' = i_0i_1\cdots i_{n-1}j_n\cdots j_{k-1}\cdots \in \Sigma\}$. It is called an $n$-cylinder of $\Sigma$. We call a cylinder an open set. The set of all cylinders forms a topological basis. Then $\Sigma$ with this topology is a topological space.

We can also introduce a metric on $\Sigma$:

$$d(w, w') = \sum_{k=1}^{\infty} \frac{|i_{k-1} - j_{k-1}|}{d^k}$$

if $w = i_0\cdots i_k\cdots$ and $w' = j_0\cdots j_{k-1}\cdots$. One can check that $d(\cdot, \cdot)$ is a metric on $\Sigma$ and the topology induced from this metric on $\Sigma$ is the same as the product topology.

One side full shift is defined as

$$\sigma : \Sigma \to \Sigma; \ \sigma(i_0i_1\cdots i_{k-1}\cdots) = i_1\cdots i_{k-1}\cdots.$$  

Then it is a continuous map. The dynamical system $\sigma$ is called a symbolic dynamical system.

Suppose $I = [0, 1]$ and $I_0 = [0, b_0], I_1 = [a_1, b_1], \cdots , I_{d-1} = [a_{d-1}, b_{d-1}]$ where

$$a_0 = 0 < b_0 < a_1 < b_1 < \cdots < a_{d-1} < b_{d-1}.$$  

Let

$$f : U = \bigcup_{i=0}^{d-1} I_i \to I$$

be a map such that $f : I_i \to I$ is a $C^1$-diffeomorphism for each $i = 0, \cdots , d-1$. We say $f$ is expanding if there are two constants $C > 0$ and $\lambda > 1$ such that

$$|(f^n)'(x)| \geq C\lambda^n$$
for all $x \in I$ and $n > 0$ such that $f^{n-1}(x)$ is defined. Let

$$\Lambda = \cap_{k=0}^{\infty} f^{-k}(I)$$

be the maximal invariant set of $f$.

**Theorem 11.** Suppose $f : U \to I$ is a $C^1$ expanding map. Then $f : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \Sigma \to \Sigma$. That is, there is a homeomorphism $h : \Sigma \to \Lambda$ such that

$$f \circ h = h \circ \sigma$$
on $\Sigma$.

**Proof.** For each $0 \leq i \leq d-1$, let $g_i = (f|I_i)^{-1} : I \to I_i$. For each $w_m = i_0i_1 \cdots i_{m-1}$, define

$$g_{wm} = g_{i_0} \circ \cdots \circ g_{i_{m-1}}$$

and

$$I_{wm} = g_{wm}(I).$$

Then for every $w = w_m \cdots \in \Sigma$,

$$\cdots \subseteq I_{wn} \subset I_{wn-1} \subset \cdots \subset I_{w1}.$$ Since $f$ is expanding, $|I_{wn}| \to 0$ as $m \to \infty$. So

$$I_w = \cap_{m=1}^{\infty} I_{wm} = \{x_w\}$$

contains only one point. Define $h : \Sigma \to \Lambda$ as $h(w) = x_w$, we have that $h$ is a homeomorphism and

$$f \circ h = h \circ \sigma \quad \Box$$
on $\Sigma$.

**Lemma 5.** All periodic points of period $m > 0$ are

$$Per_m(f) = \{x_w \mid w = (i_0 \cdots i_{m-1})^\infty\}.$$ 

**Exercise 6.** Prove that $\sigma$ (as well as $f$) is topologically transitive but not minimal.

**Exercise 7.** Find all proper closed subsets $A$ contains only finitely many points in $\Sigma$ which is forward invariant, that is, $\sigma(A) \subseteq A$. Find a proper closed subset $A$ contains infinitely many points in $\Sigma$ which is forward invariant.

**Gibbs Measures**

If a physical system of $n$ states with the energies of these states are $E_1, \cdots, E_n$. Suppose that this system is put in contact with a much larger “heat source” which is at temperature $T$. Energy is therefore
allowed to pass between the original system and the heat source. Suppose the temperature $T$ of the heat source remains constant. It is a physical fact derived in statistical mechanics that the probability $p_j$ that state $j$ occurs is given by the Gibbs distribution

$$p_j = \frac{e^{-\beta E_j}}{\sum_{i=1}^{\infty} e^{-\beta E_i}}$$

where $\beta = \frac{1}{kT}$ and $k$ is a physical constant. This is the starting point for the thermodynamical formalism. However, thermodynamical formalism is a mathematical subject which studies Gibbs measures for more general systems.

Consider the symbolic dynamical system $\sigma : \Sigma \to \Sigma$ (or $f : \Lambda \to \Lambda$). A probability measure $\mu$ on $\Sigma$ is called an invariant measure if $\mu(A) = \mu(\sigma^{-1}(A))$ for any Borel set $A$. A function $\phi : \Sigma \to \mathbb{R}$ is called positive if $\phi > 0$ and Hölder continuous if there are two constant $C > 0$ and $0 < \tau < 1$ such that

$$|\phi(w) - \phi(w')| \leq C \tau^m$$

for any $w, w' \in \Sigma$ such that their first $m$ digits are the same. For a positive and Hölder continuous function $\phi$, the function $\psi = \log \phi$ is also Hölder continuous.

**Theorem 12.** Consider the system $(\sigma, \Sigma, \log \phi)$ where $\phi$ is a positive Hölder continuous function. There is a unique probability measure $\mu = \mu_{\log \phi}$ such that

$$C^{-1} \leq \exp[-mP + \sum_{k=0}^{m-1} \log \phi(\sigma^k(w))] \leq C.$$ 

where $C > 0$ is a constant and where $[w_m]$ is any cylinder of $\Sigma$ and $w \in [w_m]$. Here $P = P(\log \phi)$ is a constant depending on $\log \phi$ and is called the pressure for the system.

**Remark 1.** If we consider $f : \Lambda \to \Lambda$ as a $C^{1+\alpha}$ expanding map, let $h : \Sigma \to \Lambda$ be the conjugacy from $\Sigma \to \Lambda$. Consider $\phi(w) = 1/|f'(h(w))|$. Then it is a positive Hölder continuous function on $\Sigma$. Consider

$$t \log \phi(w) = -t \log |f'(h(w))|.$$ 

Then the pressure $P(-t \log |f'(h(w))|)$ is a function of $t$. There is a unique number $0 < t_0 < 1$ such that $P(-t \log |f'(h(w))|) = 0$. This number is just the Hausdorff dimension of $\Lambda$. For more details about Gibbs theory, you can refer to my lecture notes (download site: http://qcpages.qc.cuny.edu/~yjiang/HomePageYJ/Download/)
Dynamics of Circle endomorphisms and Teichmüller theory

Circle endomorphisms

Let \( T = \{ z \in \mathbb{C} \mid |z| = 1 \} \) be the unit circle in the complex plane \( \mathbb{C} \). Suppose \( f : T \to T \) is an orientation-preserving covering map of degree \( d \geq 2 \). We call it a circle endomorphism. Suppose \( h : T \to T \) is an orientation-preserving homeomorphism. We call it in this paper a circle homeomorphism.

For a circle endomorphism \( f \), it has a fixed point. We will assume that \( f(1) = 1 \).

The universal cover of \( T \) is the real line \( \mathbb{R} \) with a covering map \( \pi(x) = e^{2\pi ix} : \mathbb{R} \to T \).

Then every circle endomorphism \( f \) can be lifted to an orientation-preserving homeomorphism \( F : \mathbb{R} \to \mathbb{R}, \quad F(x + 1) = F(x) + d, \quad \forall x \in \mathbb{R} \).

We will assume that \( F(0) = 0 \). Then there is a one-to-one correspondence between \( f \) and \( F \). Therefore, we also call such an \( F \) a circle endomorphism.

Every orientation-preserving circle homeomorphism \( h \) can be lifted to an orientation-preserving homeomorphism \( H : \mathbb{R} \to \mathbb{R}, \quad H(x + 1) = H(x) + 1, \quad \forall x \in \mathbb{R} \).

We will assume throughout this paper that \( 0 \leq H(0) < 1 \). Then there is a one-to-one correspondence between \( h \) and \( H \). Therefore, we also call such an \( H \) a circle homeomorphism.

A circle endomorphism \( f \) is \( C^k \) for \( k \geq 1 \) if the \( k^{th} \)-derivative \( F^{(k)} \) exists and is continuous. And, furthermore, it is called \( C^{k+\alpha} \) for some \( 0 < \alpha \leq 1 \) if \( F^{(k)} \) is \( \alpha \)-Hölder continuous, that is,

\[
\sup_{x \neq y \in \mathbb{R}} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x - y|^\alpha} = \sup_{x \neq y \in [0,1]} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x - y|^\alpha} < \infty.
\]
A $C^1$ circle endomorphism $f$ is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that
\[
(F^n)'(x) \geq C\lambda^n, \quad n = 1, 2, \ldots.
\]

**Definition 6.** A circle homeomorphism $h$ is called quasisymmetric if there is a constant $M \geq 1$ such that
\[
M^{-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq M, \quad \forall x \in \mathbb{R}, \forall t > 0.
\]
Furthermore, it is called symmetric if there is a bounded function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that
\[
\frac{1}{1 + \varepsilon(t)} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R}, \forall t > 0.
\]

**Example 4.** A $C^1$-diffeomorphism of $T$ is symmetric.

However, the class of symmetric homeomorphisms is larger than the class of $C^1$-diffeomorphisms. For example, a symmetric homeomorphism may not necessarily be absolutely continuous.

**Definition 7.** A circle endomorphism $f$ is called uniformly quasisymmetric if there is a constant $M > 0$ such that
\[
M^{-1} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq M
\]
for all $x \in \mathbb{R}$ and $t > 0$ and any $n > 0$.

**Definition 8.** A circle endomorphism $f$ is called uniformly symmetric if there is a bounded function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that
\[
\frac{1}{1 + \varepsilon(t)} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R}, \forall t > 0.
\]

**Example 5.** A $C^{1+\alpha}$, for some $0 < \alpha \leq 1$, circle expanding endomorphism $f$ is uniformly symmetric. Furthermore, $\varepsilon(t) \leq Dt^\alpha$ for some constant $D > 0$ and $0 \leq t \leq 1$.

**Proof.** Since $F(x+1) = F(x) + d$, then $F'(x+1) = F'(x)$ is a periodic function. Since $F$ is $C^{1+\alpha}$, we have a constant $C_1 > 0$ such that
\[
|F'(x) - F'(y)| \leq C_1|x - y|^\alpha, \quad \forall x, y \in \mathbb{R}.
\]
Since $F$ is expanding, we have a constant $C_2 > 0$ and $\lambda > 1$ such that
\[
(F^n)'(x) \geq C_2\lambda^n, \quad \forall x \in \mathbb{R}, \quad n > 0.
\]
For any \( x, y \in \mathbb{R} \) and \( n > 0 \), let \( x_k = F^{-k}(x) \) and \( y_k = F^{-k}(y) \), \( 0 \leq k \leq n \). Then

\[
\log \frac{(F^n)'(x)}{(F^n)'(y)} = \log \frac{(F^n)'(y_n)}{(F^n)'(x_n)} \leq \sum_{k=1}^{n} |\log F'(x_k) - \log F'(y_k)|
\]

\[
\leq \frac{1}{C_2 \lambda} \sum_{k=1}^{n} |F'(x_k) - F'(y_k)| \leq \frac{C_1}{C_2 \lambda} \sum_{k=1}^{n} |x_k - y_k|^\alpha \leq \frac{C_1}{C_2^{1+\alpha}} \sum_{k=1}^{n} \lambda^{-\alpha k} |x-y|^\alpha.
\]

Let

\[
C = \frac{C_1 \lambda^\alpha}{C_2^{1+\alpha} (\lambda^\alpha - 1) \lambda}.
\]

Then we have the following Hölder distortion property:

(1) \( e^{-C|x-y|^\alpha} \leq \frac{(F^n)'(x)}{(F^n)'(y)} \leq e^{C|x-y|^\alpha}, \quad \forall x, y \in \mathbb{R}, \forall n > 0. \)

Furthermore, let

\[
\varepsilon(t) = \begin{cases} 
0 < t \leq 1, & e^{C t^\alpha} - 1, \\
\ v(t) = 0, & t > 1.
\end{cases}
\]

Then \( \varepsilon(t) \) is a bounded function such that \( \varepsilon(t) \to 0 \) as \( t \to 0^+ \) and such that

\[
\frac{1}{1 + \varepsilon(t)} \leq \frac{(F^{-n})'(\xi)}{(F^{-n})'(\eta)} = \frac{|F^{-n}(x + t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x - t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R}, \forall t > 0,
\]

where \( \xi \) and \( \eta \) are two numbers in \([0, 1]\). Thus \( F \) is uniformly symmetric. Furthermore, one can see that \( \varepsilon(t) \leq D t^\alpha \) for some constant \( D > 0 \) and \( 0 \leq t \leq 1 \). We have proved the example. \( \square \)

**Remark 2.** The uniformly symmetric condition is a weaker condition than the \( C^{1+\alpha} \) expanding condition for any \( 0 < \alpha \leq 1 \). For example, a uniformly symmetric circle endomorphism could be totally singular, that is, it could map a set with positive Lebesgue measure to a set with zero Lebesgue measure.

Another example of a uniformly symmetric circle endomorphism is a \( C^1 \) Dini expanding circle endomorphism as follows. Suppose \( f \) is a \( C^1 \) circle endomorphism. The function

\[
\omega(t) = \sup_{|x-y| \leq t} |F'(x) - F'(y)|, \quad t > 0,
\]

is called the modulus of continuity of \( F' \). Then \( f \) is called \( C^1 \) Dini if

\[
\int_0^1 \frac{\omega(t)}{t} dt < \infty.
\]
Suppose \( f \) is a \( C^1 \) Dini expanding circle endomorphism. Let \( C > 0 \) and \( \lambda > 1 \) be two constants such that
\[
(F^n)'(x) \geq C\lambda^n, \quad x \in \mathbb{R}, \quad n \geq 1.
\]
Define
\[
\tilde{\omega}(t) = \sum_{n=1}^{\infty} \omega(C^{-1}\lambda^{-n}t).
\]
Then
\[
\tilde{\omega}(t) \leq \int_{0}^{\infty} \omega(C^{-1}\lambda^{-x}t)dx = \frac{1}{\log \lambda} \int_{0}^{C^{-1}\lambda^{-1}t} \frac{\omega(y)}{y}dy < \infty
\]
for all \( 0 \leq t \leq 1 \) and \( \tilde{\omega}(t) \to 0 \) as \( t \to 0 \).

**Example and Exercise 1.** A \( C^1 \) Dini circle expanding endomorphism \( f \) is uniformly symmetric. Furthermore, \( \varepsilon(t) \leq D\tilde{\omega}(t) \) for some constant \( D > 0 \) and \( 0 \leq t \leq 1 \). You can refer to my paper (download site: http://qcpages.qc.cuny.edu/yjiang/HomePageYJ/Download/2008TeichAndGibbsTh.pdf) “Teichmuller structures and dual geometric Gibbs type measure theory for continuous potentials”.

**Symbolic space and topological representation**

Suppose \( f \) is a circle endomorphism with \( f(1) = 1 \). Consider the preimage \( f^{-1}(1) \). Then \( f^{-1}(1) \) cuts \( T \) into \( d \) closed intervals \( J_0, J_1, \ldots, J_{d-1} \), ordered by the counter-clockwise order of \( T \). Suppose \( J_0 \) has an endpoint 1. Then \( J_{d-1} \) also has an endpoint 1. Let
\[
\varpi_0 = \{ J_0, J_1, \ldots, J_{d-1} \}.
\]
Then it is a Markov partition, that is,

i. \( T = \bigcup_{k=0}^{d-1} J_k \),

ii. the restriction of \( f \) to the interior of \( J_i \) is injective for every \( 0 \leq i \leq d - 1 \),

iii. \( f(J_i) = T \) for every \( 0 \leq i \leq d - 1 \).

Let \( I_0, I_1, \ldots, I_{d-1} \) be the lifts of \( J_0, J_1, \ldots, J_{d-1} \) in \([0,1]\). Then we have that

i) \([0,1]\) = \( \bigcup_{k=0}^{d-1} I_k \),

ii) \( F(I_i) = [i, i+1] \) for every \( 0 \leq i \leq d - 1 \).

Let
\[
\eta_0 = \{ I_0, I_1, \ldots, I_{d-1} \}.
\]
Then it is a partition of \([0,1]\).
Consider the pull-back partition $\varpi_n = f^{-n}\varpi_0$ of $\varpi_0$ by $f^n$. It contains $d^n$ intervals and is also a Markov partition of $T$. Intervals $J$ in $\varpi_n$ can be labeled as follows. Let $w_n = i_0i_1 \cdots i_{n-1}$ be a word of length $n$ of 0’s, 1’s, $\cdots$, and $(d-1)$’s. Then $J_{w_n} \in \varpi_n$ if $f^k(J_{w_n}) \subset J_{i_k}$ for $0 \leq k \leq n - 1$. Then

$$\varpi_n = \{J_{w_n} \mid w_n = i_0i_1 \cdots i_{n-1}, i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots, d-1\}.$$ 

Let $\eta_n$ be the corresponding lift partition of $\varpi_n$ in $[0, 1]$ with the same labelings. Then

$$\eta_n = \{I_{w_n} \mid w_n = i_0i_1 \cdots i_{n-1}, i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots, d-1\}.$$ 

Consider the space

$$\Sigma = \prod_{n=0}^{\infty} \{0, 1, \cdots, d-1\}$$ 

$$= \{w = i_0i_1 \cdots i_k \cdots i_{n-1} \cdots \mid i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots \}$$

with the product topology. It is a compact topological space. A cylinder for a fixed word $w_n = i_0i_1 \cdots i_{n-1}$ of length $n$ is

$$[w_n] = \{w' = i_0i_1 \cdots i_{n-1}i_{n+1}' \cdots \mid i'_{n+k} \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots \}$$

All left cylinders form a topological basis of $\Sigma$. We call it the left topology. The space $\Sigma$ with this left topology is called the symbolic space.

For any $w = i_0i_1 \cdots i_{n-1}i_n \cdots$, let

$$\sigma(w) = i_1 \cdots i_{n-1}i_n \cdots$$

be the shift map. Then $(\Sigma, \sigma)$ is called a symbolic dynamical system.

For a point $w = i_0 \cdots i_{n-1}i_n \cdots \in \Sigma$, let $w_n = i_0 \cdots i_{n-1}$. Then

$$\cdots \subset J_{w_n} \subset J_{w_{n-1}} \subset \cdots \subset J_{w_1} \subset T.$$ 

Since each $J_{w_n}$ is compact,

$$J_w = \cap_{n=1}^{\infty} J_{w_n} \neq \emptyset.$$ 

If every $J_w = \{x_w\}$ contains only one point, then we define the projection $\pi_f$ from $\Sigma$ onto $T$ as

$$\pi_f(w) = x_w.$$ 

The projection $\pi_f$ is 1-1 except for a countable set $B = \{w = i_0i_1 \cdots i_{n-1}1000 \cdots, i_0i_1 \cdots i_{n-1}0(d-1)(d-1)(d-1) \cdots \}$. From our construction, one can check that

$$\pi_f \circ \sigma(w) = f \circ \pi_f(w), \quad w \in \Sigma.$$
For any interval $I = [a, b]$ in $[0, 1]$, we use $|I| = b - a$ to mean its Lebesgue length. Let

$$
i_n,f = \max_{w_n} |I_{w_n}|,$$

where $w_n$ runs over all words of $\{0, 1, \cdots, d - 1\}$ of length $n$.

Two circle endomorphisms $f$ and $g$ are topologically conjugate if there is an orientation-preserving circle homeomorphism $h$ of $T$ such that

$$f \circ h = h \circ g.$$

The following result is first proved by Shub for $C^2$ expanding circle endomorphisms 1960’s by using the contracting mapping theorem.

**Theorem 13.** Let $f$ and $g$ be two circle endomorphisms such that both $i_{n,f}$ and $i_{n,g}$ tend to zero as $n \to \infty$. Then $f$ and $g$ are topologically conjugate if and only if their topological degrees are the same.

**Proof.** Topological conjugacy preserves the topological degree. Thus if $f$ and $g$ are topologically conjugate, then their topological degrees are the same.

Now suppose $f$ and $g$ have the same topological degree. Then they have the same symbolic space. Since both sets $J_{w,f} = \{x_w\}$ and $J_{w,g} = \{y_w\}$ contain only a single point for each $w$, we can define

$$h(x_w) = y_w.$$ 

One can check that $h$ is an orientation-preserving homeomorphism with the inverse

$$h^{-1}(y_w) = x_w.$$ 

Therefore, for a fixed degree $d > 1$, there is only one topological model $(\Sigma, \sigma)$ for dynamics of all circle endomorphisms of degree $d$ with $i_n \to 0$ as $n \to \infty$.

**Bounded geometry and uniformly quasisymmetry and quasisymmetric conjugacy**

**Definition 9.** The sequence $\{\varpi_n\}_{n=0}^{\infty}$ of nested partitions of $T$ is said to have bounded nearby geometry if there is a constant $C > 0$ such that for any $n \geq 0$ and any two intervals $I, I' \in \eta_n$ with a same endpoint or one has an endpoint 0 and the other has an endpoint 1 (in which case
we say they have a common endpoint by modulo 1),
\[ C^{-1} \leq \frac{|I'|}{|I|} \leq C. \]
The sequence \( \{\varpi_n\}_{n=0}^{\infty} \) of nested partitions of \( T \) is said to have bounded geometry if there is a constant \( C > 0 \) such that
\[ \frac{|L|}{|I|} \geq C, \quad \forall \ L \subset I, \ L \in \eta_{n+1}, \ I \in \eta_n, \ \forall \ n \geq 0. \]

The bounded nearby geometry implies the bounded geometry since each interval \( I \in \eta_n \) is divided into \( d \) subintervals in \( \eta_{n+1} \). But it is not true for the other direction.

**Theorem 14.** If \( f \) is a uniformly quasisymmetric circle endomorphisms, then the sequence \( \{\varpi_n\}_{n=0}^{\infty} \) of nested partitions of \( T \) has bounded nearby geometry and thus bounded geometry.

**Proof.** Let \( F \) with \( F(0) = 0 \) be the lift of \( f \). Define
\[ G_k(x) = F^{-1}(x + k) : [0, 1] \to [0, 1], \quad \text{for} \quad k = 0, 1, \ldots, n - 1. \]
For any word \( w_n = i_0i_1 \cdots i_{n-1} \), define
\[ G_{w_n} = G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_{n-1}}. \]
Then
\[ I_{w_n} = G_{w_n}([0, 1]) = F^{-n}([m, m + 1]), \]
where \( m = i_{n-1} + i_{n-2}d + \cdots + i_0d^{n-1} \). Suppose \( I'_{w_n} \) is an interval in \( \eta_n \) having a common endpoint with \( I_{w_n} \) modulo 1. Then
\[ I'_{w_n} = F^{-n}([m + 1, m + 2]) \quad \text{or} \quad F^{-n}([m - 1, m]). \]
Thus
\[ \frac{1}{1 + \varepsilon(1)} \leq \frac{|I_{w_n}|}{|I'_{w_n}|} \leq 1 + \varepsilon(1). \]
Let \( C = 1 + \varepsilon(1) \). Then we have that
\[ C^{-1} \leq \frac{|I|}{|I'|} \leq C \]
for any intervals \( I, I' \in \eta_n \) with a common endpoint modulo 1, \( n = 0, 1, \ldots \). This means that \( \{\varpi_n\}_{n=0}^{\infty} \) has the bounded nearby geometry. We proved the theorem. \( \square \)

**Remark 3.** The converse is also true in the above theorem. Refer to the proof of the following corollary.
Example 6. Consider $q_d(z) = z^d$, $d \geq 2$. Then it is a circle endomorphism of degree $d$. The lift of $q$ is $Q(x) = dx$. Then we have
\[
\eta_0 = \left\{ \left\{ \frac{k}{d}, \frac{k+1}{d} \right\}_{k=0}^{d-1} \right\}
\]
and
\[
\eta_{n-1} = \left\{ \left\{ \frac{k}{d^n}, \frac{k+1}{d^n} \right\}_{i=0}^{d^n-1} \right\}.
\]
Every $0 \leq k < d^{n+1}$, can be expressed as $k = i_{n-1} + i_{n-2}d + \cdots + i_0d^{m-1}$.
So for $w_n = i_0i_1\cdots i_{n-1}$, $I_{w_n} = [i/d^n, (i+1)/d^n]$.

Theorem 15. Any two uniformly quasisymmetric circle endomorphisms $f$ and $g$ of the same degree $d > 1$ are topologically conjugate and the conjugacy is a quasisymmetric homeomorphism.

Proof. From $f \circ h = h \circ g$ and $g(1) = 1$, $h(1)$ is a fixed point of $f$, that is, $f(h(1)) = h(1)$. Let $k(z) = z/h(1)$ and $\tilde{f} = k \circ f \circ k^{-1}$. Then $\tilde{f}(1) = 1$. Take $\tilde{h} = k \circ h$. We have that $\tilde{h}(1) = 1$ and $\tilde{f} \circ \tilde{h} = \tilde{h} \circ g$.
So $\tilde{h}$ is quasisymmetric if and only if $h$ is quasisymmetric. So, without loss of generality, we assume that $h(1) = 1$.

Suppose $\eta_{n,f} = \{I_{w_{n,f}}\}$ and $\eta_{n,g} = \{I_{w_{n,g}}\}$, $n = 1, 2, \cdots$ are two sequences of Markov partitions for $f$ and $g$, respectively.

From the bounded geometry property (Theorem 14), we have a constant $0 < \tau < 1$ such that
\[
\iota_{n,f} = \max_{w_n} |I_{w_{n,f}}|, \quad \iota_{n,g} = \max_{w_n} |I_{w_{n,g}}| \leq \tau^n, \quad \forall \ n = 1, 2, \cdots.
\]
Then Theorem 13 implies that $f$ and $g$ are topologically conjugate.

Suppose $h$ is the topological conjugacy between $f$ and $g$ and $H$ is its lift to $\mathbb{R}$. By adding all integers, the sequence of partitions $\eta_{n,f}$ and $\eta_{n,g}$ induce two sequences of partitions of $\mathbb{R}$, which we still denoted as $\eta_{n,f}$ and $\eta_{n,g}$. Both of these sequences of partitions have bounded nearby geometry.

Let $\Omega$ be the set of all endpoints of intervals $I \in \eta_n$, $n = 0, 1, \cdots, \infty$. Then it is dense in $\mathbb{R}$.

For $x \in \Omega$. Consider the interval $[x-t, x]$. There is a largest integer $n \geq 0$ such that there is an interval $I = [a, x] \in \eta_{n,f}$ satisfying
Suppose $J = [b, x] \in \eta_{n+1,f}$. Then $J \subseteq [x - t, x]$. Let $J' = [x, c] \in \eta_{n+1,f}$. From Theorem 14, there is a constant $C > 0$ such that

$$C^{-1} \leq \frac{|J'|}{|J|} \leq C.$$

If $|J'| > t$, we have $|J'| \leq Ct$. Let $J'_k = [x, c_k] \in \eta_{n+k+1, f}$ for $k > 0$. From the bounded geometry, there is a $0 < \tau < 1$ such that

$$|J'_k| \leq \tau^k Ct.$$

Let $k$ be the smallest integer greater than $-\log C / \log r$. Then $|J'_k| \leq t$. This implies that $J'_k \subseteq [x, x + t]$. So we have

$$\frac{|H(J'_k)|}{|H(I)|} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq \frac{|H(J')|}{|H(I)|},$$

where $H(I) \in \eta_{n,g}$, $H(J), H(J') \in \eta_{n+1,g}$, and $H(J'_k) \in \eta_{n+k+1,g}$. Now from the bounded geometry for $g$, we have a constant, still denote as $C > 0$, such that

$$C^{-1} \leq \frac{|H(J'_k)|}{|H(I)|} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq \frac{|H(J')|}{|H(I)|} \leq C.$$

If $|J'| \leq t$, we have $|J'| \geq C^{-1} t$. Let $J'_{-k} = [x, c_{-k}] \in \eta_{n-k+1, f}$ for $k \geq 0$. Then from the bounded geometry, there is a constant, which we still denote as $0 < \tau < 1$, such that $|J'_{-k}| \geq \tau^{-k} C^{-1} t$. Let $k$ be the smallest integer greater than $-\log C / \log r$. Then $|J'_{-k}| \geq t$. This implies that $J'_{-k} \supseteq [x, x + t]$. So we have

$$\frac{|H(J')|}{|H(I)|} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq \frac{|H(J'_{-k})|}{|H(J)|},$$

where $H(I) \in \eta_{n,g}$, $H(J), H(J') \in \eta_{n+1,g}$, and $H(J'_{-k}) \in \eta_{n-k+1,g}$. Now from the bounded geometry for $g$, we have a constant, which we still denote as $C > 0$, such that

$$C^{-1} \leq \frac{|H(J')|}{|H(I)|} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq \frac{|H(J'_{-k})|}{|H(J)|} \leq C.$$

For any $x \in \mathbb{R}$, since $\Omega$ is dense in $[0, 1]$, we have a sequence $x_n \in \Omega$ such that $x_n \to x$ as $n \to \infty$. For any $t > 0$, we have that

$$C^{-1} \leq \frac{|H(x_n + t) - H(x_n)|}{|H(x_n) - H(x_n - t)|} \leq C.$$

Since $H$ is uniformly continuous on $\mathbb{R}$, we get that

$$C^{-1} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq C.$$

We have proved the theorem. □
Corollary 4. Suppose $f$ is a circle endomorphism. If the sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$ has the bounded geometry. Then $f$ is uniformly quasisymmetric.

Proof. Suppose $h$ is a conjugacy from $f$ to $q$, that is, $f = h \circ q \circ h^{-1}$. Since both sequences of nested Markov partitions for $f$ and for $q$ have bounded nearby and bounded geometry. Then $h$ is quasisymmetric. Since $f^n = h \circ q^n \circ h^{-1}$, so $f$ is uniformly quasisymmetric. \qed
Thm. 4.3.1: If \( f \) and \( g \) are uniformly quasisymmetric circle endomorphisms of degree \( d > 1 \), then they are topologically conjugate by \( h \) and \( h^{-1} \) is quasisymmetric. Moreover, \( f \) and \( g \) both have bounded and bounded nearby geometry.

**Questions:**

1. If \( f \) is almost expanding circle endomorphism, then \( \text{top cong. by } h \Rightarrow \exists \text{ set } q.s. \)

2. (Open problem) \( Q \): Are \( f \) and \( g \) both almost expanding circle endomorphisms and \( C^2 \)? Then \( f \) and \( g \) are top. cong. by \( h \). Are \( h \) q.s.? 

\[
G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor : (0,1) \to (0,1] \quad \text{(cusp map)}
\]

\[
|G'(x)| = -\frac{1}{x^2} > 0, \quad x \in \left( \frac{4}{2+1}, \frac{1}{2} \right)
\]

\[
\frac{1}{x} \in \left( \frac{4}{2+1}, \frac{1}{2} \right)
\]

\[
\frac{1}{x} = a + \frac{1}{a + \frac{1}{a + \cdots}}
\]

\[
G(x) = \frac{1}{a^2 + \frac{1}{a^2 + \cdots}} = [a,0,0,\ldots]
\]

\[
G(x) \text{ is just shift map in symbolic space of } \infty \text{ letters.}
\]
This relates Q1 to Q2 (like G(x)-Gauss map)

\[ f(x) = \mu \times (1-x) \]

\[ f(x) = F(|x|^\alpha), \quad \alpha > 1 \]

\[ F : [0, 1] \rightarrow [0, f(0)] \text{ is a } C^3 \text{ diffeomorphism} \]

Schwarzsian derivative

\[ S(F(x)) = \frac{F''(x)}{F'(x)} - \frac{3}{2} \left( \frac{F''(x)}{F'(x)} \right)^2 < 0 \]

\[ S(F^{-1}) > 0 \]

Q3: \( f \) and \( g \) are both unimodal and are top. conj. by \( h \); i.e. \( f \circ h = h \circ g \). Does \( h \) q.s.?

Specific case: \( f(x) = \frac{F(-|x|^{2n})}{x} \) extend to complex map

can extend this map to: \( \mathbb{C} \rightarrow \mathbb{C} \)

\( \nabla \) \( x \mapsto |x|^3 \)

Can't do: \( x \mapsto |x|^3 \)

Can't extend to: \( \mathbb{C} \rightarrow \mathbb{C}^3 \).
Thm: \( I = [0,1] \), spce \( f : [0,1] \to [0,1] \) has a periodic point of period 3. Then, \( \forall m > 1 \), \( f \) has a periodic point of period \( m \).

(Li-York: period 3 \( \Rightarrow \) chaos)

\[
\begin{align*}
X_0 & \neq X_1 \neq X_2 \\
f(x_0) &= x_1 \\
f(x_1) &= x_2 \\
f(x_2) &= x_0
\end{align*}
\]

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_A = \sigma_{i=0}^{\infty} \alpha_i A^n
\]

Thm ( Sarkovski): Spce \( f : I \to I \) is a cont. map and has a periodic point of period \( m \). Then \( f \) has a periodic point of period \( n \) for every \( m \mid n \).

\[
\begin{cases}
\mu_0(x) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}
\end{cases}
\]

\[
\begin{align*}
\{\mu_n\} & \to \mu_\infty \\
\delta_n & = \frac{\mu_n - \mu_{n-1}}{\mu_n - \mu_{n+1}} \\
\delta_n & = \frac{\mu_n - \mu_{n-1}}{\mu_n + \mu_{n+1}}
\end{align*}
\]

Feigenbaum constant

\[
\delta = \lim_{n \to \infty} \delta_n
\]

Conj: \( \delta_n \) is universal and independent of any family as long as \( \alpha \) is fixed.

\[
\alpha = 2 \implies \text{OK}
\]
For $\mu > \frac{1}{2}$, $\mu_n \to \mu_\infty$

At $\mu = 2$

$f_{\mu_\infty}(x) \to ?$ as $n \to \infty$

Strange attractor

Note: Supse $g \to$ top conj $f_{\mu_\infty}(x) \Rightarrow \rho$, t.q.s.

Af $\exists \mu$ s.t. $f \to$ top conj $f_\mu$ on a dynamical interesting set
Let $QS$ be the space of all (orientation-preserving) quasisymmetric circle homeomorphisms. Let $Q$ be those in $QS$ fixing 1. Let $F$ be the space of all uniformly quasisymmetric circle endomorphisms of degree 2 fixing 1. (The following can be done for all degrees $d > 2$ with a little bit modification). Suppose $q(z) = z^2$. Let $\alpha : Q \to F$ be defined as

$$\alpha(h) = h \circ q \circ h^{-1}.$$

**Theorem 16.** The map $\alpha$ is bijective.

**Proof.** From Theorems 14 and 15 and Corollary 4, we have that $\alpha : Q \to F$ is a onto map. We only need to prove that $\alpha$ is injective. Suppose $\alpha(h_1) = \alpha(h_2)$ for $h_1, h_2 \in Q$. Since 1 is the only fixed point of $\alpha(h_1)$, $\alpha(h_2)$, and $q$, $h_1(1) = h_2(1) = 1$. Since

$$h_1^{-1}(q^{-1}(1)) = h_2^{-1}(q^{-1}(1)),$$

we have that $h_1(-1) = h_2(-1)$. Inductively, since

$$h_1^{-1}(q^{-n}(1)) = h_2^{-1}(q^{-n}(1)),$$

we have that $h_1(e^{2\pi i(k/2^n)}) = h_2(e^{2\pi i(k/2^n)})$ for all $0 \leq k < 2^n$. Since the set of all numbers $\{k/2^n | 0 \leq k < 2^n, n = 1, 2, \cdots \}$ is dense in $T$, we have that $h_1 = h_2$. So $\alpha$ is injective. □


**Dual symbolic space and scaling model for $Q$ and $F$.**

Suppose $f$ is a circle endomorphism of degree 2. As we have seen, we have a sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$. Here $\eta_n$ contains $2^n$ intervals, each interval has a unique label $w_n = i_0 i_1 \cdots i_{n-1}$ which we denote as $I_{w_n}$, where $i_k \in \{0, 1\}$, such that

$$f(I_{w_n}) = I_{\sigma(w_n)}.$$
Now we would like to relabel these intervals. For \( w_n = i_0 i_1 \cdots i_{n-1} \), let \( w^*_n = j_{n-1} \cdots j_1 j_0 \), where \( j_{n-1} = i_0, \ldots, j_0 = i_{n-1} \). Define the dual shift map
\[
\sigma^*(w^*_n) = j_{n-1} \cdots j_1.
\]
Then we have the following
\[
I_{w^*_n} \subset I_{w^*_n}.
\]
We call
\[
\Sigma^* = \{ w^* = \cdots j_{n-1} \cdots j_0 \} = \prod_{n=0}^{\infty} \{0,1\}
\]
with the product topology the dual symbolic space. Then the dual shift map is
\[
\sigma^* : w^* = \cdots j_{n-1} \cdots j_1 j_0 \mapsto \sigma^*(w^*) = \cdots j_{n-1} \cdots j_1.
\]
Then we call \((\Sigma^*, \sigma^*)\) the dual symbolic dynamical system. The dual cylinder for a given \( w^*_n = j_{n-1} \cdots j_1 j_0 \) is
\[
[w^*_n] = \{ w^* = \cdots j_{m-1} \cdots j'_n j_{n-1} \cdots j_0 \mid j'_k \in \{0,1\}, k \geq n \}.
\]
Then all these dual cylinders form a topological basis for \( \Sigma^* \).

Now we define another operator in \( \Sigma^* \) called the adding machine \( a(w^*) \) as follows: If \( w^* = \cdots j_{n-1} \cdots j_1 j_0 \) and \( j_0 = 0 \), then \( a(w^*) = \cdots j_{n-1} \cdots j_1(j_0 + 1) \) and if \( j_0 = 1 \), then \( j_0 + 1 = 0 \) and then consider \( j_1 + 1 \), so on. For each \( w^*_n \), we can also define the adding machine \( a \). We have that for any \( w^*_n, I_{w^*_n} \) and \( I_a(w^*_n) \) are two adjacent intervals. (Note that if \( w^*_n = 1 \cdots 1 \), then we define \( a(w^*_n) = 0 \cdots 0 \).

For each \( w^*_n \), we define two scalings, one is for bounded nearby geometry and one is for bounded geometry:
\[
bng(w^*_n) = \left| \log \frac{|I_{w^*_n}|}{|I_{a(w^*_n)}|} \right|
\]
and
\[
bg(w^*_n) = \frac{|I_{w^*_n}|}{|I_{\sigma^*(w^*_n)}|}.
\]
Thus we have two sets of scalings
\[
BNG = \{ bng(w^*_n) \mid w^*_n = j_{n-1} \cdots j_0, j_k \in \{0,1\} \}
\]
and
\[
BG = \{ bg(w^*_n) \mid w^*_n = j_{n-1} \cdots j_0, j_k \in \{0,1\} \}.
\]
These two sets can be determined to each others.
Exerice 8. Express scalings in BG in terms of scalings in BNG and vice versa.

Therefore, the sequence of nested Markov partitions $\eta = \{\eta_n\}_n=0^\infty$ has bounded nearby geometry if and only if there is a constant $C > 0$ such that

$$bng(w_n^*) \leq C, \forall w_n^*.$$  

And it has bounded geometry if and only if there is a constant $C > 0$ such that

$$bg(w_n^*) \geq C, \forall w_n^*.$$  

It is also clear that

(2)  

$$bg(w_n0) + bg(w_n1) = 1$$

which we call it the summation condition. Now let us consider the space of scalings (means positive numbers)

$$S = \{(S(w_n^*))\}$$

satisfies the summation condition (2) and the induced bounded nearby geometry ($bng(w_n^*)$) is in $l^\infty$. Then for $f \in F$, we have that

$$\gamma(f) = (bg(w_n^*)) \in S.$$  

This induces a map

$$\beta(h) = \gamma \circ \alpha \in S$$

and

Theorem 17. The maps $\beta : Q \to S$ and $\gamma : F \to S$ are both bijective.

Universal Teichmüller space and the space of uniformly quasisymmetric circle endomorphisms and space of scalings

Remember that $QS$ is the space of all quasisymmetric homeomorphisms of $T$. Let $M$ be the space of all Möbius transformations preserving the unit disk $\Delta$. Then

$$M = \{M(z) = e^{2\pi i} \frac{z - a}{1 - \overline{a}z} | |a| < 1\}$$

The quotient space

$$UT = QS/M$$

is called the universal Teichmüller space. It is a complex Banach manifold (we will discuss this later, see Remark 5).
Since each element in $\mathcal{M}$ has one complex parameter $a \in \Delta$ and one real parameter $0 \leq \theta < 1$, for any pair of triple points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ arranged in the counter-clockwise order on $T$, there is a one and only one element $M \in \mathcal{M}$ such that

$$M(z_1) = w_1, \quad M(z_2) = w_2, \quad M(z_3) = w_3.$$  

Thus the universal Teichmüller space can be thought as the space of all quasisymmetric homeomorphisms fixing three points on $T$. Let us assume that this three points are 1, $i$, $-1$. Then

$$\mathcal{U}T = \{h \in QS \mid h(1) = 1, \ h(i) = i, \ h(-1) = -1\} = Q/\mathcal{M}_1$$

where

$$\mathcal{M}_1 = \{M(z) = \frac{1 - \overline{a}}{1 - a} \frac{z - a}{1 - \overline{a}z} \mid |a| < 1\}$$

is the space of all Möbius transformations preserving the unit disk $\Delta$ fixing 1.

Now consider the space of all uniformly quasisymmetric circle endomorphisms of a fixed degree $d > 1$ conjugated by $\mathcal{M}$, which we denote as $\mathcal{U}F_d$. For the simplicity, we assume $d = 2$ and write $\mathcal{U}F = \mathcal{U}F_2$.

**Theorem 18.** The map $\alpha$ induces a bijective map $\varrho = [\alpha] : \mathcal{U}T \to \mathcal{U}F$.

**Proof.** Let $q(z) = z^2$. Then it has a unique fixed point 1 and $q^{-1} = \{1, -1\}$ and $q^{-2}(1) = \{1, i, -1, -i\}$. The space $\mathcal{U}F$ can be thought as the space of those uniformly quasisymmetric circle endomorphisms $f$ fixing 1 such that $f^{-1}(1) = \{1, -1\}$ and $f^{-2}(1) = \{1, i, -1, s\}$. It is clear that of $h \in Q$ fixing 1, $-1$, $i$ if and only if $f = \alpha(h) \in \mathcal{U}F$. Thus $\alpha$ induces a onto map $\varrho$ from $\mathcal{U}T$ to $\mathcal{U}F$. We need to prove $\varrho$ is one-to-one. Suppose $\varrho(h_1) = \varrho(h_2)$. Similar to the proof of Theorem 16, $h_1 = h_2$. \qed

Let $\mathcal{U}S$ be the space of scalings $(s(w_n^*))$ in $S$ such that $s(0) = s(1) = s(00) = 1/2$.

**Corollary 5.** The maps $\gamma$ and $\beta$ induce bijective maps $[\gamma] : \mathcal{U}F \to \mathcal{U}S$ and $[\beta] : \mathcal{U}T \to \mathcal{U}S$.

**Remark 5.** Note that

$$Q = \mathcal{U}T \times \Delta$$

where $\Delta = \{a \mid |a| < 1\}$ is the unit disk in the complex plane. For any $h \in Q$, let $t = h(i)$ and $s = h(-1)$ and

$$a = \frac{1 - i \frac{s(t-1)}{(t-s)}}{1 - i \frac{t-1}{t-s}} \in \Delta.$$
Let

\[ G_a(z) = \frac{1 - \bar{a} z - a}{1 - a \bar{1} - \bar{a} z}. \]

Then \( G_a \) maps 1, \( t, s \) to 1, \( i, -1 \). So \( G_a \circ h \) fixes 1, \( i, -1 \) and is in \( UT \).

So if we define \( \chi(h) = (G_a \circ h, a) \). Then it is a bijective map from \( Q \) to \( UT \times \Delta \). So \( Q \) as well as \( F \) and \( S \) has induced complex Banach manifold structure such that it is a complex Banach manifold. (We will discuss this later.)
Lecture 12

10:00am - 12:00 noon

by Fred Gardner.

\[ \frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}, \quad t^2 - x^2 - y^2 - z^2. \]

Minkowski space in \( \mathbb{R}^4 \).

\[ t^2 - x^2 \in \mathbb{R}^2 \quad (\cosh \alpha \sinh \beta \quad \sinh \alpha \cosh \beta \quad \frac{1}{\sqrt{1-k^2}} \quad \frac{k}{\sqrt{1-k^2}}) \]

Title: Complex structure on Teichmüller space

\[ A(x) = \begin{pmatrix} t' \\ x' \end{pmatrix}, \quad t^2 - x^2 = (t')^2 - (x')^2 \quad \text{invariant under } A. \]

In Euclidean: \( t^2 + x^2 \quad (\cosh \alpha \sinh \beta \quad \sinh \alpha \cosh \beta \quad \frac{1}{\sqrt{1-k^2}} \quad \frac{k}{\sqrt{1-k^2}}) \quad \text{invariant for } t^2 + x^2. \]

\[ \text{PSL}(2, \mathbb{C}) \cong \text{Lorentz gp} = \text{SO}^{+}(1, 3) \]

\[ \text{PSL}(2, \mathbb{R}). \]

\[ t^2 = x^2 + y^2 + z^2 \quad \text{is a cone.} \]
\[ J = \begin{pmatrix} u + iv & 0 \\ u^2 + v^2 + 1 & -2u \\ -2v \\ u^2 + v^2 - 1 \end{pmatrix} \]

\[
\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad \mathcal{E} = 4u^2 + 4v^2 + u^4 + v^4 + 1 - 2u^2 \\
\Rightarrow 2x^2 + 2u^2v^2
\]

\[ x = \begin{pmatrix} t + z \\ x - iy \\ x + iy \\ t - z \end{pmatrix} \]

is a Hermitian 2x2 matrix (i.e. \( x^* = \overline{x} \))

\[ \det = t^2 - x^2 - y^2 - z^2, \quad x, y, z, t \text{ are all real.} \]

\[ P \in SL(2, \mathbb{C}), \quad \det = 1. \]

\[
X \rightarrow PXP^t, \quad P^* = \text{conjugate transpose}.
\]

\[ t^2 - x^2 - y^2 - z^2 = c = P^t \]

\[ t^2 - x - y - z^2 = 0 \]
manifold structure
complex manifold structure

\[ S = \{ f: \Delta \to \mathbb{R}, f(0) = p, f \text{ holomorphic} \} \]
\[ f(\omega) = q \in \mathbb{R} \]

\[ \downarrow w_1 \]

\[ 0 \in \mathbb{C} \]

\[ \mathcal{S}(p) \text{ depends on local coordinate } w. \]

\[ f(p) |dp| \text{ is invariant by local coordinate } w. \]

\[ f(w_1) |dw_1| = f(w_2) |dw_2| \]

The \( f \) that realizes the minimum in (4) is the universal covering of \( \mathbb{R} \) by \( \Delta \).
\[ z \text{ as coordinate} \]

\[ f(z) = p. \]

\[ \Delta \xrightarrow{f} \text{one linear map } f_r : \Delta \to \Delta_r(p) \]

\[ \Rightarrow f_r'(z) = r. \]

Riemann mapping maps \( \Delta \to (0, 1) \), \( \{1, \infty\} \) \((-\infty, 0)\).

\[ a, b, c \to 0, 1, \infty \]

\[ \text{reflect in the triangle } \Rightarrow f : \Delta \to \mathbb{C} \setminus \{0, 1, \infty\} \text{ the universal cover of } \mathbb{C} \setminus \{0, 1, \infty\} \cup \Delta. \]

In general:

\[ l \]
Pick one closed curve and assume it 
\[ A \in \Gamma, \ A \bar{z} = \lambda z, \ \lambda > 1 \quad (0, +i\infty) \text{ in } \mathbb{H}. \]

On \( \lambda \), hyperbolic length of \( \ell_i \).

\[ A^{1/\lambda} = \lambda \]

Frizke-Tsuji-Klein. If \( R \) is of finite type, then a finite number of lengths of homotopy classes of closed curves on \( R \) determine the covering group.

\[ \Gamma_1, \ \Gamma_2 \text{ acting on } \mathbb{H}. \]

\[ w : \mathbb{H} \to \mathbb{H}, \ w \Gamma_1 w^{-1} = \Gamma_2, \quad \text{i.e.} \quad w \gamma w^{-1} = \chi(\gamma) \in \Gamma_2. \]

\( \chi \) is an isomorphism from \( \Gamma_1 \) on \( \Gamma_2 \).

\( \Lambda \) a Möbius transf. \( \in \) determined by its values at 3 pts \( \Lambda(\Gamma) = \text{limit set of } \Gamma. \)

Thought about \( \Lambda \text{ restricted to } \partial \mathbb{H} = \mathbb{R} \cup \delta \mathbb{R} \).

\[ w : \mathbb{R} \to \mathbb{R} \]
\[ \bar{w}(x+\epsilon) - \bar{w}(x) \]
\[ \leq \frac{|\bar{w}(x+\epsilon) - \bar{w}(x)|}{|ar{w}(x) - \bar{w}(x-\epsilon)|} \]

\[ \in M \]
\[ T_2 = \omega T_1 \omega^{-1} \]

\[ \omega(0) = 0, \quad \omega(1) = 1, \quad \omega(\infty) = \infty. \]

\[ \omega_{\epsilon}(z) = z + 3 V(z) + O(\epsilon) \quad z \in \mathbb{H}. \]

\[ \omega(z) = z + 4 V(z) + O(\epsilon), \quad \rho(x, \epsilon) \]

\[ M \leq \frac{|\omega(x+\epsilon) - \omega(x)|}{|\omega(x) - \omega(x-\epsilon)|} \leq M \]

If \( \epsilon = 0 \), \( \rho(x, \epsilon) = 1 \). If \( \epsilon \) small

\[ \frac{1}{1+\delta(\epsilon)} \leq \frac{|\omega_\epsilon(x+\epsilon) - \omega_\epsilon(x)|}{|\omega_\epsilon(x) - \omega_\epsilon(x-\epsilon)|} \leq (1+\delta(\epsilon)) \]

\[ \delta(\epsilon) \to 0 \quad \omega \to 0. \]

\[ \left| \frac{V(x+\epsilon) - V(x-\epsilon) - 2V(x)}{\epsilon} \right| \leq C \epsilon^s (4\epsilon) \]

\[ T_\epsilon = \omega T_\epsilon \omega^{-1} \]

\[ \frac{V(x+\epsilon)}{x(\epsilon)} - V(x) = \text{quadratic polynomial in } x (4\epsilon) \]

\[ \gamma_\delta(z) = \frac{a_{\delta} z^2 + b_{\delta}}{c_{\delta} z^2 + d_{\delta}} \quad \gamma_0(z) = z . \quad a_{\delta}, b_{\delta}, c_{\delta}, d_{\delta} \text{ are differentiable functions of } \delta. \]
\[ \lim_{\delta \to 0} \frac{1}{\delta} \left( \frac{a \delta^2 + b \delta - c \delta^2 - d \delta - \varepsilon}{c \delta^2 + d \delta} \right) = \frac{a_0 + b_0}{c_0 + d_0} \quad \text{where} \quad a_0 = 1, \quad b_0 = 1, \quad c_0 = 0, \quad d_0 = 0. \]

\[ = -c \varepsilon^2 + (a - d) \varepsilon + b. \]

\[ \Rightarrow \text{The } V \text{'s that satisfy } (\#) \text{ comprise the tangent space } T_{\text{Teich}}(\Gamma). \]

\[ Z = \text{a real vector space} \]

\[ \mathbb{I} \cdot z \to z, \quad \mathbb{I}^2 = -\text{identity} \]

\[ (a + cb) V = aV + \mathbb{I}bV, \quad V \in Z. \]

\[ e_1(c_2 V) = (c_1, c_2) V. \]

\[ (a_1 + c_1 b_1) ((a_2 + c_2 b_2) V) = (a_1 + c_1 b_1)(a_2 + c_2 b_2) \quad V \]

Bers embedding: Schwarzian derivative
Hilbert transform:

1) Harmonic conjugate

2) \( PV \int_{-\infty}^{\infty} \frac{V(t)}{t-x} \, dt \)

3) Same sort of integral to periodic functions

   \( \text{stangent (t-x) \ by \ sep \ t \ replace \ (t-x) } \)

4) Pompeiu formula.

   \( \text{Cauchy} \)

\[ V(x) = \frac{1}{\pi} \int \left( \frac{1}{x-y} + \text{convergence term} \right) \mu(s) \, ds \, dy \]

\( m \in \mathcal{L}_\infty. \) Hilbert transform is just \( I \mu, \) i.e.

\[ IV = \frac{1}{\pi} \int \int \left( \frac{1}{x-y} + \text{con. term} \right) \overline{\mu(s)} \, ds \, dy \]

\[ I\mu = \int \mu \ \text{in} \ H^1. \]

\[ -i\mu \ \text{in} \ H^1. \]

\[ \Rightarrow Z \underbrace{\text{under the Hilbert transform is a}} \text{ Complex Banach manifold}. \]
Symmetric homeomorphisms

Let $Q_0$ be the space of all symmetric homeomorphisms in $Q$, that is, $h \in Q_0$ if and only if $h(1) = 1$ and there is a bounded positive function $\epsilon(t) > 0$ such that $\epsilon(t) \to 0$ as $t \to 0$,

$$\frac{1}{1 + \epsilon(t)} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq 1 + \epsilon(t)$$

for all $x \in \mathbb{R}$ and $t > 0$.

For any $h \in Q_0$, consider $f = h \circ q \circ h^{-1}$ and the corresponding sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$ and corresponding scalings $(s(w_n^*)).

**Theorem 19.** The homeomorphism $h \in Q_0$ if and only if for any $w^* \in \Sigma^*$, the limit $S(w^*) = \lim_{n \to \infty} s(w_n^*)$ exists and is $1/2$, that is, it defines a constant function

$$S(w^*) = \frac{1}{2} : \Sigma^* \to \mathbb{R}.$$ 

Before to prove this theorem, we need to have the following concept about quasisymmetry on an interval. Suppose $I$ and $J$ are two intervals on the real line $\mathbb{R}$. Suppose $Q : I \to J$ is a homeomorphism. Let $M \geq 1$. We say that $Q$ is $M$-quasisymmetric on $I$ if

$$M^{-1} \leq \frac{|Q(x + t) - Q(x)|}{|Q(x) - Q(x - t)|} \leq M, \quad \forall x - t, x, x + t \in I, t > 0.$$ 

**Lemma 6.** Suppose $I = J = [0, 1]$. There is a function $\zeta(M) > 0$ satisfying $\zeta(M) \to 0$ as $M \to 1$ such that for any $M$-quasisymmetric homeomorphism $Q : [0, 1] \to [0, 1]$ such that $Q(0) = 0$ and $Q(1) = 1$,

$$|Q(x) - x| \leq \zeta(M), \quad \forall x \in [0, 1].$$
Proof. Consider points \( x_n = 1/2^n, \ n = 0, 1, \ldots \). \( M \)-quasisymmetry and the normalization \( Q(0) = 0, Q(1) = 1 \) imply that
\[
\frac{1}{1 + M} H(\frac{1}{2^n}) \leq Q(\frac{1}{2^n}) \leq \frac{1}{1 + M^{-1}} Q(\frac{1}{2^n-1}).
\]
Similarly,
\[
\left(\frac{1}{1 + M}\right)^n \leq Q(\frac{1}{2^n}) \leq \left(\frac{1}{1 + M^{-1}}\right)^n, \ \forall n \geq 1.
\]
Furthermore, by \( M \)-quasisymmetry and induction on \( n = 1, 2, \ldots \), yield
\[
\left(\frac{1}{1 + M}\right)^n \leq Q(\frac{i}{2^n}) - Q(\frac{i-1}{2^n}) \leq \left(\frac{1}{1 + M^{-1}}\right)^n, \ \forall n \geq 1, \ 1 \leq i \leq 2^n.
\]
Let
\[
\tau_n = \max \left\{\left(\frac{M}{M + 1}\right)^n - \frac{1}{2^n}, \frac{1}{2^n} - \left(\frac{1}{M + 1}\right)^n\right\}, \ n = 1, 2, \ldots .
\]
Then for \( n = 1 \),
\[
|Q(\frac{1}{2}) - \frac{1}{2}| \leq \tau_1 = \frac{1}{2M + 1},
\]
and for any \( n > 1 \), we have
\[
\max_{0 \leq i \leq 2^n} \left|Q(\frac{i}{2^n}) - \frac{i}{2^n}\right| \leq \max_{0 \leq i \leq 2^{n-1}} \left|Q(\frac{i}{2^{n-1}}) - \frac{i}{2^{n-1}}\right| + \tau_n.
\]
By summing over \( k \) for \( 1 \leq k \leq n \), we obtain
\[
\max_{0 \leq i \leq 2^n} \left|Q(\frac{i}{2^n}) - \frac{i}{2^n}\right| \leq \delta_n = \sum_{k=1}^{n} \tau_k.
\]
If we put \( \zeta(M) = \sup_{1 \leq n < \infty} \{\delta_n\} \), by summing geometric series, we obtain
\[
\zeta(M) = \max_{1 \leq n < \infty} \left\{M - 1 + \frac{1}{2^n} - M\left(\frac{M}{1 + M}\right)^n, 1 - \frac{1}{M} + \frac{1}{M}\left(\frac{1}{M}\right)^n - \frac{1}{2^n}\right\}.
\]
Clearly, \( \zeta(M) \to 0 \) as \( M \to 1 \), and since the dyadic points
\[
\{i/2^n \mid n = 1, 2, \ldots ; 0 \leq i \leq 2^n\}
\]
are dense in \([0, 1]\), we conclude
\[
|Q(x) - x| \leq \zeta(M) \ \forall x \in [0, 1],
\]
which proves the lemma. \(\square\)
Proof of Theorem 19. Suppose \( \epsilon(t) > 0 \) is the bounded positive function such that \( \epsilon(t) \to 0 \) as \( t \to 0 \),
\[
\frac{1}{1 + \epsilon(t)} \leq \frac{|H(x + t) - H(x)|}{|H(x) - H(x - t)|} \leq 1 + \epsilon(t)
\]
for all \( x \in \mathbb{R} \) and \( t > 0 \).

For any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( \zeta(\delta) < \epsilon \). Then we have \( \tau > 0 \) such that \( \epsilon(t) < \delta \) for any \( 0 < t < \tau \). Let \( n_0 > 0 \) be an integer such that \( 1/2^{n_0} < \tau \). Then \( H \) on any interval \( I \) with \( |I| < 1/2^{n_0} \) is a \((1 + \delta)\)-quasisymmetric homeomorphism.

Suppose \( w^* = \cdots j_{n-1} \cdots j_1 j_0 \in \Sigma^* \in \Sigma^* \) is any point. Let \( w^*_n = j_{n-1} \cdots j_1 j_0 \) and \( v^*_{n-1} = j_{n-1} \cdots j_1 \).

By definition,
\[
S(w^*_n) = \frac{|I_{w_n}|}{|I_{\sigma^*(v^*_{n-1})}|},
\]
where \( I_{w_n} \subset I_{v^*_{n-1}} \). Consider the sequence \( \{S(w^*_n)\}_{n=1}^\infty \). Then we have that, for any \( n > n_0 \),
\[
H : I_0 = \left[ \frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}} \right] \to I_{v^*_{n-1}}
\]
is a \((1 + \delta)\)-quasisymmetric homeomorphism for some integer \( 0 \leq i \leq 2^{n-1} \). Let \( j \) be another integer such that \( H([j/2^n, (j+1)/2^n]) = I_{w^*_n} \).

Let \( J_0 = [j/2^n, (j+1)/2^n] \).

From Lemma 6 (by normalizing \( I_{v^*_{n-1}} \) and \( I_0 \) to \([0, 1]\) and \( J_0 \) to \([0, 1/2]\) or \([1/2, 1]\) and \( I_{w_n} \) to \([0, x]\) or \([x, 1]\) by linear transformations), we get
\[
|S(w^*_n) - \frac{1}{2}| \leq \zeta(1 + \delta) < \epsilon.
\]
This implies that \( \{S(w^*_n)\}_{n=1}^\infty \) has a limit and the limit is \( 1/2 \).

On the other hand if all \( \{S(w^*_n)\}_{n=1}^\infty \) have limits and the limits are \( 1/2 \). Then following a similar proof to Theorem 15, we can show that for any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( H \) restricted on any interval \( I \) with \( |I| < \delta \) is a \((1 + \epsilon)\)-quasisymmetric.

The universal asymptotically affine Teichmüller space is defined as
\[
T_0 = Q_0/\mathcal{M}_1 = \{ h \in Q_0 \mid h(0) = 0, h(1) = 1, h(i) = i \}.
\]

Corollary 6. The universal asymptotically affine Teichmüller space is one-to-one corresponding to the space of scalings \( \{S(w^*_n)\} \in US \) such that \( \lim_{n \to \infty} S(w^*_n) = 1/2 \) for all \( w^* = \cdots w^*_n \in \Sigma^* \).
Thus if we denote that
\[ S_0 = \{(S(w_n^*)) \in S \mid \lim_{n \to \infty} S(w_n^*) = \frac{1}{2} \forall w^* = \cdots w_n^* \in \Sigma^*\}. \]
and
\[ US_0 = \{(S(w_n^*)) \in US \mid \lim_{n \to \infty} S(w_n^*) = \frac{1}{2} \forall w^* = \cdots w_n^* \in \Sigma^*\}. \]
Then \( \beta : Q_0 \to S_0 \) and \( \beta : T_0 \to US_0 \) are bijective.

**Remark 6.** Since \( T_0 \) is a complex Banach manifold (we will discuss this later), so \( US_0 \) is also a complex Banach manifold. Since 
\[ Q_0 = T_0 \times \Delta \quad \text{and} \quad S_0 = US_0 \times \Delta, \]
they are also complex Banach manifolds.

**Limiting quasisymmetric homeomorphisms and uniformly symmetric circle endomorphisms**

The above section gives us an idea to define the following space
\[ LQ = \{h \in Q \mid S(w^*) = \lim_{n \to \infty} S(w_n^*) \text{ exists for all } w^* \in \Sigma^*\}. \]
Then for each \( h \in LQ \), we define a function
\[ S(w^*): \Sigma^* \to \mathbb{R}. \]
The scaling function satisfies the summation condition
\[ S(w^*0) + S(w^*1) = 1 \quad \forall w^* \in \Sigma^*. \]
We call this function the scaling function for \( h \) as well as for \( f = h \circ q \circ h^{-1} \). Following a similar proof to Theorem 19 we have that

**Theorem 20.** The scaling functions for \( h \) and for \( h_0 \circ h \) (or \( h \circ h_0 \)) are the same for every \( h_0 \in Q_0 \).

Thus the space of all scaling functions is a representation of
\[ LAT = LQ / Q_0 \]
which we call the universal limiting asymptotically conformal Teichmüller space. The universal asymptotically conformal Teichmüller space is defined as
\[ AT = Q / Q_0. \]
Then we have that
\[ LAT \subset AT. \]
Remark and Problem 1. The space $\mathcal{AT}$ cannot be represented by a space of functions defined on ever points in $\Sigma^*$. We would like to use limiting Martingales to represent this space. Note that $\mathcal{AT}$ is a complex Banach manifold. (We will talk about this later.) However, we are still interested in finding the exact complex manifold structure of $\mathcal{LAT}$. It is an interesting question.

**Theorem 21.** Suppose $f = h \circ q \circ h^{-1}$ is a uniformly symmetric circle endomorphism. Then its scaling function

$$S(w^*) : \Sigma^* \to \mathbb{R}^+$$

exists and is a continuous function. Furthermore, if $f$ is $C^{1+\alpha}$, then $S(w^*)$ is Hölder continuous. Actually when $f$ is $C^1$ Dini expanding, the modulus of continuity of $S(f)(w^*)$ is controlled by $\tilde{\omega}(t)$.

**Proof.** Suppose $w^* = \cdots j_{n-1} \cdots j_1 j_0 \in \Sigma^*$. Let

$$w^*_n = j_{n-1} \cdots j_1 j_0 \quad \text{and} \quad v^*_{n-1} = j_{n-1} \cdots j_1.$$

By definition,

$$S(w^*_n) = \frac{|I_{w^*_n}|}{|I_{v^*_{n-1}}|},$$

where $I_{w^*_n} \subset I_{v^*_{n-1}}$. Consider the sequence $\{S(w^*_n)\}_{n=1}^\infty$.

Let $0 < \tau < 1$ be a constant such that

$$i_n = \max_{w^*_n} |I_{w^*_n}| \leq \tau^n, \quad \forall n \geq 1.$$

For any $\epsilon > 0$, let $n_0 > 0$ be an integer such that $\zeta(1 + \epsilon(\tau^{n-1})) \leq \epsilon$ for all $n > n_0$. Then for any $m > n > n_0$, we have that

$$F^{m-n}(I_{v^*_{m-1}}) = I_{v^*_{n-1}} \quad \text{and} \quad F^{m-n}(I_{w^*_m}) = I_{w^*_n}.$$

Since $F^{-(m-n)}|I_{v^*_{n-1}}$ is a $(1+\epsilon(\tau^{n-1}))$-quasisymmetric homeomorphism, from Lemma 6 (by normalizing $I_{v^*_{n-1}}$ to $[0, 1]$ and $I_{w^*_m}$ to $[0, x]$ by a linear transformation),

$$|S(w^*_m) - S(w^*_n)| = \left| \frac{|F^{-(m-n)}(I_{w^*_m})|}{|F^{-(m-n)}(I_{v^*_{n-1}})|} - \frac{|I_{w^*_n}|}{|I_{v^*_{n-1}}|} \right| \leq \zeta(1 + \epsilon(\tau^{n-1})) \leq \epsilon.$$

This implies that $\{S(w^*_n)\}_{n=1}^\infty$ is a Cauchy sequence. Thus the limit

$$S(w^*) = \lim_{n \to \infty} S(w^*_n)$$

exists.

Now consider two points

$$w^* = \cdots j_{m-1} \cdots j_n j_{n-1} \cdots j_0 \quad \text{and} \quad \tilde{w}^* = \cdots j_{m-1} \cdots j'_n j_{n-1} \cdots j_0.$$
Let \( w_m^* = j_{m-1} \cdots j_n j_{n-1} \cdots j_0 \) and \( \tilde{w}_m^* = j_{m-1} \cdots j_n' j_{n-1} \cdots j_0 \). Then \( w_n^* = \tilde{w}_n^* \). For any \( m > n \),

\[
|S(w_m^*) - S(\tilde{w}_m^*)| \\
\leq |S(w_m^*) - S(w_n^*)| + |S(\tilde{w}_m^*) - S(w_n^*)| \leq 2\zeta(1 + \varepsilon(\tau^{n-1})).
\]

So by taking a limit,

\[
|S(w^*) - S(\tilde{w}^*)| \leq 2\zeta(1 + \varepsilon(\tau^{n-1})).
\]

Thus we have that

\[
S(w^*) : \Sigma^* \to \mathbb{R}^+
\]

is a continuous function whose modulus of continuity is bounded by \( 2\zeta(1 + \varepsilon(\tau^{n-1})) \).

Moreover, if \( f \) is a \( C^{1+\alpha} \) expanding circle endomorphism for some \( 0 < \alpha \leq 1 \), from the Hölder distortion property (1), there is a constant \( C > 0 \) such that

\[
|S(w^*) - S(\tilde{w}^*)| \leq C\tau^{\alpha(n-1)}.
\]

This implies that the scaling function \( S(w^*) \) is Hölder continuous.

When \( f \) is \( C^1 \) Dini, then there is a constant \( C > 0 \) such that

\[
|S(w^*) - S(\tilde{w}^*)| \leq C\tilde{\omega}(\tau^{n-1}).
\]

Thus the scaling function \( S(w^*) \) is continuous and its modulus of continuity is controlled by \( \tilde{\omega}(\tau^{n-1}) \). We have proved the theorem. \( \square \)

Define

\[
\mathcal{UA}(q) = \{ h \in \mathcal{Q} \mid f = h \circ q \circ h^{-1} \text{ is uniformly symmetric} \}
\]

Then it can be represented by all scalings \( S(w_n^*) \) such that the limiting scaling functions \( S(w^*) \) exists and is continuous. Since the limiting scaling functions are invariant under \( \mathcal{Q}_0 \), we have that the Teichmüller space of uniformly symmetric circle endomorphisms

\[
\mathcal{AT}(q) = \mathcal{UA}(q)/\mathcal{Q}_0
\]

is represented exactly by the space of all continuous scaling functions

\[
\mathcal{CS} = \{ S(w^*) \mid S : \Sigma^* \to \mathbb{R} \text{ is a continuous scaling function} \}.
\]

Define

\[
\mathcal{HA}(q) = \{ h \in \mathcal{Q} \mid f = h \circ q \circ h^{-1} \text{ is } C^{1+\alpha} \text{ expanding for some } 0 < \alpha \leq 1 \}
\]

Then it can be represented by all scalings \( S(w_n^*) \) such that the limiting scaling functions \( S(w^*) \) exists and is Hölder continuous. Since the limiting scaling functions are invariant under \( \mathcal{Q}_0 \), we have that the Teichmüller space of uniformly symmetric circle endomorphisms

\[
\mathcal{HAT}(q) = \mathcal{HA}(q)/\mathcal{Q}_0 = \mathcal{HA}(q)\mathcal{D}
\]
is represented exactly by the space of all Hölder continuous scaling functions

\[ \mathcal{H} = \{ S(w^*) \mid S : \Sigma^* \to \mathbb{R} \text{ is a Hölder continuous scaling function} \}, \]

where \( \mathcal{D} \) is the space of all circle diffeomorphisms.

Then we have that

\[ \mathcal{H}(q) \subset \mathcal{U}(q) \subset \mathcal{L}(q) \subset \mathcal{Q} \]

\[ \mathcal{H}(q) \subset \mathcal{A}(q) \subset \mathcal{L}(q) \subset \mathcal{A}(q) \]

**Remark 7.** Under the Teichmüller metric (we will talk later), the completion of \( \mathcal{H}(q) \) is \( \mathcal{A}(q) \).

### Quasiconformal maps and quasisymmetric homeomorphisms

**Theorem 22** (Riemann Mapping Theorem). Suppose \( D \) is a simply connected domain in the complex plan \( \mathbb{C} \) such that its complement contains at least one point. Then there is a conformal map \( f \) from \( D \) onto the open unit disk \( \Delta \). Furthermore, if we specific a point \( z_0 \) in \( D \) such that \( f(z_0) = 0 \) and \( f'(z_0) > 0 \), then \( f \) is unique.

In the theorem \( f : D \to \Delta \) is called a Riemann mapping.

**Theorem 23** (Caratheodory’s Extension Theorem). If \( D \) is a Jordan domain, then the Riemann map \( f : D \to \Delta \) can be extended to the boundary of \( D \) such that \( f : \partial D \to S^1 = \partial \Delta \) is a homeomorphism.

Now we specific four points \( z_1, z_2, z_3, z_4 \) on the boundary of \( D \) and other four points \( w_1, w_2, w_3, w_4 \) on the boundary of \( \Delta \). Could you find a conformal \( f : D \to \Delta \) such that its extension to the boundary maps \( z_i \) to \( w_i \) for \( i = 1, 2, 3, 4 \)? This is Grötzsch’s problem. Answer to this question is no in general.

Let \( w = f(z) \) be a \( C^1 \) map. let \( z = x + iy \) and \( w = u + iv \). Then

\[ du = u_x dx + u_y dy \]
\[ dv = v_x dx + v_y dy. \]

If we write it into complex version, we have

\[ dw = f_z dz + f_{\overline{z}} d\overline{z} \]

where

\[ f_z = \frac{1}{2} (f_x - if_y) \quad \text{and} \quad f_{\overline{z}} = \frac{1}{2} (f_x + if_y). \]
Suppose $f$ is orientation-preserving. Then the Jacobian

$$J(f)(z) = |f_z|^2 - |f_{\bar{z}}|^2 = u_x v_y - u_y v_x > 0.$$ 

So $|f_{\bar{z}}| < |f_z|$. This implies that

$$(|f_z| - |f_{\bar{z}}|) |dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|) |dz|.$$ 

Thus $dw$ maps the unit circle to an ellipse whose longest axis is $|f_z| + |f_{\bar{z}}|$ and whose shortest axis is $|f_z| - |f_{\bar{z}}|$. So the ratio of the major to minor axis is

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1.$$ 

This is called the dilatation of $f$ at $z$. let

$$d_f = \frac{|f_{\bar{z}}|}{|f_z|} \leq 1.$$ 

Then

$$D_f = \frac{1 + d_f}{1 - d_f} \quad \text{and} \quad d_f = \frac{D_f - 1}{D_f + 1}.$$ 

The mapping $f$ is conformal iff $D_f = 1$, $d_f = 0$.

Suppose $R = [0, a] \times [0, b]$ and $R' = [0, a'] \times [0, b']$ are two rectangles. Suppose $f : R \to R'$ is a $C^1$ map such that it maps the four corner points of $R$ to the four corner points of $R'$ in order. Assume that $m = a/b \leq m' = a'/b'$. A curve is called a $a$-side if its two endpoints are in $[0, a] \times \{0\}$ and $[0, a] \times \{1\}$, respectively. So $f$ must take any $a$-side to a $a'$-side. This implies that

$$a' \leq \int_0^a |df(x + iy)| \leq \int_0^a (|f_z| + |f_{\bar{z}}|) dx.$$ 

So

$$a'b \leq \int_0^a \int_{0,b} (|f_z| + |f_{\bar{z}}|) dx dy$$

and

$$(a'b)^2 \leq \int_0^a \int_{0,b} |f_z| + |f_{\bar{z}}| dx dy \int_0^a \int_{0,b} (|f_z|^2 - |f_{\bar{z}}|^2) dx dy = a'b' \int_0^a \int_{0,b} D_f dx dy.$$ 

This implies that

$$\frac{m'}{m} \leq \frac{ab}{\int_0^a \int_{0,b} D_f dx dy}.$$ 

Therefore,

$$\frac{m'}{m} \leq \sup D_f.$$
This says that if \( m' \neq m \), then \( f \) can not be conformal. Let \( K = \sup D f \geq 1 \). We call \( f \) a \( K \)-quasiconformal map from \( R \) to \( R' \). This says that \( K \geq m'/m \). The minimum is attained for the affine mapping

\[
f(z) = \frac{1}{2} \left( \frac{a'}{a} = \frac{b'}{b} \right) z + \frac{1}{2} \left( \frac{a'}{a} - \frac{b'}{b} \right) \bar{z}.
\]

A quadrilateral \( Q \) is a Jordan domain \( D \), together with a pair of disjoint closed arcs on the boundary \( \partial D \) (called the \( b \)-arcs). Then there is a conformal map \( f \) maps \( Q \) to a rectangle \( R = [0, a] \times [0, b] \) such that the \( b \)-arcs are mapped to the \( b \)-sides of \( R \). The number \( m(Q) = a/b \) is unique. We call \( m(Q) \) the module of \( Q \).

Suppose \( \Omega \) is a region in the complex plane. \( f : \Omega \to f(\Omega) \) is a orientation-preserving homeomorphism. Then \( f \) maps any quadrilateral \( Q \) to a quadrilateral \( f(Q) \) in \( f(\Omega) \).

**Definition 10** (Geometric Definition). The map \( f \) is \( K \)-quasiconformal if

\[
\frac{1}{K} \leq \frac{m(f(Q))}{m(Q)} \leq K
\]

for any quadrilateral \( Q \) in \( \Omega \).

We say a real function \( u(x, y) \) is ACL (absolutely continuous on lines) in the region \( \Omega \) if for any closed rectangle \( R \subset \Omega \) with sides parallel to the \( x \)-axis and to the \( y \)-axis, \( u(x, y) \) is absolutely continuous on a.e. horizontal lines and a.e. on vertical lines. Such a function has partial derivatives a.e. in \( \Omega \).

**Definition 11** (Analytic Definition). The map \( f \) is \( K \)-quasiconformal if

1) \( f \) is ACL in \( \Omega \);
2) \( f_x \leq k |f_z| \), a.e. in \( \Omega \) (where \( k = (K - 1)/(K + 1) \)).

**Theorem 24.** The geometric definition and the analytic definition are equivalent.