Topics in One-Dimensional and Holomorphic Dynamics

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(This note is used only for this class, not for distribution)

Lecture 1. Introduction

- Topology: $f : X \to Y$, where the domain X and the range Y are two spaces and f is a map.
- Dynamics: $f : X \to X$, where the domain and the range are the same space X and f is a self-map.

Therefore, the study of dynamics can be thought as a part of topology. However, since we can iterate f to get

$$f^{\circ n} = \underbrace{f \circ \cdots \circ f}_{n}$$

for each integer n in dynamics, we have a group $\mathcal{G}_f = \{f^n\}_{n=-\infty}^{\infty}$ if f is one-to-one and onto or a semi-group $\mathcal{SG}_f = \{f^n\}_{n=0}^{\infty}$ if f is non-invertible. We simple use f^n to denote $f^{\circ n}$ and f^0 always means the identity map from X onto itself. The group $\mathcal{G}_f$ or the semi-group $\mathcal{SG}_f$ is called a (discrete) dynamical system. The space X is called a phase space.

One can also consider a continuous dynamical system $\{f^t\}_{t\in\mathbb{R}}$ where f^t is a family of one-to-one and onto map from X into itself parameterized by real number t such that $f^{t+s} = f^t \circ f^s$. A continuous dynamical system is a solution of an ordinary differential equation. For example, suppose $X = \mathbb{R}^n$ and suppose f^t is differentiable at $t = 0$, then we have a vector field

$$V(x) = \lim_{t\to 0} \frac{f^t(x) - x}{t}, \quad x \in X.$$

The continuous dynamical system $\{f^t\}$ is a solution of the ordinary differential equation

$$\dot{x} = V(x),$$

that is,

$$\dot{f}^t(x) = \frac{df^t(x)}{dt} = \lim_{\Delta t\to 0} \frac{f^{t+\Delta t}(x) - f^t(x)}{\Delta t}$$
$$= \lim_{\Delta t\to 0} \frac{f^{\Delta t}(f^t(x)) - f^t(x)}{\Delta t} = V(f^t(x)), \quad t \in \mathbb{R}, \ \ x \in X.$$

Conversely, a solution of ordinary differential equation above is a continuous dynamical system.

Given a continuous dynamical system $\{f^t\}_{t\in\mathbb{R}}$, we can consider the time-one map $f = f^1$. Then $\mathcal{G}_f = \{f^n\}_{n=-\infty}^{\infty}$ is a part of the continuous dynamical system $\{f^t\}_{t\in\mathbb{R}}$. However, it is not true, in general, that a group $\mathcal{G}_f = \{f^n\}_{n=-\infty}^{\infty}$ can be always thought as a group generated by the time-one map of a continuous dynamical system. It is actually an interesting problem:

Problem 1. *Under what conditions, can a one-to-one and onto map $f: X \to X$ be embedded into a continuous dynamical system as the time-one map?*

It is a hard problem if we insist to work on the same phase space X. However, if we consider another phase space with one-dimension higher than X, we can always embed f into a continuous dynamical system. This is called suspension technique. Consider $X \times [0,1]$ and glue $(x,1)$ with $(f(x),0)$. We get a new phase space $M_f = X \times [0,1]/\sim$ which has one dimension higher than X. Consider the vertical vector field $V(x,t) = \partial/\partial t$ on X_f. The suspension flow $f^t(x,t)$ is just the solution of the ordinary differential equation

$$\dot{(x,t)} = V(x,t) = (0,1)$$

on X_f. So $f^1(x,0) = (x,1) = (f(x),0)$. Then $\{f^t\}_{t\in\mathbb{R}}$ is a continuous dynamical system and f is its time-one map. Basically, we can use a discrete dynamical system to study a continuous dynamical system. One can refer to the following book for the embedding problem.

- P. Hartman, Ordinary Differential Equations (second edition), Birkhäuser, Boston (1982)

In this class, I will concentrate on a discrete dynamical system. Moreover, we will mostly concentrate on a semi-group $\mathcal{SG} = \{f^n\}_{n=0}^{\infty}$ where f is a one-dimensional map from an interval I of the real line into itself (in this case, we call it a one-dimensional dynamical system) or a holomorphic map from a region of the complex plane into itself (in this case, we call it a complex dynamical system or a holomorphic dynamical system).

A basic problem in dynamics is to understand the future of the orbit $\mathcal{O}(x_0) = \{x_n = f^n(x_0)\}_{n=0}^{\infty}$ for a given initial data x_0.

Definition 1. *The ω-limit set of x_0 is the set of all limiting points of convergent subsequences of $\mathcal{O}(x_0)$.*

If X is a good topological space (for example, a smooth Riemannian manifold), we can prove that $\omega(x_0)$ is a closed subset of X. The set $\omega(x_0)$ is the future of the orbit $\mathcal{O}(x_0)$.

Definition 2. *A point $x_0 \in X$ is called a* **fixed point** *if $f(x_0) = x_0$.*

It is clear that if x_0 is a fixed point, then $\omega(x_0) = \{x_0\}$.

Definition 3. *A point $x_0 \in X$ is called a* **periodic point** *of period $n > 0$ if $f^n(x_0) = x_0$ and $f^i(x_0) \neq x_0$ for all $1 \leq i < n$.*

It is clear that if x_0 is a periodic point of period n, then $\omega(x_0) = \{x_0, f(x_0), \cdots, f^{n-1}(x_0)\}$. In general, $\omega(x_0)$ could be a very complicate subset of X.

To better understand the future, we need to obtain knowledge of the history of x_0, that is, the set of $\cup_{n=0}^{\infty} f^{-n}(x_0)$. In particular, when f is non-invertible, $f^{-n}(x_0)$ may contain more than one points. Then we can define a life tree $Tree_f(x_0)$ as follows. The starting point is x_0. The first level vertices are all points y in $f^{-1}(x_0)$ and then use an edge to connect x_0 to y. Suppose the n^{th}-level of tree is defined. Then the $(n+1)^{th}$-level of tree is defined by all points in $f^{-1}(y)$ for all y in the n^{th}-level and use an edge to connect y and z if $f(z) = y$.

The life trees for all $x_0 \in X$ define an inverse limiting space

$$X_{\infty,f} = \{\ldots x_n x_{n-1} \ldots x_1 x_0 \mid x_n \in X,\ f(x_n) = x_{n-1}, n \geq 0\},$$

which gives the full history of the life of x_0. Therefore, we can use the knowledge of the full history of the life to study the future $\omega(x_0)$ for every $x_0 \in X$.

Example 1. *Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle where $\mathbb{C}$ is the complex plane. Suppose $f(z) = z^2 : S^1 \to S^1$. Find the inverse limiting space $S^1_{\infty,f}$.*

Proof. Since $|z| = 1$, we write $z = e^{2\pi i\theta}$ where $0 \leq \theta < 1$. Then

$$f^{-1}(z) = \{z_0 = e^{\pi i\theta}, z_1 = e^{\pi i(\theta+1)}\}.$$

Thus

$$S^1_{2,f} = \{z_0 z, z_1 z\}$$

becomes a 2-fold of circle. Similarly,

$$f^{-1}(z_0) = \{z_{00} = e^{\frac{\pi i\theta}{2}}, z_{10} = e^{\frac{\pi i(\theta+2)}{2}}\}$$

and

$$f^{-1}(z_1) = \{z_{01} = e^{\frac{\pi i(\theta+1)}{2}}, z_{11} = e^{\frac{\pi i(\theta+3)}{2}}\}.$$

Thus

$$S^1_{3,f} = \{z_{00}z_0z, z_{10}z_0z, z_{01}z_1z, z_{11}z_1z\}$$

becomes a 4-fold of circle. Similarly, $S^1_{n,f}$ is a 2^{n-1}-fold of circle for any $n \geq 1$. They are still connected. The inverse limiting space $S^1_{\infty,f}$ is actually very complicated (for example, it is disconnected). For every point $z \in S^1$, the life tree $Tree_f(z)$ is a uncountable and totally disconnected (we will prove it later), thus it is a Cantor set which can be identified with the symbolic space

$$\Sigma^- = \prod_{n=-\infty}^{-1} \{0,1\} = \{w^- = \ldots j_n j_{n+1} \ldots j_{-2} j_{-1} \mid j_n = 0 \text{ or } 1\}.$$

Note that $\{0,1\}$ is a set with discrete topology and Σ^- has the product topology (we will construct this product topology later).

For any $0 \leq \theta \leq 1$, we have a 2-adic expansion

$$\theta = \sum_{m=0}^{\infty} \frac{i_m}{2^{m+1}}, \quad i_m = 0 \text{ or } 1.$$

Then we consider the symbolic space (with the product topology)

$$\Sigma^+ = \prod_{m=0}^{\infty} \{0,1\} = \{w^+ = i_0 i_1 \ldots i_{m-1} i_m \ldots\}$$

and a map

$$\pi_+(w^+) = e^{2\pi i \theta}, \quad \theta = \sum_{m=0}^{\infty} \frac{i_m}{2^{m+1}}.$$

Then π_+ is one-to-one except for $000\ldots$ and $111\ldots$, $1000\ldots$ and $0111\ldots$, and $w_m^+ = i_0 \ldots i_{m-1} i_m 100\ldots$ and $v_m^+ = i_0 i_1 \ldots i_{m-1} i_m 011\ldots$ on which π_+ is two-to-one (for example, $000\ldots$ and $111\ldots$ are mapped to 1, $1000\ldots$ and $0111\ldots$ are mapped -1, and $w_0^+ = i_0 100\ldots$ and $v_0^+ = i_0 011\ldots$ are mapped i if $i_0 = 0$ and $-i$ if $i_0 = 1$). Thus we can think $S^1 = \Sigma^+/\{000\ldots \sim 111\ldots, 1000\ldots \sim 0111\ldots, w_m^+ \sim v_m^+\}$. This gives us a picture of the inverse limiting space S^1_∞ which can be written as

$$S^1_{\infty,f} = \Sigma/\{000\ldots \sim 111\ldots, 1000\ldots \sim 0111\ldots, w_m^+ \sim v_m^+\}$$

where

$$\Sigma = \Sigma^- \bigoplus \Sigma^+ = \{w = w^-.w^+ = \ldots j_n j_{n+1} \ldots j_{-2} j_{-1}.i_0 i_1 \ldots i_{m-2} i_{m-1} \ldots\},$$

where $n < 0$ and $m \geq 0$. We call S^1_∞ a solenoid. □

Define $\sigma : \Sigma \to \Sigma$ as

$$\begin{aligned}\sigma(w) &= \ldots j'_n j'_{n+1} \ldots j'_{-2} j'_{-1}.i'_0 i'_1 \ldots i'_{m-2} i'_{m-1} \ldots \\ &= \ldots j_n j_{n+1} \ldots j_{-2} j_{-1} i_0.i_1 \ldots i_{m-2} i_{m-1} \ldots.\end{aligned}$$

Then σ is invertible. As you can see, from a non-invertible dynamical system $f(z) = z^2$, we construct an invertible dynamical system which contains the full history and the future for every $z \in S^1$.

Exercise 1. *Let* $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ *be the unit circle where* $\mathbb{C}$ *is the complex plane. Suppose* $q_d(z) = z^d : S^1 \to S^1$, $d \geq 3$. *Find the inverse limiting space* S^1_{∞, q_d}.

Definition 4. *Given a (discrete) dynamical system* $f : X \to X$. *A subset* $\Lambda \subset X$ *is called an invariant subset of* f *if* $f(\Lambda) \subseteq \Lambda$.

If Λ is an invariant subset of f, then $f : \Lambda \to \Lambda$ is a new (discrete) dynamical system.

Definition 5. *An invariant subset* Λ *is called completely invariant if* $f(\Lambda) = f^{-1}(\Lambda) = \Lambda$.

Definition 6. *An invariant set* Λ *is called minimal if there is no proper subset of* Λ *which is also invariant.*

Example 2. *Suppose* $f : X \to X$. *Suppose* x_0 *is a periodic point of period* $n \geq 1$. *Then the set* $\Lambda = \{x_0, \ldots, f^{n-1}(x_0)\}$ *is an invariant set and minimal. But it may not be a complete invariant set. However, if we consider* $f : \Lambda \to \Lambda$ *as a new dynamical system, then* Λ *is a completely invariant set.*

Example 3. *Let* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. *Suppose* $f(z) = z^d : \mathbb{C}^* \to \mathbb{C}^*$. $d > 1$. *Then* S^1 *is a completely invariant set of* f *but it is not minimal.*

Lecture 2. Markov partition and counting problem and zeta function

Given a dynamical system $f : X \to X$. Let P_n denote the set of all periodic points of period $n \geq 1$. Let F_n be the set of all fixed points of f^n. Then it is clear that

$$P_n = F_n \setminus (\cup_{1 \leq p < n, p|n} F_p).$$

An important problem is to count numbers of F_n, therefore, numbers of P_n. Actually, we are more interested in the asymptotical growth rate of numbers of F_n which can be defined as

$$h(f) = \lim_{n \to \infty} \frac{1}{n} \log \#(F_n).$$

(Later, this number is also called the topological entropy.) What we can do is to put all numbers $\#(F_n)$ into a power series and define the (unweighted) dynamical zeta function

$$\zeta(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{\#(F_n)}{n} z^n \Big).$$

(It definitely has some relation with the classical Riemann zeta function. Refer to

- Dieter H. Mayer, The thermodynamic formalism approach to Selberg's zeta function for $PSL(2, \mathbb{Z})$, Bull. Amer. Math. Soc. (N.S.), Volume 25, Number 1 (1991), 55-60.

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Then function $\zeta(z)$ is holomorphic in a disk centered at 0 of radius $r \geq 0$. the logarithm of the reciprocal of the convergence radius r_{max} of $\zeta(z)$ is the growth rate $h(f)$. One of most important and interesting question is that

Problem 2. *Can this zeta function be extended to an meromorphic or entire function on the whole complex plane?*

To study this problem, we should first study an important method in dynamics called *Markov partitions*.

Suppose X is a compact metric space and $d(\cdot, \cdot)$ is the metric.

Definition 7. *A partition $X = X_0 \cup \ldots \cup X_{d-1}$ is called a Markov partition for f if*

(1) *Each X_i is a non-empty compact subset of X;*
(2) *$X_i \cap X_j = \emptyset$ for any $0 \leq i \neq j \leq d - 1$;*
(3) *$f|X_i$ is one-to-one for every $0 \leq i \leq d - 1$;*

(4) $f(X_i) = \cup_{k=1}^{m_i} X_{i_k}$ *for every* $0 \leq i \leq d-1$.

More generally, we replace (2) and (3) in the above definition as

Definition 8. *A partition* $X = X_0 \cup \ldots X_{d-1}$ *is called a Markov partition for* f *if*

(1) *Each* X_i *is a non-empty compact subset of* X*;*
(2') *mathring*$X_i \cap X_j = \emptyset$ *for any* $0 \leq i \neq j \leq d-1$*;*
(3') $f|\mathring{X}_i$ *is one-to-one for every* $0 \leq i \leq d-1$*;*
(4) $f(X_i) = \cup_{k=1}^{m_i} X_{i_k}$ *for every* $0 \leq i \leq d-1$*,*

where $\mathring{X}_i$ *means the interior of* X_i*.*

In this case, we have to take care of more on the boundary of X_i.

Example 4. *Let* $I = [0,1]$ *and* $I_0 = [0, 1/3]$ *and* $I_1 = [2/3, 1]$*. Let* $f(x) = 3x$ *for* $x \in I_0$ *and* $f(x) = 3(x - 2/3)$ *for* $x \in I_1$*. Let* $\Lambda = \{x \in I \mid f^n(x) \in I, \ \forall n > 0\}$*. The set* Λ *is called the non-escaping set for* f*. Then* $f : \Lambda \to \Lambda$ *is a dynamical system. Let* $\Lambda_0 = \Lambda \cap I_0$ *and* $\Lambda_1 = \Lambda \cap I_1$*. The* $\{\Lambda_0, \Lambda_1\}$ *gives a Markov partition for* f*.*

Example 5. *Let* $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ *be the unit circle of the complex plane. Let* $q_d(z) = z^d : S^1 \to S^1$*. Then it is a dynamical system. Let* $I_k = [e^{2\pi i k/d}, e^{2\pi i (k+1)/d}]$ *for* $0 \leq k < d$*. Then* $\{I_0, \ldots, I_{d-1}\}$ *gives us a Markov partition for* q_d *in the meaning of Definition 8.*

If $f : X \to X$ has a Markov partition, we use

$$\eta_0 = \{X_0, \ldots, X_{d-1}\}$$

to denote the initial partition. Associate with this initial partition, we have a $d \times d$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1, & \text{if } f(X_i) \supset X_j; \\ 0, & \textit{otherwise.} \end{cases}$$

We also have a symbolic space

$$\Sigma_A = \{w = i_0 i_1 \ldots i_{k-1} i_k \ldots \mid i_k \in \{0, \ldots, d-1\}, a_{i_{k-1} i_k} = 1, \ \forall k \geq 1\}$$

and a shift map

$$\sigma_A : w = i_0 i_1 \ldots i_{k-1} i_k \ldots \to \sigma_A(w) = i_1 \ldots i_{k-1} i_k \ldots .$$

Here $w = i_0 i_1 \ldots i_{k-1} i_k \ldots$ is called an admissible sequence (with respect to A) of $\{0, \ldots, d-1\}$.

We can also get a sequence of nested pull-back partitions from the initial partition, $\eta_k = f^{-k}(\eta_0)$ which consists of all non-empty compact subsets Y of X such that $f^k : Y \to X_i$ is one-to-one and onto (or f^k :

$\mathring{Y} \to \mathring{X}_i$ is one-to-one and onto in Definition 8) for some $0 \leq i \leq d-1$. Therefore, we can label subsets in η_k as

$$\eta_k = \{X_{i_0 i_1 \ldots i_l \ldots i_{k-1} i_k}\}$$

so that $f^l(X_{i_0 i_1 \ldots i_l \ldots i_{k-1} i_k}) \subset X_{i_l}$ for all $0 \leq l \leq k$. Note that each η_k is also a Markov partition. Thus we have a sequence of Markov partitions with the property that

$$f(X_{i_0 i_1 \ldots i_l \ldots i_{k-1} i_k}) = X_{i_1 \ldots i_l \ldots i_{k-1} i_k}$$

and

$$X_{i_0 i_1 \ldots i_l \ldots i_{k-1} i_k} \subset X_{i_0 i_1 \ldots i_l \ldots i_{k-1}}.$$

Since

$$X_{i_0 i_1 \ldots i_l \ldots i_{k-1} i_k} \subset X_{i_0 i_1 \ldots i_l \ldots i_{k-1}} \subset X_{i_0 i_1 \ldots i_l \ldots i_{k-2}} \subset \cdots X_{i_0 i_1} \subset X_{i_0} \subset X$$

and since each subset is non-empty and compact, for any admissible sequence $w = i_0 i_1 \ldots i_{k-1} i_k \ldots$, we have a non-empty subset

$$X_w = \cap_{k=0}^{\infty} X_{i_0 i_1 \ldots i_l \ldots i_{k-1} i_k}.$$

Definition 9. *We say $f : X \to X$ is topologically expansive if every $X_w = \{x_w\}$ contains only one point.*

For every finite admissible sequence $w_k^0 = i_0^0 \ldots i_{k-1}^0 i_k^0$, where $i_i \in \{0, \ldots, d-1\}$, $0 \leq i \leq k$, we call

$$[w_k^0] = \{w = w_k^0 i_{k+1} \ldots i_l \ldots \mid i_l \in \{0, 1, \ldots, d-1\}, l \geq k+1\}$$

a cylinder. Since $\{0, \ldots, d-1\}$ has the discrete topology, Σ_A has the product topology, that is, the topology generated by all cylinders $[w_k^0]$.

We also can put a metric on Σ_A as

$$d(w, w') = \sum_{k=0}^{\infty} \frac{|i_k - i'_k|}{d^{k+1}}$$

where $w = i_0 i_1 \ldots i_{k-1} i_k \ldots$ and $w' = i'_0 i'_1 \ldots i'_{k-1} i'_k \ldots$. The topology induced by this metric on Σ_A is the same topology generated by the set of cylinders.

For a topologically expansive $f : X \to X$ with the initial Markov partition η_0 and the associative matrix A, we then have a homeomorphism

$$\pi_f(w) = x_w : \Sigma_A \to X$$

if we are in Definition 7. (However, if we are in Definition 8, π_f is one-to-one and onto except for all boundary points of X_i.) Moreover, $\sigma_A : \Sigma_A \to \Sigma_A$ and $f : X \to X$ are topologically conjugate by π_f in

Definition 7 (or they are topologically semi-conjugate in Definition 8), that is,

$$f \circ \pi_f = \pi_f \circ \sigma_A.$$

Definition 10. *Two dynamical systems $f : X \to X$ and $g : Y \to Y$ are said to be topologically conjugate if there is a homeomorphism $h : Y \to X$ such that*

$$h \circ g = f \circ h.$$

Here h is called the conjugacy between f and g. If h is just continuous and onto, we call f and g are topologically semi-conjugate. In this case, h is called the semi-conjugacy.

Proposition 1. *Suppose $f : X \to X$ and $g : Y \to Y$ are topologically conjugate and h is the conjugacy. Then $p \in Y$ is a periodic point of period $n \geq 1$ of g if and only if $q = h(p)$ is a periodic point of period $n \geq 1$ of f. Thus $\#(P_n(f)) = \#(P_n(g))$ and $\#(F_n(f)) = \#(F_n(g))$.*

Proof. The point $p \in Y$ is a periodic point of period $n \geq 1$ of g if $g^j(p) \neq p$ for $1 \leq j < n$ but $g^n(p) = p$. Since $f = h \circ g \circ h^{-1}$, we have that $f^n(q) = h(g^n(h^{-1}(h(p)))) = h(g^n(p)) = h(p) = q$ and $f^j(q) = h(g^j(h^{-1}(h(p)))) = h(g^j(p)) \neq h(p) = q$ for $1 \leq j < k$. Thus q is a periodic point of period $k \geq 1$ of f. By using h^{-1}, we get the "if and only if". We have bijective maps

$$h : P_{n,g} \to P_{n,f} \quad \text{and} \quad h : F_{n,g} \to F_{n,f}.$$

This implies $\#(P_{n,g}) = \#(P_{n,f})$ and $\#(F_{n,g}) = \#(F_{n,f})$. □

For an expansive $f : X \to X$ with the initial Markov partition η_0 (in Definition 7) and the transitive matrix A, we have that

$$\#(F_{n,f}) = \#(F_{n,\sigma_A}) = trace(A^n).$$

Use this relation, we can have a full calculation of the dynamical zeta function of f as follows:

$$\zeta(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{\#(F_n)}{n} z^n\Big) = \exp\Big(\sum_{n=1}^{\infty} \frac{trace(A^n)}{n} z^n\Big).$$

By using the relation $det(B) = \exp(trace(\log B))$ for a square matrix, we have that

$$\zeta(z) = \exp(-trace(\log(I - zA))) = \frac{1}{det(I - zA)}.$$

This implies that $\zeta(z)$ can be extended analytically to a rational map.

In the calculation of $\zeta(z)$ we have a non-negative matrix. So let us to learn some facts in theory of matrices.

Suppose

$$\mathcal{V} = \{\mathbf{v} = (v_1, \ldots, v_n)^T \mid v_i \in \mathbb{R}\}$$

is the n-dimensional vector space. We will use the norm

$$\|\mathbf{v}\| = \sum_{i=1}^{n} |v_i|, \quad \mathbf{v} = (v_1, \cdots, v_n)^T \in \mathcal{V}. \tag{1}$$

For any linear map $L : \mathcal{V} \to \mathcal{V}$, let A be the corresponding $n \times n$-matrix for L, that is, $L(\mathbf{v}) = A\mathbf{v}$. Define

$$\|A\| = \sup_{\|\mathbf{v}\|=1} \|A\mathbf{v}\|.$$

The spectral radius $\rho(A)$ of A can be calculated as

$$\rho(A) = \lim_{n\to\infty} \sqrt[n]{||A^n||} \geq 0.$$

A square matrix $A = (a_{ij})_{n\times n}$ is said to be positive if all $a_{ij} > 0$ for $1 \leq i, j \leq n$. An eigenvalue is called simple if the corresponding eigenspace is one-dimensional.

Theorem 1 (Perron-Frobenius Theorem)**.** *If a $n \times n$ matrix A is positive, then A has a unique, simple, positive, maximal eigenvalue λ with a positive eigenvector $\mathbf{v}_0 = (v_1, \cdots, v_n)^T$, i.e, $v_i > 0$ for all $1 \leq i \leq n$. Here the term "unique" means that all other eigenvalues μ of A satisfy that*

$$|\mu| < \lambda.$$

Proof. Let

$$S = \{\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathcal{V} \mid \|\mathbf{v}\| = 1, v_i \geq 0, 1 \leq i \leq n\}$$

be a convex compact subset in $\mathcal{V}$. Define

$$f(\mathbf{v}) = \frac{A\mathbf{v}}{\|A\mathbf{v}\|} : S \to S.$$

The Brouwer fixed point theorem: every continuous function which maps a convex compact subset into itself must have a fixed point, implies that f has a fixed point $\mathbf{v}_0 \in S$, that is, $f(\mathbf{v}_0) = \mathbf{v}_0$. Then we have that $\lambda = \|A\mathbf{v}_0\|$ is an eigenvalue with an corresponding eigenvector $\mathbf{v}_0$. Suppose $\mathbf{v}_0 = (v_1, \ldots, v_n)^T$. Then

$$\lambda v_i = \sum_{j=1}^{n} a_{ij} v_j.$$

This implies that $\lambda > 0$ and $v_i > 0$ for all $0 \leq i \leq n$. Clearly, $\lambda = \rho(A)$ (note that $\|A\mathbf{v}\| \leq \|A|\mathbf{v}|\|$ for every $\mathbf{v} = (v_1, \ldots, v_n)$ where $|\mathbf{v}| = (|v_1|, \ldots, |v_n|)^T$.

Now we show the eigenspace E_λ is one-dimensional. Suppose $\mathbf{u} = (u_1, \ldots, u_n)^T \neq 0 \in E_\lambda$. Let $\mathbf{w} = t\mathbf{v}_0 - \mathbf{u} \geq 0$ and at least one component of $\mathbf{w}$ be zero, that is,

$$t = \min\Big\{\frac{u_i}{v_i} \;\Big|\; 1 \leq i \leq n\Big\}.$$

Then from $A\mathbf{w} = \lambda\mathbf{w}$, we get $\mathbf{w} = 0$. That is $\mathbf{u} = t\mathbf{v}_0$. So $E_\lambda = \mathrm{span}\{\mathbf{v}_0\}$. Similarly, we also showed that no other positive vector not in E_λ is an eigenvector.

There is no negative eigenvalue r such that $|r| = \lambda$. Otherwise, $A\mathbf{u} = -r\mathbf{u}$ and $A^2\mathbf{u} = r^2\mathbf{u}$ for some $\mathbf{u} \neq 0$ and $\mathbf{u} \notin E_\lambda$. Since A^2 is also a positive matrix and λ^2 is a positive eigenvalue with the eigenspace E_λ, this is a contradiction. This implies that all real eigenvalues $r \neq \lambda$ of A must have $|r| < \lambda$.

There is one more property about f as follows: For any $\mathbf{v} \in S$, $f^k(\mathbf{v}) \to \mathbf{v}_0$ as $k \to \infty$. This property can be proved by using the Hilbert projective metric on the positive cone

$$\mathbb{V}_+ = \{\mathbf{v} = (v_1, \ldots, v_n)^T \mid v_i \geq 0,\ 1 \leq i \leq n\}.$$

Refer to, for example,

- G. Birkhoff, Extensions of Jentzchs theorem. Trans. Amer.Math.Soc., 85 (1957), 219-227.
- Y. Jiang, Nanjing Lecture Notes In Dynamical Systems. Part One: Transfer Operators in Thermodynamical Formalism. FIM Preprint Series, ETH-Zurich, June 2000.
- C. MaCarthy, The Hilbert projective metric, multi-type branching processes and mathematical biology: a model of the evolution of resistance. Ph.D thesis, CUNY Graduate Center, 2010.

Now we use this property to show that A has no complex eigenvalue z such that $|z| = \lambda$. We prove it by contradiction. First, the property implies that $A^k\mathbf{x}$ tends to E_λ as k goes to ∞ for any $\mathbf{x} > 0$. Suppose we have a complex number z with $|z| = \lambda$ such that $A\mathbf{u} = z\mathbf{u}$ for a complex vector $\mathbf{u} = \mathbf{x} + i\mathbf{y}$. Let $z = \lambda e^{2\pi i\theta}$ where $0 < \theta < 1$. Let t_1 be a real number such that $t_1\mathbf{v}_0 - \mathbf{x} \geq 0$ with at least one zero component and t_2 be another real number such that $t_2\mathbf{v}_0 - \mathbf{y} \geq 0$ with at least one zero component. Note that $\tilde{\mathbf{x}} = t_1\mathbf{v}_0 - \mathbf{x}$ and $\tilde{\mathbf{w}} = t_2\mathbf{v}_0 - \mathbf{y}$ can not be in E_λ (otherwise, from $A\tilde{\mathbf{x}} = \lambda\tilde{\mathbf{x}}$, we get $\tilde{\mathbf{x}} = \mathbf{0}$ and a similar argument holds for $\tilde{\mathbf{y}}$, this implies that $A\mathbf{u} = \lambda\mathbf{u} = z\mathbf{u}$ but $z \neq \lambda$). Consider $A^k\tilde{\mathbf{x}}$ and $A^k\tilde{\mathbf{y}}$, we have the equation,

$$A^k\tilde{\mathbf{x}} + iA^k\tilde{\mathbf{y}} = \lambda^k(t_1\mathbf{v}_0 - e^{2\pi ki\theta}\mathbf{x}) + \lambda^k(t_2\mathbf{v}_0 - e^{2\pi ki\theta}\mathbf{y})i.$$

The real part and the imaginary part of the left side both tend to the line E_λ. Now let us examine the real part and the imaginary part of the right side. If $\theta = p/q$ is a rational number, then we have a sequence of integers $k_l = lq$ such that $e^{2\pi i k_l} = 1$. If θ is an irrational number, then the set $\{k\theta \pmod 1 \mid k \in \mathbb{Z}\}$ is dense in the interval $[0,1]$. We concluded that for any θ, we have a sequence of integers k_l tending to ∞ such that $e^{2\pi i k_l \theta} \to 1$ as $l \to \infty$. This implies that the real part of the right side tends to the line spanned by $\tilde{\mathbf{x}} = t_1\mathbf{v}_0 - \mathbf{x}$ as $l \to \infty$, which is not the line E_λ. Similar, the imaginary part of the right side tends to the line spanned by $\tilde{\mathbf{y}} = t_2\mathbf{v}_0 - \mathbf{y}$ as $l \to \infty$, which is not the line E_λ. This is a contradiction. The contradiction implies that for any complex eigenvalue z of A, we have that $|z| < \lambda$. $\square$

One can check references (the third reference will be also used later) for other proofs of the Perron-Frobenius theorem and its generalization to infinite-dimensional transfer operators.

1. F. R. Gantmacher, Theory of matrices. Chelsea, 1959.
2. G. Strang, Linear Algebra and Its Applications, third edition, 1988. (A simple proof for the finite-dimensional case.)
3. Y. Jiang, A proof of existence and simplicity of a maximal eigenvalue for Ruelle-Perron-Frobenius operators. Letters in Math. Phys. 48 (1999), no. 3, 211-219. (A simple proof for the infinite-dimensional case.)

A square matrix $A = (a_{ij})_{n\times n}$ is said to be non-negative if all $a_{ij} \geq 0$ for $1 \leq i,j \leq n$. A non-negative $n \times n$ matrix A is called irreducible if no permutation of the indices places the matrix in a block lower-triangular form. More precisely, there is no permutation matrix P, which is a matrix consisting of 0 and 1 such that each row or each column contains one and only one 1, such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices.

If A is a non-negative but reducible, then we can find a permutation matrix P such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & * & * & \cdots & * \\ 0 & A_{22} & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & * \\ 0 & 0 & \cdots & 0 & A_{kk} \end{pmatrix},$$

and such that all A_{ii} are irreducible matrices for $1 \leq i \leq k$. The set of eigenvalues of A is the union of sets of eigenvalues of A_{ii}, $1 \leq i \leq k$.

For example,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is reducible with one positive maximal eigenvalue $\lambda = 1$ of multiple 2.

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is irreducible with one positive simple maximal eigenvalue 1 and one negative simple eigenvalue -1. Note that $J^2 = I$.

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

is irreducible with one positive simple maximal eigenvalue 2 and one simple eigenvalue 0.

An equivalent definition of irreducibility is that for any $1 \leq i,\ j \leq n$, there exists a $0 \leq q = q(i,j) \leq n$ such that the ij-th entry of A^q is positive.

Theorem 2 (Perron-Frobenius Theorem)**.** *If A is irreducible, $\rho(A)$ is a simple, positive, maximal eigenvalue with a positive eigenvector $\mathbf{v} = (v_1, \cdots, v_n)$, i.e, $v_i > 0$ for all $1 \leq i \leq n$.*

Remark 1. *If A is a non-negative matrix, then*

$$PAP^{-1} = \begin{pmatrix} A_{11} & * & * & \cdots & * \\ 0 & A_{22} & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & * \\ 0 & 0 & \cdots & 0 & A_{kk} \end{pmatrix},$$

Then we can apply the Perron-Frobenius theorem on A_{ii} to get the simple, positive, maximal eigenvalue $\rho(A_{ii})$. Then the maximal eigenvalue of A is $\rho(A) = \max_{1 \leq i \leq k} \rho(A_{ii})$.

However, even if A is irreducible, there may exist another eigenvalue $\mu \neq \rho(A)$ but $|\mu| = \rho(A)$. For example, consider

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is an irreducible matrix. The spectral radius is 1 which is a simple, positive, maximal eigenvalue with an eigenvector $\mathbf{v}_1 = (1,1)$. However, -1 is also an eigenvalue with an eigenvector $\mathbf{v} = (1,-1)$.

Now let us return to zeta function $\zeta(z)$. If f is topologically expansive with a Markov partition η_0, then $\zeta(z)$ can be extended analytically to a rational function. The smallest pole of $\zeta(z)$ is the reciprocal of the maximal eigenvalue $\rho(A)$ of A. Notice that according to the Perron-Frobenius theorem A has a positive maximal eigenvalue $\rho(A)$ since all entries of A are non-negative. Then we have that the growth rate of numbers of $F_{n,f}$ is $h(f) = \log \rho(A)$. In particular, when A is a positive matrix, then $1/\rho(A)$ is the simple smallest pole of $\zeta(z)$. For example, if

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\zeta(z) = 1/(1-z)^2$. If

$$A = J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$\zeta(z) = 1/(1-z^2) = 1/(1-z)(1+z)$. And if

$$A = K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$\zeta(z) = 1/(1-2z)$.

Example 6. *Suppose $I = [0,1]$ and $I_0 = [0,a]$ and $I_1 = [b,1]$ for $0 < a < b < 1$. Suppose $f : I_0 \cup I_1 \to I$ is a map satisfying that $f_0 = f|I_0 : I_0 \to I$ is a homeomorphism $f_1 = f|I_1 : I_1 \to I$ is a homeomorphism and that $f(0) = f(b) = 0$ and $f(a) = f(1) = 1$. The non-escaping set for f is, by definition,*

$$\Lambda = \{x \in [0,1] \mid f^n(x) \in [0,1],\ \forall n \geq 0\}.$$

It is clearly,

$$\Lambda = \cap_{n=0}^{\infty} f^{-n}(I).$$

Then $f : \Lambda \to \Lambda$ is a Markovian dynamical system with the initial partition $\eta_0 = \{\Lambda_0, \Lambda_1\}$ where $\Lambda_0 = \Lambda \cap I_0$ and $\Lambda_1 = \Lambda \cap I_1$ and the transitive matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We assume that f is expansive. So $\#(F_{n,f}) = trac(A^n) = 2^n$. Then maximal positive eigenvalue of A is 2. The growth rate of numbers of F_n is $\log 2$. Furthermore, we have that

$$\frac{1}{det(I - zA)} = \frac{1}{1-2z}.$$

Example 7. *Suppose $S^1 = \{z = e^{2\pi\theta};\ 0 \leq \theta \leq 1\}$ and $f(z) = z^2$. $I_0 = \{z = e^{2\pi\theta};\ 0 \leq \theta \leq 1/2\}$ and $I_1 = \{z = e^{2\pi\theta};\ 1/2 \leq \theta \leq 1\}$. Then f is* *a Markovian dynamical system with the initial partition* $\eta_0 = \{I_0, I_1\}$ *in Definition 8. The transitive matrix is still*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The map $\pi_f : \Sigma_A \to S^1$ *is one-to-one except for all endpoints* I_{w_k} *for* $w_k = i_0 i_1 \ldots i_l \ldots i_{k-1} i_k$ *for* $k \geq 0$.

So $\#(F_{n,f}) = \#(\sigma_A^n) - 1 = trac(A^n) - 1 = 2^n - 1$ *for* $n \geq 1$. *Then maximal positive eigenvalue of* A *is still* 2. *The growth rate of numbers of* F_n *is* $\log 2$. *Furthermore, we have that*

$$\begin{aligned} \zeta(z) &= \exp\Big(\sum_{n=1}^{\infty} \frac{\#(F_n)}{n} z^n\Big) = \exp\Big(\sum_{n=1}^{\infty} \frac{trac(A^n) - 1}{n} z^n\Big) \\ &= \exp\Big(\sum_{n=1}^{\infty} \frac{trac(A^n)}{n} z^n\Big) \cdot \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{n} z^n\Big) \\ &= \frac{1}{det(I - zA)} \exp\Big(\log(1 - z)\Big) = \frac{1 - z}{1 - 2z}. \end{aligned}$$

Definition 11. *A dynamical system* $f : X \to X$ *is said to be metrically expanding if there are constants* $0 < a < 1$, $C > 0$, *and* $\lambda > 1$ *such that*

$$d(f^n(x), f^n(y)) \geq C\lambda^n d(x, y)$$

for all $x, y \in X$ *with* $d(f^i(x), f^i(y)) \leq a$, $0 \leq i \leq n - 1$ *and* $n > 0$.

Theorem 3. *Suppose* (X, d) *is a compact metric space. Any metrically expanding dynamical system* $f : X \to X$ *has a Markov partition.*

Proof. Suppose $0 < \epsilon \leq a/2$ is a real number such that f is injective on any ball $B(x, \epsilon)$ of radius ϵ and centered at x. Let $\delta < 2C(\lambda - 1)\epsilon/\lambda$. Consider a cover of X by balls $\{B_1, \cdots, B_n\}$ of radius δ, that is, $X = \cup_{k=1}^n B_k$. For $f(B_1)$ let $\{B_{1j_k}\}$ be its minimal covers and $S_1 = \cup_{j_k} B_{j_k}$. Let B_{11} be the preimage of S_1. Then the diameter of B_{11} is $\leq \delta + \delta/(C\lambda)$. Now consider new B_2 which is the old B_2 minus $B_{11} \cup f(B_{11})$. Consider the minimal cover $\{B_{2j_k}\}$ of B_2. Let $S_2 = \cup_k B_{2j_k}$. Let B_{22} be the preimage of S_2. Do this for all B_i we get B_{ii} with diameter $\leq \delta + \delta/(C\lambda)$. Inductive, we can construct B_{i^n} with diameter $\leq \sum_{i=0}^{n-1} \delta/(C\lambda^i)$. By taking limit, we have a partition of X by $X_i = B_{i^\infty}$ such that $\mathring{X}_i \cap X_j = \emptyset$ for $1 \leq i \neq j \leq n$ and $f(X_i)$ is just a union of some X_j. Since the diameter of $x_i \leq \sum_{i=0}^{\infty} \delta/(C\lambda^i) = (\delta/C) \cdot \lambda/(\lambda - 1) \leq 2\epsilon \leq a$, we have that f is injective on X_i. And $f(X_i) = \cup_{k=1}^{m_i} X_{i_k}$. □

Remark 2. *Thus if a map $f : X \to X$ is metrically expanding, then it is topologically expanding.*

Lecture 3. Non-linear dynamical systems and distortions

Refer to Chapter 1 of the following book for this lecture.

- Y. Jiang, Renormalization and Geometry in One-Dimensional and Complex Dynamics. Advanced Series in Nonlinear Dynamics, Vol. 10 (1996) World Scientific Publishing Co. Pte. Ltd., River Edge, NJ, xvi+309 pp. ISBN 981-02-2326-9.

Suppose $f : I_0 \cup I_1 \to I$ is an expanding map. Let $G = (a, b)$ be the complement of $I_0 \cup I_1$ in I. A number x in I is said to be escaping to G if $f^{\circ k}(x)$ is in G for some integer $k \geq 0$ (where $f^{\circ 0}$ is the identity). The set $\Omega \subseteq I$ of escaping points is an open subset of the real line. The complement Λ of Ω in I is called the non-escaping set under f. It is a compact (closed and bounded) subset of the real line $\mathbb{R}$.

Example 8 ($\frac{1}{3}$-Cantor set)**.** *Suppose $a = 1/3$ and $b = 2/3$. Define*

$$f(x) = \begin{cases} 3x, & 0 \leq x \leq a; \\ 3x - 2, & b \leq x \leq 1. \end{cases}$$

Then f is an expanding map for which the non-escaping set Λ under f is the famous $\frac{1}{3}$-Cantor set.

One can see that $f : \Lambda \to \Lambda$ is a Markov partition with the initial partition

$$\Lambda_0 = [0, 1/3] \cap \Lambda \quad \text{and} \quad \Lambda_1 = [2/3, 1] \cap \Lambda.$$

In general we have Λ_{w_k} for all $w_k = i_0 \ldots i_{k-1} i_k$. The length $|\Lambda_{w_k}| = 1/3^{k+1}$. Clearly Λ has zero Lebesgue measure since $|\Lambda| \leq \sum_{w_k} |\Lambda_{w_k}| \leq 2^{k+1}/3^{k+1}$ for all $k \geq 0$. Let $s > 0$ be real number such that

$$\sum_{w_k} |\Lambda_{w_k}|^s = 1.$$

Then we can see that $2^{k+1} 3^{-s(k+1)} = 1$. This implies that

$$s = \frac{\log 2}{\log 3}.$$

This number is called the Hausdorff dimension of Λ (we will define it later). The reason we can do all these calculations is that this is a *linear dynamical system.*

Now we are going to see how to study a typical non-linear dynamical system. Let X be an interval $I = [a, b]$ or the circle $S^1 = \{z = e^{2\pi i \theta}\}$. Let $f(x)$ be a self-map of X. The dynamical system generated by f is the semigroup $\{f^{\circ n}\}_{n=0}^{\infty}$ of iterations of f. For semplice, we call f is a dynamical system. Linear dynamical systems are among simplest ones to study. For a dynamical, an important problem one would like to study is to estimate how far is it derived from linear dynamical systems? An important analytic characterization of a linear dynamical system f is

$$\log\Big|\frac{(f^{\circ n})'(x)}{(f^{\circ n})'(y)}\Big| = 0, \quad x, y \in I, n \geq 0.$$

Therefore, for a dynamical system f, we call the set of ratios

$$\Big\{ \log\Big|\frac{(f^{\circ n})'(x)}{(f^{\circ n})'(y)}\Big|\Big\}$$

the distortion. The distortion of a linear dynamical system contains only zero. Indeed, estimating the distortion of a dynamical system is useful and important. In this series of lectures, we will study dynamics of circle homeomorphisms and dynamics of interval mappings. Through this study we will also learn several important techniques in one-dimensional dynamical system which are widely used to estimate distortions of one-dimensional dynamical systems.

One technique, going back to Denjoy, estimates the distortion of a one-dimensional dynamical system without critical points. We can summarize this technique as follows.

Let f be a function defined on a set U of the real line $\mathbb{R}$. It is said to be C^1 (or $C^{1+\alpha}$ for $0 < \alpha \leq 1$ or C^{1+bv}) if it can be extended to a differentiable function defined on an open set containing U and if the derivative of the extension is continuous (or is α-Hölder continuous or is of bounded variation). Suppose f is a C^1 function on a set $U \subset \mathbb{R}$ and $X = \{x_i\}_{i=1}^n$ and $Y = \{y_i\}_{i=1}^n$ are two sequences of points in U. The number

$$\log\Big|\prod_{i=1}^n \frac{f'(x_i)}{f'(y_i)}\Big|$$

is called the distortion of f along X and Y.

Lemma 1 (Nävie Distortion Lemma)**.** *If*

$$\kappa = \inf_{x\in U} |f'(x)| > 0,$$

then the distortion of f along X and Y can be estimated as

$$\Big|\log\Big|\prod_{i=1}^n \frac{f'(x_i)}{f'(y_i)}\Big|\Big| \leq \frac{1}{\kappa}\sum_{i=1}^n |f'(x_i) - f'(y_i)|.$$

Moreover if f is of $C^{1+\alpha}$ for some $0 < \alpha \leq 1$, let

$$\iota = \sup_{x\neq y\in U}\Big(\frac{|f'(x) - f'(y)|}{|x-y|^\alpha}\Big) < \infty,$$

then the distortion of f along X and Y is bounded by $(\iota/\kappa)\sum_{i=1}^{n}|x_i - y_i|^{\alpha}$*, that is,*

$$\Big|\log\Big|\prod_{i=1}^{n}\frac{f'(x_i)}{f'(y_i)}\Big|\Big| \leq \frac{\iota}{\kappa}\sum_{i=1}^{n}|x_i - y_i|^{\alpha}.$$

Lemma 2. *If f is of C^{1+bv}, then there is a constant $C > 0$ so that the distortion of f along X and Y is bounded by C whenever the open intervals I_i, bounded by x_i and y_i, for $i = 1, \ldots, n$, are pairwise disjoint, that is,*

$$\Big|\log\Big|\prod_{i=1}^{n}\frac{f'(x_i)}{f'(y_i)}\Big|\Big| \leq C.$$

These distortion results are not difficult-they are just obtained from some direct calculations from Calculus. However, using them wisely in a research problem is a big challenge in the study of one-dimensional dynamical systems.

Suppose $I = [0,1]$ is the unit interval and $I_0 = [0,a]$ and $I_1 = [b,1]$ are two subintervals where $0 < a < b < 1$. A C^1 map f defined on $I_0 \cup I_1$ is said to be (degree two, for any degree it is similar) *expanding* if $f|I_i$ from I_i to I is bijective for $i = 0, 1$ and if there are constants $C > 0$ and $\lambda > 1$ so that

$$|(f^{\circ n})'(x)| \geq C\lambda^n$$

whenever $f^{\circ i}(x)$ are in $I_0 \cup I_1$ for all $i = 0, 1, \ldots, n-1$.

A C^1 map f defined on an interval I is said to have Hölder continuous derivative if there are constant $0 < \alpha \leq 1$ and $C > 0$ such that

$$|f'(x) - f'(y)| \leq C|x - y|^{\alpha}, \quad \forall x, y \in I.$$

More precisely, we call f $C^{1+\alpha}$. A C^1 map f of I is expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$(f^n)'(x)| \geq C\lambda^n, \quad \forall x \in I, n > 0.$$

Remark 3. *If a C^1 map $f : I_0 \cup I_1 \to I$ is expanding, then $f : \Lambda \to \Lambda$ is metrically expanding, thus topologically expanding.*

Topologically, a Cantor subset of the real line $\mathbb{R}$ is compact, uncountable, totally disconnected, and perfect. We will see later that Cantor sets or Cantor-like sets will appear again and again in the study of dynamical systems.

Theorem 4 (Cantor set)**.** *Suppose $f : I_0 \cup I_1 \to I$ is a topologically expansive map. Then the non-escaping set $\Lambda = \Lambda_f$ under f is a Cantor set.*

Proof. Let f_i be the restriction of the function f to I_i, and $g_i = f_i^{-1} : I \to I_i$ be the inverse of f_i for $i = 0$ or 1. We can consider compositions

$$g_{w_n} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_n}$$

for all strings $w_n = i_0 i_1 \ldots i_n$ of $0's$ and $1's$.

Suppose $w_n = i_0 i_1 \ldots i_n$ is a string of $0's$ and $1's$. Let $I_{w_n} = g_{w_n}(I)$ be the image of I under g_{w_n}, and let $G_{w_n} = g_{w_n}(G)$ be the set of all the points escaping to G under $g_{w_n}^{-1}$. The union $\cup_{w_n} I_{w_n}$ is the set of all points not escaping to G under the iterates $f^{\circ k}$ for $k = 0, 1, \ldots, n$, where w_n runs over all the strings of $0's$ and $1's$ of length $n+1$. The set $\{I_{w_n}\}$ is a collection of pairwise disjoint closed intervals and one to one correspondence with the set $\{w_n\}$ of all the strings of $0's$ and $1's$ of length $n+1$. Hence $\Lambda = \cap_{n=0}^{\infty} \cup_{w_n} I_{w_n}$, where w_n runs over all the strings of $0's$ and $1's$ of length $n+1$.

Let us first prove that Λ is uncountable. For a string $w_n = i_0 i_1 \ldots i_n$ of $0's$ and $1's$ and a digit $i_{n+1} = 0$ or 1, $I_{w_n i_{n+1}} \subseteq I_{w_n}$ since $I_{i_{n+1}} \subseteq I$. This implies that

$$\cdots \subseteq I_{i_0 i_1 \ldots i_n} \subseteq \cdots \subseteq I_{i_0 i_1} \subseteq I_{i_0}$$

and that

$$I_w = \cap_{n=0}^{\infty} I_{i_0 i_1 \ldots i_n}$$

is a non-empty closed subset for any infinite string $w = i_0 i_1 \ldots$ of $0's$ and $1's$. Hence the set $\{I_w\}$ is a collection of pairwise disjoint non-empty closed subsets and is in one to one correspondence with the uncountable set $\{w = i_0 i_1 \ldots\}$ of all infinite strings of $0's$ and $1's$. Hence the set $\{I_w\}$ is uncountable. So is the set Λ because $\Lambda = \cup_w I_w$ where $w = i_0 i_1 \ldots$ runs over all infinite strings of $0's$ and $1's$.

Since f is topologically expansive, $I_w = \{x_w\}$ contains only one point. The map $\pi(w) = x_w$ from $\{w\}$ to Λ is bijective. We use this to prove that Λ is totally disconnected, that is, every (connected) component Π of Λ contains only one number. Suppose there is a component Π of Λ which contains two different numbers x_w and $x_{w'}$ where $w = i_0 i_1 \ldots i_n i_{n+1} \ldots$ and $w' = i_0 i_1 \ldots i_n i'_{n+1} \ldots$ where $i_{n+1} \neq i'_{n+1}$. Both x_w and $x_{w'}$ are in I_{w_n} where $w_n = i_0 i_1 \ldots i_n$. The set I_{w_n} is the union of an open interval G_{w_n} and two closed intervals $I_{w_n i_{n+1}}$ and $I_{w_n i'_{n+1}}$ which are on different sides of G_{w_n}. The numbers x_w and $x_{w'}$ are in $I_{w_n i_{n+1}}$ and $I_{w_n i'_{n+1}}$, respectively. Take a point z in G_{w_n}. Then

$$\Pi = \Big(\Pi \cap (-\infty, z)\Big) \cup \Big(\Pi \cap (z, \infty)\Big).$$

This contradicts the statement that Π is a component of Λ and proves that Λ is totally disconnected.

Since Λ is closed, the set Λ' of limit points of Λ is contained in Λ. To prove that Λ is a perfect set, we only need to show that Λ is contained in Λ'. Let x_w be a number in Λ and $w = i_0 i_1 \ldots i_n i_{n+1} \ldots$. Let $r(i) = i + 1 \pmod 2$; $r(i)$ is 1 for $i = 0$ and 0 for $i = 1$. Take $w^{(n)} = i_0 \ldots i_{n-1} r(i_n) i_{n+1} \ldots$; $w^{(n)}$ differs from w at $(n+1)^{th}$ position. Then $x_{w^{(n)}} \neq x_w$ and both of them are in $I_{i_0 \ldots i_{n-1}}$. Since the length of $I_{i_0 \ldots i_{n-1}}$ tends to zero as n goes to infinity, $x_{w^{(n)}}$ tends to x_w as n goes to infinity. This says that x_w is a limit point of Λ. So Λ is contained in Λ'. Hence Λ is a Cantor set. $\square$

Exercise 2. *Construct a topologically expansive map $f : I_0 \cup I_1 \to I$ such that Λ has positive Lebesgue measure.*

Theorem 5 ($C^{1+\alpha}$-hyperbolic Cantor set, Lebegue measure)**.** *Suppose $f : I_0 \cup I_1 \to I$ is a $C^{1+\alpha}$ expanding map for some $0 < \alpha \leq 1$. Then the non-escaping set $\Lambda = \Lambda_f$ under f has zero Lebesgue measure.*

Proof. Let $m(\cdot)$ mean the Lebesgue measure and let $|J|$ mean the length of an interval. An inequality which can be easily obtained is

$$m(\Lambda) \leq \sum_{w_n} |I_{w_n}| < C 2^{n+1} \mu^n,$$

where w_n runs over all the strings of $0's$ and $1's$ of length $n+1$. This inequality is true because $\{I_{w_n}\}$ is a cover of Λ and the total number of the strings of $0's$ and $1's$ of length $n+1$ is 2^{n+1}. If $\mu < 1/2$, it is much easier to see $m(\Lambda) = 0$. However, to prove that the Lebesgue measure of Λ is zero for any $0 < \mu < 1$, we need help from Lemma 1

Suppose $w_n = i_0 \ldots i_n$ is a string of $0's$ and $1's$ of length $n+1$. The map f^{n+1} from I_{w_n} to I is a monotone function and its inverse is g_{w_n}. For any two numbers x and y in I_{w_n}, let $x_i = f^{\circ(n-i+1)}(x)$ and $y_i = f^{\circ(n-i+1)}(y)$ for $i = 0, 1, \ldots, n+1$. By the mean value theorem and the chain rule, $|x_i - y_i| < C\mu^i$ and $\sum_{i=0}^{n+1} |x_i - y_i|^\alpha < C/(1-\mu^\alpha)$. According to Lemma 1, the distortion of f along $X = \{x_i\}$ and $Y = \{y_i\}$ is bounded by the constant $C' = (\iota/\kappa)\big(C/(1-\mu^\alpha)\big)$, that is,

$$\Big|\log\Big|\frac{\Big(f^{\circ(n+1)}\Big)'(x)}{\Big(f^{\circ(n+1)}\Big)'(y)}\Big|\Big| \leq C',$$

where ι is the Hölder constant of f' on $I_0 \cup I_1$ and $\kappa = \inf_{x \in I_0 \cup I_1} |f'(x)|$. This implies that

$$\frac{|G_{w_n}|}{|I_{w_n}|} \geq c = e^{-C'} |G|,$$

since $G = f^{\circ(n+1)}(G_{w_n})$ and $I = f^{\circ(n+1)}(I_{w_n})$. Now we have that

$$|I_{w_n0}| + |I_{w_n1}| \leq (1-c)|I_{w_n}|$$

because $I_{w_n} = I_{w_n0} \cup G_{w_n} \cup I_{w_n1}$; moreover,

$$m(\Lambda) \leq \sum_{w_{n+1}} |I_{w_{n+1}}| = \sum_{w_n}(|I_{w_n0}|+|I_{w_n1}|) \leq (1-c)\sum_{w_n} |I_{w_n}| \leq \cdots \leq (1-c)^{n+1}$$

for all positive integers n. Hence the Lebesgue measure of Λ is zero. □

Problem 3. *Can you construct a C^1 expanding map $f : I_0 \cup I_1 \to I$ such that Λ has positive Lebesgue measure?*

Lecture 4. The Mandelbrot set and hyperbolic holomorphic dynamical systems

4.1. Quadratic polynomials.

Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plan, that is, the Riemann sphere. Suppose $f(z) = az^2 + bz + d$ is a quadratic polynomial, $a \neq 0$. By a linear conjugacy $h(z) = (1/a)(z - b/(2a))$, we can assume $q_c(z) = h^{-1} \circ f \circ h(z) = z^2 + c$, where $c = d - b^2/(4a)$. The map $q_c(z)$ has a unique critical point 0 in the complex plane. We consider the critical orbit $\mathcal{O}(0) = \{q_c^n(0)\}_{n=0}^{\infty}$. The Mandelbrot set

$$\mathcal{M} = \{c \mid \mathcal{O}(0) \text{ is bounded.}\}.$$

Theorem 6. *The set $\mathcal{M}$ is a closed bounded simply connected subset contained in the disk of radius 2 centered 0 in the complex plane. And $\mathcal{M} \cap \mathbb{R} = [-2, 1/4]$.*

Proof. If $|c| > 2$, then $|q_c^2(0)| = |q_c(c)| = |c^2 + c| \geq |c|(|c| - 1) > |c|$. Inductively, $|q_c^n(0)| \geq |c|(|c| - 1)^{2^{n-1}}$. This implies that

$$\mathcal{M} \subseteq D_2 = \{|c| \leq 2\}.$$

Thus it is bounded. Now consider $|c| \leq 2$ and there is an integer $m > 0$ such that $|q_c^m(0)| = 2 + t$, $t > 0$. Then we have that $|q_c^{m+1}(0)| \geq (2 + t)^2 - 2 \geq 2 + 4t$. Inductively, we have that $|q_c^{m+n}(0)| \geq 2 + 4^n t$. This implies $q_c^n(0) \to \infty$ as $n \to \infty$. So $c \notin \mathcal{M}$. This implies that $\mathcal{M}$ is closed. This also implies that $\mathbb{C} \setminus \mathcal{M}$ has no bounded component (by the maximum principle) and thus is connected. This implies that $\mathcal{M}$ is simply connected.

Consider $q_c(z) = z^2 + c$, we have $z^2 - z + c = 0$. This implies

$$z = \frac{1 \pm \sqrt{1 - 4c}}{2}$$

So q_c always has two fixed points

$$p_0 = \frac{1 + \sqrt{1 - 4c}}{2} \quad \text{and} \quad p_1 = \frac{1 - \sqrt{1 - 4c}}{2}.$$

When $c = 1/4$, this becomes a one fixed point of multiplier 2. When c and x are real and $c > 1/4$, q_c has no real fixed point. $q_c(x) > x$ for all $x > 0$. So $q_c^n(0) \to \infty$ as $n \to \infty$. When c is real and $-2 \leq c \leq 1/4$, the largest fixed point $p_0 \geq |c| = |q_c(0)|$. If $|q_c^m(0)| \leq p_0$, then $|q_c^{m+1}(0)| \leq p_0^2 + c = p_0$. Thus $c \in \mathcal{M}$. We get $\mathcal{M} \cap \mathbb{R} = [-2, 1/4]$. $\square$

4.2. Local dynamics of holomorphic maps, easy cases.

Let U be a domain in the complex plane $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be a holomorphic map. Suppose $z_0 \in U$ is a fixed point of f, i.e., $f(z_0) = z_0$. Then $\lambda = f'(z_0)$ is called the multiplier of f at z_0. The number λ is a conformal invariant, that is, if $g : V \to \mathbb{C}$ is another holomorphic map and w_0 is a fixed point of g and if $h : U \to V$ is a conformal map such that $w_0 = h(z_0)$ and such that $g = h \circ f \circ h^{-1}$, then $g'(w_0) = f'(z_0)$.

We can classify z_0 by λ as follows:

(1) Attracting if $0 < |\lambda| < 1$;
(2) Repelling if $|\lambda| > 1$;
(3) Super-attracting if $\lambda = 0$;
(4) Rational neutral if $\lambda = e^{2\pi \frac{p}{q}}$ where p and q are integers;
(5) Irrational neutral if $\lambda = e^{2\pi\theta}$ where $0 < \theta < 1$ is an irrational number.

Conjugating by a simple linear transformation $w \to w - z_0$, that is, by considering $g(w) = f(w + z_0) - z_0$, we will assume that $z_0 = 0$. We use

$$\Delta_\delta = \{z \in \mathbb{C} \mid |z| < \delta\}$$

to denote the disk centered at 0 with radius $\delta > 0$. In particular, $\Delta = \Delta_1$ is the unit disk.

4.2.1. Attracting case.

Suppose $0 \in U$ is an attracting fixed point of f. Let $\lambda = f'(0)$ with $0 < |\lambda| < 1$. Then $f(z) = \lambda z$ or its Taylor expansion of f at 0 is

$$f(z) = \lambda z + a_p z^p + \cdots$$

where $a_p \neq 0$ and $p \geq 2$. In the first case $f(z) = \lambda z$ is a linear map whose dynamics is easy to understand. In the second case, we will see that under an appropriate holomorphic change of coordinate, it is again a linear map.

Let $|\lambda| < \rho < 1$. There is a domain $0 \in U' \subset U$ such that

$$|f(z)| \leq \rho |z|, \quad z \in U'.$$

This implies

$$|f^n(z)| \leq \rho^n |z|, \quad n > 0, z \in U'.$$

Theorem 7 (Königs' Theorem). *There is a conformal map ϕ defined on Δ_δ for some $\delta > 0$ such that $\phi(0) = 0$ and*

$$\phi \circ f \circ \phi^{-1}(z) = \lambda z, \quad z \in \phi(\Delta_\delta).$$

The conjugacy ϕ is unique up to multiplication by a non-zero factor.

Proof. Let us first prove the "uniqueness". Suppose ϕ_1 defined on Δ_δ is another conjugacy such that $\phi_1(0) = 0$ and

$$\phi_1 \circ f \circ \phi_1^{-1}(z) = \lambda z, \quad z \in \phi_1(\Delta_\delta).$$

Then

$$\psi = \phi \circ \phi_1^{-1} : \phi_1(\Delta_\delta) \to \phi(\Delta_\delta)$$

is also a conformal map and commutes with λz, i.e.,

$$\psi(\lambda z) = \lambda \psi(z), \quad z \in \phi_1(\Delta_\delta).$$

So

$$\psi'(z) = \psi'(\lambda^n z) = \psi'(0), \quad z \in \phi_1(\Delta_\delta).$$

This implies that $\psi = c$ is a constant function. Thus $\phi = c\phi_1$.

Now let us prove the "existence". Define

$$\phi_n(z) = \lambda^{-n} f^n(z), \quad z \in U'.$$

We have

$$\phi_n \circ f(z) = \lambda^{-n} f^{n+1}(z) = \lambda \phi_{n+1}(z).$$

If $\phi_n \to \phi$ as $n \to \infty$ in some Δ_δ, then $\phi \circ f = \lambda\phi$.

So we need to show that $\{\phi_n\}_{n=0}^{\infty}$ is a Cauchy sequence on $\Delta_\delta \subset U'$ for some $\delta > 0$. First there are constants $\delta' > 0$ and $C' > 0$ such that

$$|f(z) - \lambda z| \leq C'|z|^2, \quad \text{for all} \quad |z| < \delta'.$$

This implies

$$|f(z)| \leq |\lambda||z| + C'|z|^2 \leq (|\lambda| + C'\delta')|z|.$$

By taking δ' small, we assume that $\rho = |\lambda| + C'\delta' < 1$. So

$$|f(z)| < \rho|z|.$$

And,

$$|f^n(z)| < \rho^n|z|, \quad n \geq 0, \ |z| < \delta'.$$

Again by taking δ' small, we assume $\mu = \rho^2/|\lambda| < 1$. Let $C = C'(\delta')^2/\lambda$. Then

$$\begin{aligned}
|\phi_{n+1}(z) - \phi_n(z)| &= \Big|\frac{f^{n+1}(z)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n}\Big| = \frac{|f^{n+1}(z) - \lambda f^n(z)|}{|\lambda^{n+1}|} \\
&\leq \frac{C'|f^n(z)|^2}{|\lambda^{n+1}|} \leq \frac{\rho^{2n}C'|z|^2}{|\lambda|^{n+1}} = C\mu^n, \quad |w| < \delta.
\end{aligned}$$

So ϕ_n converges uniformly to an analytic map ϕ on $\Delta_{\delta'}$. Since $\phi \circ f(z) = \lambda\phi(z)$ on $\Delta_{\delta'}$, ϕ is not a constant. Since $\phi_n'(0) = 1$, so $\phi'(0) = 1$. There is a $0 < \delta \leq \delta'$ such that ϕ is conformal on Δ_δ. $\square$

4.2.3. Repelling case.

If 0 is a repelling fixed point of $f : U \to \mathbb{C}$. Then there is another domain $V \subset U$ such that $f|V$ is injective. Its inverse f^{-1} on $f(V)$ is also holomorphic and 0 is an attracting fixed point of $f^{-1} : f(V) \to \mathbb{C}$. Let $\lambda = f'(0)$ be the multiplier of f at 0. Then λ^{-1} is the multiplier of f^{-1} at 0. Following the theorem in the attracting case, there is a conformal map ϕ defined on $\Delta_\delta \subset f(V)$ such that $\phi \circ f^{-1} \circ \phi^{-1}(z) = \lambda^{-1} z$. So

$$\phi \circ f \circ \phi^{-1}(z) = \lambda z, \quad z \in \phi(f^{-1}(\Delta_\delta)).$$

Since $\phi(f^{-1}(\Delta_\delta))$ is an open domain containing 0, we have

Theorem 8. *There is a conformal map ϕ defined on Δ_{δ_0} for some $\delta_0 > 0$ such that $\phi(0) = 0$ and*

$$\phi \circ f \circ \phi^{-1}(z) = \lambda z, \quad z \in \phi(\Delta_{\delta_0}).$$

The conjugacy ϕ is unique up to multiplication by a non-zero factor.

4.2.4. Super-Attracting case.

Suppose 0 is an isolated super-attracting fixed point of $f : U \to \mathbb{C}$. Then $f(z) = a_p z^p$ or the Taylor expansion of f at 0 is

$$f(z) = a_p z^p + a_q z^q + \cdots, \quad a_p, a_q \neq 0, \quad q > p \geq 2.$$

By changing the coordinate $\xi = \phi_0(z) = bz$ with $b^{p-1} = a_p$, we have

$$\phi_0 \circ f \circ \phi_0^{-1}(\xi) = \xi^p \quad \text{or} \quad \phi_0 \circ f \circ \phi_0^{-1}(\xi) = \xi^p + a_q b^{-q+1} \xi^q + \cdots.$$

So we assume

$$f(z) = z^p + a_q z^q + \cdots, \quad a_q \neq 0, \quad q > p > 1.$$

Theorem 9 (Böttcher's Theorem)**.** *There is a conformal map ϕ defined on $\Delta_\delta \subset U$ such that*

$$\phi \circ f \circ \phi^{-1}(z) = z^p, \quad w \in \phi(\Delta_\delta).$$

The conjugacy ϕ is unique up to multiplication by a $(p-1)^{th}$-root of the unity.

Proof. Let us first prove the "existence". Consider

$$f(z) = z^p(1 + O(z))$$

Then

$$f^n(z) = f(f^{n-1}(z)) = (f^{n-1}(z))^p(1 + O(f^{n-1}(z))) = \ldots$$
$$= z^{p^n}(1 + O(z))^{p^{n-1}} \cdots (1 + O(f^{n-2}(z)))^p(1 + O(f^{n-1}(z)))$$

There are a $0 < \rho < 1$ and Δ_δ for some $\delta > 0$ such that

$$|f(z)| \leq \rho |z|, \quad z \in \Delta_\delta.$$

This implies

$$|f^n(z)| \leq \rho^n |z|, \quad n \geq 1, \quad |z| < \delta.$$

So $f^n(z)$ tends to 0 as n goes to infinity. The image of Δ_δ under every $(1 + O(f^k(z)))$ avoids 0. Thus we has a well-defined branch $(1 + O(f^k(z)))^{p^{-k-1}}$ on Δ_δ for all $k > 0$. Let

$$g_n(z) = (1 + O(z))^{p^{-1}} \cdots (1 + O(f^{n-2}(z)))^{p^{-n+1}} (1 + O(f^{n-1}(z)))^{p^{-n}}$$

be a fixed branch and

$$\phi_n(z) = z g_n(z).$$

Then ϕ_n is holomorphic and

$$\phi_n(f(z)) = (\phi_{n+1}(z))^p, \quad z \in \Delta_{\delta'}.$$

If ϕ_n tends ϕ uniformly as n goes to infinity, then $\phi \circ f = \phi^p$. To show $\phi_n \to \phi$ uniformly, we note that

$$\phi_{n+1}(z) = \big(\phi_1(f^n(z))\big)^{p^{-n}} = \big(f^n(z)(1 + O(|f^n(z)|))\big)^{p^{-n}}.$$

So there are constants $C', C'' > 0$ such that

$$\Big|\frac{\phi_{n+1}(z)}{\phi_n(z)}\Big| \leq \big(1+C'|f^n(z)|\big)^{p^{-n}} \leq \big(1+C'\rho^n|z|\big)^{p^{-n}} \leq 1+C''\rho^n p^{-n}|z|, \quad |z| \leq \delta'.$$

Thus we have another constant $C > 0$ such that

$$\Big|\frac{\phi_{n+1}(z)}{\phi_n(z)}\Big| \leq 1 + Cp^{-n}, \quad |z| \leq \delta'.$$

Therefore,

$$\phi_n(z) = \phi_1(z) \prod_{n=1}^{\infty} \frac{\phi_{n+1}(z)}{\phi_n(z)} \to \phi(z) \quad n \to \infty$$

uniformly on $\Delta_{\delta'}$. Since $\phi_n(0) = 0$ and $|\phi_n'(0)| = 1$, we have $\phi(0) = 0$ and $|\phi'(0)| = 1$. So ϕ is conformal on Δ_δ for some $0 < \delta \leq \delta'$.

Now let us see the "uniqueness". If ϕ_1 and ϕ_2 are both conjugacies, i.e.,

$$\phi_1 \circ f = \phi_1^p \qquad \text{and} \qquad \phi_2 \circ f = \phi_2^p,$$

then

$$\phi_1 \phi_2^{-1}(z^p) = \big(\phi_1 \phi_2^{-1}(z)\big)^p.$$

Consider the Taylor expansion

$$\phi_1 \phi_2^{-1}(z) = az + a_2 z^2 + \cdots .$$

We have

$$az^p + a_2 z^{2p} + \cdots = (az)^p + \cdots .$$

This implies that $a_1 = a_1^p$ and $a_i = 0$ for all $i \geq 2$. So $\phi_1 = a_1 \phi_2$ with $a_1^{p-1} = 1$. $\square$

4.3. Basins of ∞ and Julia sets.

Given a quadratic polynomial $q_c(z) = z^2 + c$. Then ∞ is a super-attractive fixed point. According to Theorem 9, there is a conformal map ϕ_c from the neighborhood U_0 of ∞ to a neighborhood V_0 of ∞ such that

$$\phi_c \circ q_c \circ \phi_c^{-1}(z) = z^2 = q_0(z), \quad z \in V_0.$$

Let $V_1 = q_0^{-1}(V_0)$ and, inductively, let $V_n = q_0^{-1}(V_{n-1})$ for $n > 0$. Then it is clear

$$\mathbb{C} \setminus \Delta = \cup_{n=0}^{\infty} V_n.$$

Now let $U_1 = q_c^{-1}(U_0)$ and, inductively, let $U_n = q_c^{-1}(U_{n-1})$ for $n > 0$. Define

$$A_c(\infty) = \cup_{n=0}^{\infty} U_n \quad \text{and} \quad K_c = \mathbb{C} \setminus A_c(\infty).$$

Definition 12. *The domain $A_c(\infty)$ is called the basin of ∞ for q_c. Then set K_c is called the filled Julia set of q_c. The boundary*

$$J_c = \partial K_c = \partial A_c(\infty)$$

is called the Julia set of q_c.

If $0 \notin U_n$, then U_n is simply connected and we can use

$$\phi_c(z) = q_0^{-1}(\phi_c(q_c(z))),$$

to extend ϕ_c to U_n. That is, using the diagram

$$\begin{array}{ccc} U_n & \xrightarrow{\phi_c} & V_n \\ q_c \downarrow & & q_0 \downarrow \\ U_{n-1} & \xrightarrow{\phi_c} & V_{n-1} \end{array}$$

to lift ϕ_c on U_{n-1} to U_n since in this case $q_c : U_n \to U_{n-1}$ and $q_0 : V_n \to V_{n-1}$ are both covering maps. Then we need to discuss the following two cases:

(1) $c \in \mathcal{M}$ and
(2) $c \notin \mathcal{M}$.

In the first case, $0 \notin U_n$ for all $n \geq 0$. Then $A_c(\infty) = \cup_{k=0}^{\infty} U_n$ is a simply connected domain and we have a conformal map

$$\phi_c(z) : A_c(\infty) \to \mathbb{C} \setminus \overline{\Delta}$$

such that $\phi_c(\infty) = \infty$ and $\phi_c'(\infty) = 1$ and such that

$$\phi_c \circ q_c \circ \phi_c^{-1}(z) = z^2.$$

Recall the Riemann mapping theorem in complex analysis. A domain Ω is an open and (path) connected set in $\mathbb{C}$. A domain Ω is simply connected if every closed curve γ in Ω can be deformed in Ω to a point.

Theorem 10 (Riemann Mapping Theorem). *There is a conformal mapping ϕ from Ω onto the unit disk Δ for every simply connected domain Ω having at least two boundary points.*

The conformal mapping ϕ_c we constructed above is a Riemann mapping.

Definition 13. *The analytic curve*

$$S_r = \phi_c^{-1}(\{c = re^{2\pi i\theta} \in \mathbb{C}\} \mid 0 \leq \theta < 1\}),\ r > 1,$$

is called the equipotential curve for q_c and the analytic ray

$$E_\theta = \phi_c^{-1}(\{c = te^{2\pi i\theta} \in \mathbb{C} \mid t > 1\}),\ 0 \leq \theta < 1,$$

is called the external ray for q_c.

We have the following property

Proposition 2.

$$q_c(S_r) = S_{r^2}, \quad \text{and} \quad q_c(E_\theta) = E_{2\theta \pmod 1}.$$

Now the problem is that

Problem 4. *Can the inverse $\phi_c^{-1} : \Delta \to A_c(\infty)$ of the Riemann mapping ϕ_c be extended to the boundary J_c continuously?*

This problem is related with the topological property of the set J_c. In the case $c \in \mathcal{M}$, K_c is connected and thus J_c is also connected.

Definition 14. *A connected set $S \in \widehat{\mathbb{C}}$ is called locally connected if for every point $p \in S$ and any neighborhood V about p, there is a neighborhood $p \in U \subset V$ such that $S \cap U$ is connected.*

Example 9. *The set*

$$S_1 = \overline{\{x \sin(1/x) \mid 0 < x \leq 1/2\pi\}}$$

is locally connected. The set

$$S_2 = \overline{\{\sin(1/x) \mid 0 < x \leq 1/2\pi\}}$$

is not locally connected.

Theorem 11 (Carathéodory's Theorem). *Suppose Ω is a simply connected domain having at least two boundary points. The inverse ϕ^{-1} : $\Delta \to \Omega$ of the Riemann mapping $\phi : \Omega \to \Delta$ can be extended to a continuous map $\overline{\phi} : \overline{\Delta} \to \overline{\Omega}$ if and only if $S = \partial\Omega$ is locally connected.*

Thus Problem 4 is equivalent to the following problem.

Problem 5. *For which $c \in \mathcal{M}$, the Julia set J_c is locally connected.*

We say an external ray $E_\theta(t)$ lands at J_c if $\lim_{t\to 1^+} E_\theta(t) \in J_c$ exists. Thus if J_c is locally connected, then every external ray lands at J_c.

Definition 15. *A quadratic polynomial $q_c(z) = z^2 + c$ is called hyperbolic if there is a neighborhood U of J_c and two constants $C > 0$ and $\lambda > 1$ such that*

$$|(q_c^n)'(z)| \geq C\lambda^n$$

for any $z \in U$ such that $q_c^k(z) \in U$, $1 \leq k < n-1$. This is equivalent to say that there is a constant $0 < a \leq 1$ such that for any $z, w \in U$ with $|q_c^k(z) - q_c^k(w)| \leq a$ for all $0 \leq k \leq n-1$, then

$$|q_c^n(z) - q_c^n(w)| \geq C\lambda^n |z - w|.$$

Example 10. *If $|c|$ is small, then q_c is hyperbolic and J_c is a Jordan circle.*

More general, we define the main cardioid

$$M_0 = \{c \in \mathcal{M} \mid q_c \text{ has an attractive fixed point}\}.$$

For any $c \in M_0$, q_c is hyperbolic and J_c is a Jordan circle.

Theorem 12. *Suppose q_c is hyperbolic. Then $\phi_c^{-1} : \Delta \to A_c(\infty)$ can be extended to a continuous map $\overline{\phi_c} : \overline{\Delta} \to A_c(\infty) \cup J_c$ such that $\overline{\phi}_c(S^1) = J_c$. Thus J_c is locally connected.*

Proof. Without lost of generality, we assume $C = 1$. Otherwise, we consider an integer $k > 1$ such that $\sqrt[k]{C}\lambda > 1$ and consider q_c^k instead of q_c. Fix a real number $R > 1$. Consider the equipotential curve parametrized as $\gamma_n(\theta) = \phi_c^{-1}(R^{2^{-n}} e^{2\pi i\theta})$. Then $q_c(\gamma_n(\theta)) = \gamma_{n-1}(2\theta)$. This implies that

$$|\gamma_n(\theta) - \gamma_{n+1}(\theta)| \leq \lambda^{-1} |\gamma_{n-1}(2\theta) - \gamma_n(2\theta)|.$$

Thus γ_θ converges uniformly to a continuous function $\gamma(\theta)$ from S^1 to J_c. The function γ extends ϕ_c^{-1} continuously to the boundary of the open unit disk. □

Later, after we learn the global theory, we will give a characterization of hyperbolic quadratic polynomial as that

Theorem 13. *A quadratic polynomial is hyperbolic if it has an attractive periodic point.*

The above proof can be applied to a slightly more general quadratic polynomial called a sub-hyperbolic quadratic polynomial.

Definition 16. *A quadratic polynomial $q_c(z) = z^2 + c$ is called sub-hyperbolic if its critical orbit $\{q_c^n(0)\}_{n=0}^{\infty}$ is finite but is not a periodic orbit.*

In the sub-hyperbolic case, 0 is not periodic but some $p_k = q_c^k(0)$, $k > 0$, is periodic, that is, $q_c^{k+m}(0) = q_c^k(0)$. Let $p_i = q_c^i(0)$. Then the critical orbit is

$$0 \to p_1 \to \cdots \to p_k \to p_{k+1} \to \cdots \to p_{k+m-1} \to p_k.$$

In this case all $p_i \in J_c$. Let U be a neighborhood of J_c. Let $\rho(z)$ be a function smooth at all point in U but p_i's. At a small neighborhood about p_i, let $\rho(z) = 1/\sqrt{|z - p_i|}$. Then $\rho(z)|dz|$ is a new metric on U equivalent to $|dz|$. The expanding condition as defined in Definition 15 holds for this metric. Replace the Lebesgue metric $|\cdot|$ by $\rho(z)|dz|$ in the proof of Theorem 12, we have

Theorem 14. *Suppose q_c is sub-hyperbolic. Then $\phi_c^{-1} : \Delta \to A_c(\infty)$ can be extended to a continuous map $\overline{\phi_c} : \overline{\Delta} \to A_c(\infty) \cup J_c$ such that $\overline{\phi}_c(S^1) = J_c$. Thus J_c is locally connected.*

Example 11. *Let $q_{-2}(z) = z^2 - 2$. Then $p_1 = -2$ and $p_2 = q_{-2}(p_1) = 2$ which is fixed by q_{-2}. Thus q_{-2} is sub-hyperbolic. The Julia set $J_{-2} = [-2, 2]$ which is clearly locally connected. Then $\rho(z) = 1/\sqrt{|z^2 - 4|}$. One can check that the absolute value of the derivative of q_{-2} with respect to the new metric $\rho(z)|dz|$ is 2, that is*

$$|D_\rho(q_{-2})(z)| = \frac{\rho(q_{-2}(z))q'_{-2}(z)}{\rho(z)} = 2.$$

The study of Problem 5 (equivalently, Problem 4) can be divided into the following cases according to $c \in \mathcal{M}$:

(1) hyperbolic point;
(2) sub-hyperbolic point;
(3) rational neutral point;
(4) irrational neutral Cremer point
(5) irrational neutral Siegel point;
(6) finitely renormalizable point;
(7) infinitely renormalizable point.

A lot of research has been done. However, for 7) there are still some points which have not too good and not too bad combinatorics which are not fully understood. Refer to

- J. Milnor, Dynamics in One Complex Variable: Introductory Lectures. Third Edition. Annals of Mathematics Studies.

- J. H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's Sixtieth Birthday (1993).
- Y. Jiang, Renormalization and Geometry in One-Dimensional and Complex Dynamics, Advanced Series in Nonlinear Dynamics 10, World Sci. Publ., 1996.
- Y. Jiang, Infinitely renormalizable quadratic polynomials. Trans. Amer. Math. Soc., 352(2000), no. 11, 5077-5091.
- M. Lyubich, Dynamics of quadratic polynomials, I-II. Volume 178, Issue 2 , pp 185-297.
- G Levin and S Van Strien, Local connectivity of the Julia set of real polynomials. Annals of mathematics, 1998, 471-541.

4.4. Structures of the complement of the Mandelbrot set.

Now let us consider $c \notin \mathcal{M}$. From the proof of Theorem 9, we can write ϕ_c as

$$\phi_c(z) = \lim_{n\to\infty} q_c(z)^{2^{-n}} = z\prod_{n=1}^{\infty}\Big(\frac{q_c^n(z)}{(q_c^{n-1}(z))^2}\Big)^{2^{-n}} = z\prod_{n=0}^{\infty}\Big(1+\frac{c}{(q_c^n(z))^2}\Big)^{2^{-n-1}}.$$

The functions $\phi_c(z)$ is holomorphic on both variables z and c and is defined up to the domain U_n such that $c \in U_n$. Thus $\Phi(c) = \phi_c(c)$ is defined for $c \in \mathbb{C} \setminus \mathcal{M}$. Since

$$\Phi(c) = c\prod_{n=0}^{\infty}\Big(1+\frac{c}{(q_c^n(c))^2}\Big)^{2^{-n-1}},$$

it has a simple pole at ∞ and $\Phi'(\infty) = 1$. Douady and Hubbard, independently, Sibony, proved that $\Phi : \mathbb{C} \setminus M \to \mathbb{C} \setminus \overline{\Delta}$ is a conformal mapping (by using some properties of Green's function $G(c) = |\Phi(c)|$ which can be extended to the whole $\mathbb{C}$ as a harmonic function such that $G(c) = 0$ for $c \in \mathcal{M}$ which we will study later). Refer to

- L. Carleson and T. Gamelin, Complex Dynamics, Springer 1993.
- A. Douady and J. Hubbard, Itèration des polynômes quadratiques complexes, C.R. Acad. Sc. Paris, t. 294 (18 janvier 1982).

This implies that

Theorem 15. *The Mandelbrot set M is connected.*

Another way to prove the above theorem topologically is to consider

$$\Phi_n(c) = q_c^{n+1}(0).$$

It is a polynomial of c of degree 2^n. Remember that we use $D_2 = \{c \in \mathbb{C} \mid |c| \leq 2\}$ to denote the closed disk of radius 2 centered at 0. Define

$$M_n = \Phi_n^{-1}(D_2).$$

Then from the proof of Theorem 6, we knew that

$$\mathcal{M} = \cap_{n=1}^{\infty} M_n.$$

Kahn showed that M_n is connected for all $n > 0$. Thus $\mathcal{M}$ is connected. Refer to

- J. Kahn, The Mandelbrot set is Connected: A topological proof. Preprint.

The analytic curve

$$\mathcal{S}_R = \Phi^{-1}(\{c = Re^{2\pi i\theta} \in \mathbb{C} \mid 0 \leq \theta < 1\}), \quad R > 1,$$

is called the equipotential curve of $\mathcal{M}$ and the analytic ray

$$\mathcal{E}_\theta = \Phi^{-1}(\{c = te^{2\pi i\theta} \in \mathbb{C} \mid t > 1\}), \quad 0 \leq \theta < 1,$$

is called the external ray of $\mathcal{M}$. We say an external ray $E_\theta(t)$ lands at $\mathcal{M}$ if $\lim_{t\to 1^+} E_\theta(t) \in \mathcal{M}$ exists.

Conjecture 1. *Every external ray $E_\theta(t)$ lands.*

Moreover, we have that

Conjecture 2. *The inverse*

$$\Phi^{-1}(c) : \widehat{\mathbb{C}} \setminus \overline{\Delta} \to \widehat{\mathbb{C}} \setminus \mathcal{M}$$

of the Riemann mapping can be extended to the circle S^1 continuously.

Equivalently, we have

Conjecture 3 (MLC conjecture)**.** *The Mandelbrot set $\partial\mathcal{M}$ is locally connected.*

Another conjecture as a conclusion of the MLC conjecture is the following famous hyperbolicity conjecture. Suppose

$$\mathcal{H} = \{c \in \mathcal{M} \mid q_c \text{ is hyperbolic}\} = \{c \in \mathcal{M} \mid q_c \text{ has an attractive periodic point}\}$$

It is clear that $\mathcal{H}$ is an open set in $\mathbb{C}$.

Conjecture 4 (Hyperbolicity Conjecture)**.** *The set $\mathcal{H}$ is dense in $\mathcal{M}$.*

The study of the MLC conjecture was divided into the following cases:

(1) rational neutral points;

(2) irrational neutral points;
(3) finitely renormalizable points;
(4) infinitely renormalizable points.

A lot of research has been done in this direction but there are still some remaining unknown points which are infinitely renormalizable points. Refer to

- J. H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's Sixtieth Birthday (1993).
- Y. Jiang, Y. Jiang, Renormalization and Geometry in One-Dimensional and Complex Dynamics, Advanced Series in Nonlinear Dynamics 10, World Sci. Publ., 1996.
- Y. Jiang, Local connectivity of the Mandelbrot set at certain infinitely renormalizable points. Complex Dynamics and Related Topics, New Studies in Advanced Mathematics, 2004, The International Press, 236-264.
- J. Kahn and M. Lyubich, A priori bounds for some infinitely renormalizable quadratics, I. Bounded primitive combinatorics, ArXiv Preprint; II. Decorations. Annales Scientifiques de lÉcole Normale Superieure; III. Molecules. Complex Dynamics, the proceedings of the conference in honor of J. H. Hubbard's 60th birthday.

4.6. Julia sets for $c \notin \mathcal{M}$.

For $c \notin \mathcal{M}$, let

$$S_{R(c)} = \phi_c^{-1}(\{R(c)e^{2\pi i\theta} \mid 0 \leq \theta < 1\})$$

be the equipotential curve passing c. Let

$$E_{\theta(c)} = \phi_c^{-1}(\{te^{2\pi i\theta(c)} \mid t \geq R(c)\})$$

be the external ray landing at c. The preimage of $S_{R(c)}$ under q_c is a 8-shape analytic curve

$$S_{\sqrt{R(c)}} = \phi_c^{-1}(\{\sqrt{R(c)}e^{2\pi i\theta} \mid 0 \leq \theta < 1\})$$

intersecting at 0. Two external rays

$$E_{\theta(c)/2} = \phi_c^{-1}(\{te^{\pi i\theta(c)} \mid t \geq \sqrt{R(c)}\})$$

and

$$E_{(\theta(c)+1)/2} = \phi_c^{-1}(\{te^{\pi i(\theta(c)+1)} \mid t \geq \sqrt{R(c)}\})$$

land at the intersection point 0.

Now consider another equipotential curve $S_{R(c)-\epsilon}$ for $\epsilon > 0$. Let V be the open domain bounded by $S_{R(c)-\epsilon}$. Let $U = q_c^{-1}(V)$. We have a small $\epsilon > 0$ such that $\overline{U} \subset V$. The set U has two disjoint simply connected domains V_0 and V_1 such that $q_c|V_0$ and $q_c|V_1$ are one-to-one and onto V. Let $J = J_c$ be the non-escaping set for $q_c : U = V_0 \cup V_1 \to V$. The $J = J_c = \partial K_c$ is the non-escaping set and the Julia set.

The map $q_c : J \to J$ is a Markovian dynamical system with the initial partition $\eta_0 = \{J_0, J_1\}$, where $J_0 = J \cap V_0$ and $J_1 = J \cap V_1$, such that $f(J_0) = J$ and $f(J_1) = J$. Thus we have a homeomorphism $\pi = \pi_c : \Sigma^+ = \prod_{n=0}^{\infty}\{0,1\} \to J$ such that $q_c \circ \pi = \pi \circ \sigma$.

Before a further study of this dynamical system, we study some properties in complex analysis and in hyperbolic geometry. Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk. Let $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$ for $0 < r < 1$.

Lemma 3 (Schwarz Lemma)**.** *For all* $|z| < 1$, *suppose that* $f(z) = a_1 z + a_2 z^2 + \cdots$ *converges, and that* $|f(z)| \leq 1$. *Then for all* $0 < |z| < 1$,

$$|f(z)| \leq |z| \qquad \textit{and} \qquad |f'(0)| = |a_1| \leq 1$$

with equalities unobtainable except if $f(z) = e^{i\alpha} z$ *for some real number* α.

Proof. Define the function

$$g(z) = \frac{f(z)}{z} = a_1 + a_2 z + \dots .$$

By the maximum modulus principle, $g(z)$ cannot achieve a maximum of $|g(z)|$ in Δ_r. Thus

$$|g(z)| \leq \frac{1}{r}$$

for z in Δ_r. This holds for any $z \in \Delta$ and any $|z| < r < 1$. Therefore,

$$|g(z)| \leq 1,$$

that is,

$$|f(z)| \leq |z| \qquad \text{and} \qquad |a_1| = |g(0)| \leq 1$$

for $|z| < 1$.

If the equality sign holds for some $0 < |z_0| < 1$ (or if $|a_1| = 1$), then $|f(z_0)| = |z_0|$ (or $|g(0)| = 1$). Unless g is a constant function, g must map a neighborhood about z_0 (or 0) onto a neighborhood about $g(z_0)$ (or $g(0)$) since an analytic function maps an open set to an open set. Since $|g(z_0)| = 1$ (or $|g(0)| = 1$), this would imply that there are points z arbitrarily close to z_0 (or 0) for which $|g(z)| > 1$, which would contradict the first part of this argument. Hence, if the equality sign holds anywhere in $0 < |z| < 1$ (or if $|a_1| = 1$), then g must be a constant function. In this case, $|g(z)| = 1$ everywhere, whence $f(z) = e^{i\alpha}z$ on Δ for some real number α. $\square$

An analytic function f from Δ into $\mathbb{C}$ is called a schlicht function (or a univalent function or a conformal mapping) if it is one-to-one. This is equivalent to saying $f'(z) \neq 0$ for all z in Δ.

Lemma 4. *Every conformal map f from Δ onto itself is a Möbius transformation*

$$f(z) = a\frac{z+b}{1+\bar{b}z}$$

where a and b are complex numbers with $|a| = 1$ and $|b| < 1$.

Proof. Suppose $f(0) = 0$; by applying Lemma 3 to f and to f^{-1}, we get

$$|f(z)| \leq |z| \text{ and } |z| \leq |f(z)|.$$

Thus,

$$f(z) = az$$

with $|a| = 1$. In general, consider

$$g = \gamma \circ f,$$

where

$$\gamma(z) = \frac{z - f(0)}{1 - \overline{f(0)}z}.$$

Then g is a schlicht function from Δ onto Δ such that $g(0) = 0$. Thus $g(z) = az$ with $|a| = 1$, whence

$$f(z) = \gamma^{-1}(az) = \frac{az + f(0)}{1 + a\overline{f(0)}z} = a\frac{z+b}{1+\bar{b}z}$$

where $b = \overline{a}f(0)$. $\square$

Definition 17. *The hyperbolic disk, which is still denoted as Δ, is the open unit disk Δ equipped with the metric*

$$d_H s = \rho(z)|dz| = \frac{|dz|}{1 - |z|^2}.$$

The corresponding distance d_H is

$$d_H(z_1, z_2) = \inf_l \int_l d_H s = \inf_l \int_l \frac{|dz|}{1 - |z|^2}$$

where l are over all C^1 curves in Δ connecting z_1 and z_2.

Remark 4. *The metric $d_H s = \rho(z)|dz|$ is called the Poincaré metric (or hyperbolic metric). The distance d_H is called the (induced) hyperbolic distance.*

Theorem 16. *Suppose f is an analytic function from Δ into Δ. Then for all z_1 and z_2 in Δ,*

$$d_H\big(f(z_1), f(z_2)\big) \leq d_H(z_1, z_2)$$

with equality unobtainable except if

$$f(z) = a\frac{z+b}{1+\bar{b}z}$$

for complex numbers a and b with $|a| = 1$ and $|b| < 1$.

Proof. For a fixed z in Δ, let

$$g(\xi) = \frac{f(\frac{\xi+z}{1+\bar{z}\xi}) - f(z)}{1 - \overline{f(z)} f(\frac{\xi+z}{1+\bar{z}\xi})}.$$

Then $g(0) = 0$ and $|g(\xi)| \leq 1$. Thus we have $|g'(0)| \leq 1$, whence

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}.$$

The equality sign holds if and only if $g(\xi) = c\xi$ with $|c| = 1$, which is equivalent to saying f takes a Möbius transformation. Take two points z_1 and z_2 in Δ. For any C^1 curve l in Δ connecting z_1 and z_2,

$$d_H(f(z_1), f(z_2)) \leq \int_{f(l)} \frac{|dz|}{1-|z|^2} = \int_l \frac{|f'(z)|}{1-|f(z)|^2}\,|dz| \leq \int_l \frac{|dz|}{1-|z|^2}.$$

Therefore,

$$d_H\big(f(z_1), f(z_2)\big) \leq d_H(z_1, z_2).$$

The equality sign holds for some z_1 and z_2 if and only if f is a Möbius transformation. $\square$

Suppose Ω is a simply connected domain. Let $g : \Omega \to \Delta$ be the Riemann map which maps a basepoint $z_0 \in \Omega$ to 0. The hyperbolic metric $d_{H,\Omega}s$ on Ω is

$$d_{H,\Omega}s = \rho(g(z))|g'(z)||dz|.$$

All results (with appropriate constants) in this section applies to an analytic function defined on a simply connected domain Ω whose boundary $\partial\Omega$ contains at least two points. In particular, if

$$\Omega = \mathbb{H} = \{z = x + yi \in \mathbb{C} \mid y > 0\}$$

is the upper-half plane, then the hyperbolic metric is

$$d_{H,\mathbb{H}}s = \frac{|dz|}{|\overline{z}-z|} = \frac{|dz|}{2y}.$$

Any analytic function $f : \mathbb{H} \to \mathbb{H}$ contracts this hyperbolic metric.

For the disk Δ_r, the hyperbolic metric

$$d_{H,\Delta_r}s = \frac{r|dz|}{r^2 - z^2}$$

and the hyperbolic distance is $d_{H,\Delta_r}(\cdot,\cdot)$.

Now for any $c \notin \mathcal{M}$, let $S_{R(c)}$ be the equipotential curve passing c. Let $D(c)$ be the domain bounded by $S_{R(c)}$. Let V be a domain bounded by an equipotential curve $S_{R(c)-\epsilon}$ for some small $\epsilon > 0$. Consider g_0 and g_1 are two inverse branches of q_c^{-1} on $D(c)$. Then we have

$$g_0 : D(c) \to g_0(D(c)) \subset D(c)$$

and

$$g_1 : D(c) \to g_1(D(c)) \subset D(c).$$

It is clear that both g_0 and g_1 are not Möbius transformations. Thus g_0 and g_1 contract the hyperbolic distance $d_{H,D(c)}(\cdot,\cdot)$, that is,

$$d_{H,D(c)}(g_i(z), g_i(w)) < d_{H,D(c)}(z,w), \quad z, w \in D(c)$$

for $i = 0, 1$. Since $\overline{V}$ is a compact subset of $D(c)$, we have a constant $0 < \tau < 1$ such that

$$d_{H,D(c)}(g_i(z), g_i(w)) \leq \tau d_{H,D(c)}(z,w), \quad z, w \in V.$$

for $i = 0, 1$. Thus we have that

$$d_{H,D(c)}(g_{w_n}(z), g_{w_n}(w)) \leq \tau^{n+1} d_{H,D(c)}(z,w), \quad z, w \in V.$$

where $w_n = i_0 \dots i_{n-1} i_n$ and $g_{w_n} = g_{i_0} \cdot \dots \cdot g_{i_{n-1}} \cdot g_{i_n}$. On V, $d_{H,D(c)}(\cdot,\cdot)$ is equivalent to the Lebesgue metric, that is, there is a constant $C > 0$ such that

$$C^{-1}|z-w| \leq d_{H,D(c)}(z,w) \leq C|z-w|, \quad \forall z, w \in V.$$

This implies that

$$|g'_{w_n}(z)| \leq C\tau^{n+1}, \quad \forall z \in V,\ \forall w_n,\ \forall n \geq 0,$$

for some positive constant which we still denote as C. Equivalently, we say that $q_c : J \to J$ is a hyperbolic dynamical system, that is, there are constants $C > 0$ and $\lambda > 1$ such that

$$|(q_c^{n+1})'(z)| \geq C\lambda^{n+1}, \quad \forall z \in J, n \geq 0.$$

Similar to the proof of Theorem 4

Theorem 17. *For any $c \notin \mathcal{M}$, q_c is hyperbolic and its Julia set J_c is a Cantor set.*

Next we will study the Lebesgue measure for the hyperbolic dynamical system $q_c : J \to J$ for $c \notin \mathcal{M}$. This study works for any hyperbolic rational map which we will introduce later. For this study, we need some results in complex analysis which has been obtained in 60's, refer to Chapter V of

- G.M.Goluzin, Geometric Theory of Functions of a Complex Variable, Translation of Mathematical Monographs, Vol 26, AMS, 1969.

A domain D in the Riemann sphere $\widehat{\mathbb{C}}$ is called double connected if its complement $\widehat{\mathbb{C}} \setminus D$ consists of two disjoint connected sets.

Theorem 18. *Suppose D is a double connected domain. Then there is a conformal map f from D to one of the following domains:*

(1) $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$;
(2) $\Delta^* = \Delta \setminus \{0\}$;
(3) $A_R = \{z \in \mathbb{C} \mid 1 < |z| < R\}$.

And no one of the above three domains is conformally equivalent to another one.

Proof. If $\widehat{\mathbb{C}} \setminus D$ contains only two points $\{a, b\}$, then we have a Möbius transformation f which maps a to 0 and b to ∞. It is the first case.

If one component of $\widehat{\mathbb{C}} \setminus D$ contains one point a and the other E contains at least two points. By the Rimeann mapping theorem, there is a conformal map $f_1 : \widehat{\mathbb{C}} \setminus E \to \Delta$. Then we have a Möbius transformation $f_2 : \Delta \to \Delta$ such that $f_2(f_1(a)) = 0$. Then $f = f_2 \circ f_1$ is a conformal map from $\widehat{\mathbb{C}} \setminus D$ to Δ^*. This is the second case.

Now let us prove the third case: both components E_1 and E_2 of $\widehat{\mathbb{C}} \setminus D$ contain more than two points. By the Rimeann mapping theorem, there is a conformal map $f_1 : \widehat{\mathbb{C}} \setminus E_2 \to \Delta$. Again, we have a conformal map $f_2 : \widehat{\mathbb{C}} \setminus f_1(E_1) \to \widehat{\mathbb{C}} \setminus \overline{\Delta}$. Then $f_2(S^1)$ is an analytic closed curve $\gamma(t)$ in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. The unit circle and the closed curve bounded a doubly connected domain D'.

The analytic map $f_3(z) = e^z$ maps a region $\tilde{D}'$ bounded by the imaginary axis and an analytic curve $\tilde{\gamma}(t)$ in the right-plane into D', where $\tilde{\gamma}(t) \to -i\infty$ as $t \to -\infty$ and $\tilde{\gamma}(t) \to i\infty$ as $t \to \infty$. It is a universal covering. Using the Riemann mapping theorem again, we have a conformal mapping f_4 from $\tilde{D}'$ to the strip $S_d = \{z = x+iy \mid 0 < y < d\}$ such that it can be extended to the boundary $\partial\tilde{D}'$ with $f_4(0) =$

0, $f_4(-i\infty) = -i\infty$ and $f_4(i\infty) = i\infty$. By picking an appropriate d, we can have $f_4(2\pi i) = 2\pi i$. The conformal map $f_4(z+2\pi i) - 2\pi i$ from $\tilde{D}'$ to S_d also fixes 0, $-i\infty$,and $i\infty$. By the uniqueness in the Riemann mapping theorem, we have $f(z+2\pi i) = f(z) + 2\pi i$.

Now f_3 maps the strip S_d to the round annulus A_R for $R = \log d$. Now $f = f_3 \circ f_4 \circ f_3^{-1} \circ f_2 \circ f_1$ is a conformal map from D onto A_R, where $f_3^{-1} = \log z$ is the inverse of $f_3 : \tilde{D}' \cap \{z = x + iy \mid 0 \le y < 2\pi\} \to D'$.

It is clear that three of domains are not conformally equivalent each other. □

Remark 5. *By using the Schwarz reflection theorem, A_R is unique for a given D in the third case.*

Definition 18. *In the third case, $m(D) = (1/2\pi)\log R$ is called the modulus of D. In the first case and the second case, $m(D) = \infty$.*

Remark 6. *The modulus is a conformal invariant, that is, there is a conformal map $f : D_1 \to D_2$ between two doubly connected domains D_1 and D_2 if and only if $m(D_1) = m(D_2)$.*

Remark 7. *The proof of Theorem 18 is only for the doubly connected planar domain where only the Riemann mapping theorem is used in the proof. It actually holds for any double connected Riemann surface. A Riemann surface is double connected if its fundamental group is $\mathbb{Z}$. However, the known proof of Theorem 18 for any Riemann surface using the uniformization theorem which says that any simply connected Riemann surface is conformally equivalent to either the Riemann sphere $\mathbb{C}$, or the complex plane $\mathbb{C}$, or the open unit disk Δ. Refer to*

- *W. Abikoff, The Uniformization Theorem. The American Mathematical Monthly, Vol. 88, No. 8 (Oct., 1981), pp. 574-592.*

Given a round annulus A_R. Suppose f is an analytic map on A_R. Then we have the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

Let $\Gamma_r = \{z \in \mathbb{C} \mid 1 < |z| = r < R\}$ be the circle in A_R. Then $f(\Gamma_r)$ is a closed curve in $\mathbb{C}$. Let Ω_r be the region bounded by $f(\Gamma_r)$ and $\zeta = f(z)$. Then from Green's theorem,

$$Area(\Omega_r) = \frac{1}{2i}\oint_{\partial\Omega_r} \overline{\zeta} d\zeta = \frac{1}{2i}\oint_{|z|=r} \overline{f(z)} f'(z) dz$$

$$= \frac{1}{2i} \oint_{|z|=r} (\sum_{n=-\infty}^{\infty} \overline{c}_n \overline{z}^n)(\sum_{n=-\infty}^{\infty} n c_n z^{n-1}) dz$$

$$= \frac{1}{2} \int_0^{2\pi} \sum_{n,k=-\infty}^{\infty} (n c_n \overline{c}_k r^{n+k} e^{2\pi(n-k)i\theta}) d\theta = \pi \sum_{n=-\infty}^{\infty} n |c_n|^2 r^{2n}.$$

Now suppose Ω_1 and Ω_2 are two bounded simply connected domains. Suppose $\overline{\Omega_1} \subset \Omega_2$. Then $D = \Omega_2 \setminus \overline{\Omega_1}$ is a doubly connected domain. Let $m(D) = (1/(2\pi)) \log R$ be the modulus of D.

Lemma 5.

$$R^2 \leq \frac{Area(\Omega_2)}{Area(\Omega_1)}.$$

The equality holds if and only if

$$\Omega_1 = \{z \in \mathbb{C} \mid |z| < R_1\} + c_0 \quad \textit{and} \quad \Omega_2 = \{z \in \mathbb{C} \mid |z| < R_2\} + c_0$$

for $R_1 < R_2$ and $R = R_2/R_1$.

Proof. Let $f : A_R \to D$ be an conformal map from A_R onto D. Let $1 < r < s < R$ be two real numbers. Let

$$\Gamma_r = \{z \in \mathbb{C} \mid |z| = r\} \quad \text{and} \quad \Gamma_s = \{z \in \mathbb{C} \mid |z| = s\}$$

be circles in A_R. Then $f(\Gamma_r)$ and $f(\Gamma_s)$ are closed curves in D. Let Ω_r be the region bounded by $f(\Gamma_r)$ and let Ω_s be the region bounded by $f(\Gamma_s)$ Then

$$\frac{Area(\Omega_s)}{Area(\Omega_r)} - \frac{s^2}{r^2} = \frac{\sum_{n=-\infty}^{\infty} n|c_n|^2 s^{2n}}{\sum_{n=-\infty}^{\infty} n|c_n|^2 r^{2n}} - \frac{s^2}{r^2}$$

$$= \frac{r^2 \sum_{n=-\infty}^{\infty} n|c_n|^2 s^{2n} - s^2 \sum_{n=-\infty}^{\infty} n|c_n|^2 r^{2n}}{r^2 \sum_{n=-\infty}^{\infty} n|c_n|^2 r^{2n}}$$

$$= \frac{\sum_{n=-\infty}^{\infty} n|c_n|^2 r^2 s^2 (s^{2(n-1)} - r^{2(n-1)})}{r^2 \sum_{n=-\infty}^{\infty} n|c_n|^2 r^{2n}} > 0.$$

Let $r \to 1$ and $s \to R$. We have that $Area(\Omega_s) \to Area(\Omega_2)$ and $Area(\Omega_r) \to Area(\Omega_1)$. Then

$$\frac{Area(\Omega_2)}{Area(\Omega_1)} - R^2 \geq 0.$$

This proves the inequality.

The equality holds for all $n \neq 0, 1$, $c_n = 0$. That is $f(z) = c_1 z + c_0$. □

We study a more general case than the quadratic case, that is, $d = 2$ in the following notation.

Suppose Ω is a bounded simply connected region. Suppose Ω_0, ..., Ω_{d-1} are bounded simply connected regions such that

$$\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset, \ \ \forall\, 0 \leq i \neq j \leq d-1$$

and

$$\cup_{i=0}^{d-1} \overline{\Omega}_i \subset \Omega.$$

Let $f : \cup_{i=0}^{d-1} \overline{\Omega}_i \to \overline{\Omega}$ be a holomorphic map, that means that f is holomorphic on a neighborhood of $\cup_{i=0}^{d-1} \overline{\Omega}_i$. Suppose $f|\overline{\Omega}_i : \overline{\Omega}_i \to \overline{\Omega}$ is conformal for every $0 \leq i \leq d-1$. Let

$$J = \cap_{n=0}^{n} f^{-n}(\overline{\Omega})$$

be the non-escaping set of f (we also call it the Julia set later). Then from the hyperbolic geometry we know that $f : J \to J$ is an expanding Markovian map with the initial partition

$$\eta_0 = \{J_0, \ldots, J_{d-1}\}$$

where $J_i = J \cap \overline{\Omega}_i$. Then we have a homeomorphism

$$\pi = \pi_f : \Sigma_d = \prod_{n=0}^{\infty} \{0, \ldots, d-1\} \to J.$$

Thus J is a Cantor set (by using the same proof as the proof of Theorem 4).

Theorem 19. *The Lebesgue measure $Area(J) = 0$.*

Proof. Let $B \subset \Omega$ be a simply connected region such that

$$\cup_{i=0}^{d-1} \overline{\Omega}_i \subset B \quad \text{and} \quad \overline{B} \subset \Omega.$$

Then $G = \Omega \setminus \overline{B}$ is a doubly connected domain. Let

$$m(G) = \frac{1}{2\pi} \log R$$

be the modulus where $R > 1$. Let

$$g_i = (f|\overline{\Omega}_i)^{-1} : \overline{\Omega} \to \overline{\Omega}_i, \quad 0 \leq i \leq d-1.$$

For any $w_n = i_0 \ldots i_{n-1}$ of a sequence of $\{0, \ldots, d-1\}$,let

$$g_{w_n} = g_{i_0} \circ \ldots g_{i_{n-1}} : \overline{\Omega} \to \overline{\Omega}_{w_n} = g_{w_n}(\overline{\Omega}).$$

Then we have

$$\overline{B}_{w_n} = g_{w_n}(\overline{B}) \subset \Omega_{w_n} \quad \text{and} \quad \cup_{i=0}^{d-1} \overline{\Omega}_{w_n i} \subset B_{w_n} = g_{w_n}(B).$$

The region $G_{w_n} = \Omega_{w_n} \setminus \overline{B}_{w_n}$ is a double connected domain with the same modulus of G, that is,

$$m(G_{w_n}) = m(G) = \frac{1}{2\pi} \log R.$$

From Lemma 5,

$$Area(J) \leq \sum_{i=0}^{d-1} Area(\overline{\Omega}_{w_n i}) \leq Area(B_{w_n}) \leq \frac{1}{R^2} Area(\Omega_{w_n}) = \frac{1}{R^2} Area(\overline{\Omega}_{w_n}).$$

Inductively, we have that

$$Area(J) \leq \frac{1}{R^{2(n+1)}} Area(\Omega).$$

Thus $Area(J) = 0$. $\square$

More general, suppose $U = \cup_{k=1}^n U_k$ is a union of pairwise disjoint connected regions. Suppose f on U is a holomporphic map such that $V = f(U) \supset \overline{U}$. Then we can define the non-escaping set $K = \{z \mid f^n(z) \in U, \ \forall n > 0\}$. We say that $f : K \to K$ is expanding if there are two constants $C > 0$ and $\lambda > 1$ such that

$$(f^n)'(z) \geq C\lambda^n, \ \ \forall n > 0$$

Following a similar proof of Theorem 19, we have that

Corollary 1. *Suppose $f : K \to K$ is expanding. Then it is an Markovian dynamical system and $J = \partial K = K$ and $Area(J) = 0$.*

For example, for $c \in \mathcal{M}_0$, the main cardioid of the Mandelbrot set. The Julia set $J = J_c$ of q_c is a Jordan circle. Then we have a doubly connected domain $U \supset J$ such that $V = q_c(U) \supset \overline{U}$. We have a homeomorphism $h : S^1 \to J_c$ such that $q_c \circ h = h \circ q_0$. Let

$$J_0 = h(\{e^{2\pi i\theta} \mid 0 \leq \theta \leq 1/2\}) \quad \text{and} \quad J_1 = h(\{e^{2\pi i\theta} \mid 1/2 \leq \theta \leq 1\}).$$

Then

$$\eta_0 = \{J_0, J_1\}$$

is a Markov partition for $q_c : J \to J$. Let

$$\eta_1 = f^{-1}(\eta_0) = \{J_{00}, J_{01}, J_{10}, J_{11}\}$$

be the second Markov partition. Let $U_{i_1 i_0}$ be a simply connected domain containing $J_{i_1 i_0}$. Then $V_{i_0} = q_c(U_{i_1 i_0}) \supset \overline{U}_{i_1 i_0}$ and $q_c : U_{i_1 i_0} \to V_{i_0}$ is conformal. Let $B_{i_0} \supset J_{i_1 i_0}$ be a simply connected domain such that $\overline{B}_{i_0} \subset V_{i_0}$. Then $D_{i_0} = V_{i_0} \setminus \overline{B}_{i_0}$ is a doubly connected domain. Then using the exact the same argument as that of Theorem 19, we have that $Area(J) = 0$.

In general, we can consider a polynomial of degree $d \geq 2$,

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0, \quad a_d \neq 0.$$

Since

$$P(z) = a_d z^d \Big(1 + \frac{a_{d-1}}{a_d z} + \cdots + \frac{a_1}{a-dz^{d-1}} + \frac{a_0}{a_d z^d} \Big),$$

we have a disk $U = \Delta_r$ for $r >$ large enough such that $V = f(U) \supset \overline{U}$. The filled Julia set

$$K = \cap_{n=1}^{\infty} P^{-n}(\overline{U})$$

is a compact subset. The boundary $J = \partial K$ is the Julia set of P.

Let C be the set of critical points of P, that is,

$$P'(c) = a_d c^d + a_{d-1} c^{d-1} + \cdots + a_1 c + a_0 = 0, \quad c \in C.$$

Then P has $d-1$ critical points in the complex plane (counted by multiplicity) and ∞ is another critical point with the multiplicity $d-1$ So P has $2d-2$ critical points (counted by multiplicity). We write

$$C = \{c_1, \ldots, c_{d-1}\}$$

be the set of all critical point in the complex plane. Consider critical orbits $\mathcal{O}_i = \{c_{i,n} = P^n(c_i)\}_{n=0}^{\infty}$. If all $c_{i,n} \to \infty$ as $n \to \infty$, then we can see that, from the hyperbolic geometry, $J = K$ is a Cantor set and P on J is an expanding Markovian hyperbolic dynamical system, that is, there are two constants $C > 0$ and $\lambda > 1$ such that

$$(P^n)'(z) \geq C\lambda^n, \quad \forall\, z \in J,\ n > 0$$

and we have a partition $J = J_1 \cup \cdots \cup J_n$ such that $f|J_i$ is one-to-one for every $1 \leq i \leq n$ and $f(J_i) = \cup_{i=1}^{s} J_{k_i}$. Indeed, Let D be a large open disk containing J such that $\overline{D} \subset P(D)$. Choose an integer $m > 0$ such that $P^m(c) \in \mathbb{C} \setminus \overline{D}$ for every critical point c of P. Then D contains no critical values $P^n(C)$ for $n \geq m$. All inverse branches R^{-n} are defined in D and map D conformally into D. Let D_i^n, $0 \leq i < d^n$, be the components of $R^{-n}(D)$. Then each D_i^n is a topological disk such that $\overline{D_i^n} \subset D$ and $R^n : D_i^n \to D$ is conformal. Suppose $z \in J$, then $P^n(z) \in J$. We define f_n as the inverse branch of P^n which maps $P^n(z)$ to z. Then $\{f_n\}_{n\geq m}$ are uniformly bounded in a neighborhood of $\overline{D}$. So it is a normal family. Since $d(f_n(w), J) \to 0$ as $n \to \infty$ for any $w \in D \cap B_\infty$, any limiting function f of a convergent subsequence of $\{f_n\}_{n\geq m}$ maps $D \cap B_\infty$ into J. But J has no interior point, so f is a constant function. This implies that $|f_n(\overline{D})| \to 0$ as $n \to \infty$. Since $f_n(\partial D) \cap J = \emptyset$, z must be a connected component of J. So J is totally disconnected. But we know J is perfect. So J is a Cantor set. Following a similar proof of Theorem 19, we have that

Theorem 20. *If all critical orbits of P tend to ∞, then $P|J$ is an expanding Markovian dynamical system and the Julia set J is a Cantor set with $Area(J) = 0$.*

Suppose there is no finite critical point of P in B_∞. Then we can extended the Böttcher coordinate to the whole B_∞ as we have discussed for a quadratic polynomial. Let $\phi : B_\infty \to \widehat{\mathbb{C}} \setminus \overline{\Delta}$ be the Böttcher coordinate, that is, it is the 1-1 analytic (or conformal) conjugacy with $\phi(\infty) = \infty$ and $\phi'(\infty) = 1$ such that

$$\phi(P(z)) = \phi(z)^d, \quad z \in B_\infty$$

Then $G(z) = \log|\phi(z)|$ is the Green function of (B_∞, ∞) satisfying

$$G(P(z)) = dG(z), \quad z \in B_\infty.$$

The Green function G can be also defined as

$$G(z) = \max\{0, \lim_{n\to\infty} \frac{1}{d^n} \log|P^n(z)|\}.$$

Define

$$S_r = \{z \in \mathbb{C} \mid G(z) = \log r\} = \phi^{-1}(\{z \in \mathbb{C}) \mid |z| = r\}, \quad 1 < r < \infty$$

and

$$E_\theta = \{z \in \mathbb{C} \mid \arg\phi(z) = 2\pi\theta\} = \phi^{-1}(\{z = re^{2\pi i\theta} \mid 1 < r < \infty\}), \quad 0 \le \theta < 1.$$

We call S_r the equipotential circle of radius r and E_θ the external ray of angle θ. It is clear that

$$P(S_r) = S_{r^d} \quad \text{and} \quad P(E_\theta) = E_{d\theta \pmod 1}.$$

If $\mathcal{B}_\infty$ contains no finite critical point of P. Then $\mathcal{B}_\infty$ is simply connected. Thus we have that

Theorem 21. *The filled-in Julia set K (or the Julia set J) is connected if and only if there is no finite critical point of P in B_∞.*

Lecture 5. Koebe distortions

Let $\mathbb{C}$ be the complex plane and let

$$\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$$

be the open unit disk of the complex plane. A function defined on Δ is analytic if it can be written as a convergent power series, that is,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for z in Δ. We first prove a theorem obtained by Koebe in 1907.

Theorem 22 (Koebe's $\frac{1}{4}$-Theorem)**.** *Suppose $f : \Delta \to \mathbb{C}$ is a schlicht function. Then $f(\Delta)$ contains an open disk centered at $f(0)$ with radius $|f'(0)|/4$.*

First we prove a lemma.

Lemma 6. *Suppose*

$$f(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n z^n : \Delta \to \mathbb{C}$$

is a schlicht function. Then

$$|a_2| \leq 2.$$

Proof. Suppose $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$ for $0 < r < 1$. Then $G_r = f(\Delta_r)$ is a bounded subset of the complex plane $\mathbb{C}$. Let G_r' be the complement of G_r. For $w = Re^{\Theta i}$ in G_r',

$$\int\int_{G_r'} w^t \overline{w}^t R \, dR \, d\Theta > 0$$

provided the integral converges, whence

$$\lim_{r \to 1} \int\int_{G_r'} w^t \overline{w}^t R \, dR \, d\Theta \geq 0.$$

The integral certainly converges if $t < -1$. Let us set $z = re^{\theta i}$ and $w = f(z)$. On the boundary of G_r', which is the image of $|z| = r$ under f, we set $R = R(\theta)$ and $\Theta = \Theta(\theta)$. Then

$$\int\int_{G_r'} w^t \overline{w}^t R \, dR \, d\Theta = \int\int_{G_r'} R^{2t+1} \, dR \, d\Theta = -\int_0^{2\pi} \frac{R^{2t+2}}{2t+2} \frac{d\Theta}{d\theta} \, d\theta > 0.$$

We have

$$\Theta = \arg w = \frac{\log w - \log \overline{w}}{2i}.$$

Therefore
$$\frac{d\Theta}{d\theta} = \frac{w'\overline{w}z + \overline{w}'w\overline{z}}{2w\overline{w}},$$
and
$$\int_0^{2\pi} \frac{R^{2t+2}}{2t+2}\frac{d\Theta}{d\theta}\,d\theta = \int_0^{2\pi} \frac{R^{2t}(w'\overline{w}z + \overline{w}'w\overline{z})}{4(t+1)}\,d\theta < 0.$$
This is easily transformed into
$$\int_0^{2\pi} \Big(\overline{w}^{t+1}z\frac{dw^{t+1}}{dz} + w^{t+1}\overline{z}\frac{d\overline{w}^{t+1}}{d\overline{z}}\Big)\,d\theta < 0.$$
But
$$w^{t+1} = z^{t+1}\Big(1 + (t+1)a_2z + \Big((t+1)a_3 + \frac{t(t+1)}{2}a_2^2\Big)z^2 + \cdots\Big).$$
We substitute this series into the last integral and integrate term by term, observing that for any non-zero integer k, $\int_0^{2\pi} e^{ki\theta}d\theta = 0$. Thus we obtain
$$1 + (t+1)(t+2)|a_2|^2r^2 + (t+1)(t+3)\Big|a_3 + \frac{t}{2}a_2^2\Big|^2 r^4 + \cdots > 0.$$
For $r \to 1$, this yields
$$1 + (t+1)(t+2)|a_2|^2 + (t+1)(t+3)\Big|a_3 + \frac{t}{2}a_2^2\Big|^2$$
$$+ \cdots + (t+1)(t+k)\Big|a_k + g_k(a_2, \ldots, a_{k-1})\Big|^2 + \cdots \geq 0.$$
In particular, we take $t = -3/2$; then
$$1 - \frac{|a_2|^2}{4} \geq 0,$$
whence
$$|a_2| \leq 2.$$
□

Remark 8. *In general, $|a_n| \leq n$ for all $n \geq 2$. This was the famous Bieberbach conjecture formulated in 1916 and proved by De Branges in 1984.*

Proof of Theorem 22. Let $f(z) \neq c$ for all $|z| < 1$. The function
$$f_1(z) = \frac{f(z) - f(0)}{f'(0)} = z + a_2z^2 + \cdots \neq \frac{c - f(0)}{f'(0)}$$
for all $|z| < 1$. Set
$$f_2(z) = \frac{f(0) - c}{f'(0)}\frac{f(z) - f(0)}{f(z) - c} = z + \Big(a_2 - \frac{f'(0)}{f(0) - c}\Big)z^2 + \cdots.$$

Both of f_1 and f_2 are schlicht functions from Δ into $\mathbb{C}$. Thus

$$|a_2| \leq 2 \quad \text{and} \quad \Big|a_2 - \frac{f'(0)}{f(0)-c}\Big| \leq 2.$$

Therefore,

$$|c - f(0)| \geq \frac{|f'(0)|}{4}.$$

We have thus proved that no boundary point, of the image of $|z| < 1$ under the mapping f, has distance from $f(0)$ less than $|f'(0)|/4$. $\square$

Lemma 7. *Suppose* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n : \Delta \to \mathbb{C}$ *is a schlicht function. Then*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Proof. For a point z in Δ, let

$$g(\xi) = \frac{f(\frac{\xi+z}{1+\overline{z}\xi}) - f(z)}{f'(z)(1-z\overline{z})}.$$

This is a schlicht function from Δ into $\mathbb{C}$ and has a power series expansion valid in $|\xi| < 1$:

$$g(\xi) = \xi + b_2\xi^2 + \dots ,$$

where

$$b_2 = \frac{1}{2}\Big(\frac{f''(z)(1-z\overline{z})}{f'(z)} - 2\overline{z}\Big).$$

Therefore,

$$\Big|\frac{f''(z)(1-z\overline{z})}{f'(z)} - 2\overline{z}\Big| \leq 4.$$

Hence

$$\Big|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\Big| \leq \frac{4|z|}{1-|z|^2}.$$

Thus

$$\frac{2|z|^2 - 4|z|}{1-|z|^2} \leq \Re\Big(\frac{zf''(z)}{f'(z)}\Big) \leq \frac{2|z|^2 + 4|z|}{1-|z|^2}.$$

Since

$$\Re\Big(\frac{zf''(z)}{f'(z)}\Big) = |z|\frac{\partial}{\partial|z|}\Big(\Re\big(\log f'(z)\big)\Big) = |z|\frac{\partial}{\partial|z|}\Big(\log|f'(z)|\Big),$$

we obtain

$$\frac{2|z|-4}{1-|z|^2} \leq \frac{\partial}{\partial|z|}\Big(\log\big(|f'(z)|\big)\Big) \leq \frac{2|z|+4}{1-|z|^2}.$$

Integration now yields

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

□

Theorem 23 (Koebe Distortion Theorem)**.** *Suppose* $f : \Delta \to \mathbb{C}$ *is a schlicht function. Then*

$$\Big(\frac{1-r}{1+r}\Big)^4 \leq \frac{|f'(z_1)|}{|f'(z_2)|} \leq \Big(\frac{1+r}{1-r}\Big)^4$$

for all $|z_1| \leq r < 1$ *and all* $|z_2| \leq r < 1$.

Proof. Let $g(z) = (f(z) - f(0))/f'(0)$. Then $g'(z) = f'(z)/f'(0)$. Thus, this theorem follows from Lemma 7. □

Corollary 2. *Suppose* $f(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots : \Delta \to \mathbb{C}$ *is a schlicht function, then*

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2}.$$

Remark 9. *The above theorems, lemmas, corollary are are the sharpest possible. One sees this with the example*

$$f(z) = \frac{z}{(1-z)^2}.$$

Lecture 6. Theory of Fatou and Julia

Recall that $\widehat{\mathbb{C}}$ is the Riemann sphere. Let

$$d_{SP}s = \frac{2|dz|}{1+|z|^2}$$

is the spherical metric. Then the spherical distance $d_{SP}(z_1, z_2)$ is

$$d_{SP}(z_1, z_2) = \inf_l \int_l d_{SP}s$$

where inf takes all smooth curves connecting z_1 and z_2.

Suppose $\mathcal{F}$ is a family of meromorphic functions defined on a domain U in $\widehat{\mathbb{C}}$. Then $\mathcal{F}$ is said to be *normal* if every sequence $\{f_n\}_{n=0}^{\infty}$ in $\mathcal{F}$ contains a subsequence that converges uniformly in the spherical metric on compact subsets of U. Following Arzelá-Ascoli theorem, the family $\mathcal{F}$ is normal if and only if it is equicontinuous on every compact subset of U (under spherical metric). Note that we allow $f_n \to \infty$ in the definition of a normal family. Thus it is more convenient to consider the function $f \equiv \infty$ as a meromorphic function.

Theorem 24. *A family $\mathcal{F}$ of analytic functions on a domain U in $\mathbb{C}$ which is bounded by some fixed constant is a normal family.*

Proof. Since U can be covered by disks, it is sufficient to prove the theorem for a disk. We may assume that U is the unit disk. Suppose M is a fixed constant such that $|f(z)| \leq M$ for all $f \in \mathcal{F}$ and $|z| < 1$. Cauchy's theorem implies that $|f'(z)| \leq M/(1-r)$ on each closed disk $\{z||z| \leq r < 1\}$. Therefore, $\mathcal{F}$ is equicontinuous on every compact set of U. So $\mathcal{F}$ is normal. $\square$

Let $\mathcal{U}$ be the set of all univalent functions f on the unit disk $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ such that $f(0) = 0$ and $f'(0) = 1$.

Corollary 3. *The family $\mathcal{U}$ is normal and any limit of any sequence in $\mathcal{U}$ is still in $\mathcal{U}$.*

Proof. By the Koebe distortion theorem,

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}, \quad z \in \Delta.$$

So $\mathcal{U}$ is uniformly bounded on any compact set of Δ. Suppose $\{f_n\}$ is a convergent sequence in $\mathcal{U}$. Let f be its limiting function. Following Hurwitz's theorem and the normalization $f_n'(0) = 1$, f is either univalent or a constant function. But $|f'(z)| \geq (1-|z|)/(1+|z|)^3$. So f is univalent. $\square$

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Theorem 25 (Montel's Theorem)**.** *Let $\mathcal{F}$ be a family of meromorphic functions on a domain U of $\widehat{\mathbb{C}}$. If there are three fixed values that are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

Proof. Following the same reason as that in the proof of the previous theorem, we assume that U is a disk. Furthermore, by conjugating by a Möbius transformation, we assume that the functions in $\mathcal{F}$ do not take the values 0, 1, and ∞. Let $V = \mathbb{C} \setminus \{0, 1\}$. Then V is a hyperbolic surface. Its universal cover is the unit disk Δ. Let $\phi : \Delta \to V$ be a universal covering map. Let $\tilde{f} : \Delta \to \Delta$ be a lift of $f \in \mathcal{F}$, i.e., $\tilde{f} \circ \phi = f$. Then the family $\tilde{\mathcal{F}} = \{\tilde{f}; f \in \mathcal{F}\}$ is normal. So $\mathcal{F}$ is normal too. $\square$

Now suppose

$$f(z) = \frac{p(z)}{q(z)}$$

is a rational function, where

$$p(z) = a_n z^n + \cdots + a_0 \quad \text{and} \quad q(z) = b_m z^m + \cdots + b_0$$

are two complex polynomials of degrees n and m, respectively. The degree of f is $d = \max\{n, m\}$. Let f^n mean the n^{th}-iterate of f.

Definition 19. *A point $z \in \widehat{\mathbb{C}}$ is called a* Fatou point *if there is a neighborhood U about z such that $\{f^n|U\}_{n=0}^{\infty}$ is a normal family. The set F of all Fatou points is called the Fatou set of f. The complement $J = \widehat{\mathbb{C}} \setminus F$ is called the Julia set of f.*

From the definition, the Fatou set F is an open set and the Julia set J is a compact set.

Theorem 26. *If the degree $d > 1$, then the set J is not empty.*

Proof. Since the degree d of f is greater than one, suppose $J = \emptyset$, then $\{f^n\}$ is normal on the whole $\widehat{\mathbb{C}}$ and there is a subsequence $\{f^{n_i}\}$ converges to a rational function f_0 on $\widehat{\mathbb{C}}$. If f_0 is a constant function, then $f^{n_i}(\widehat{\mathbb{C}})$ is contained in a small neighborhood of c, but it is impossible since $R^{n_i} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is onto. If f_0 is not a constant function, then f^{n_i} has the same number of zeros as f_0 for j large enough (apply the argument principle), which is also impossible since f^{n_i} has degree d^{n_i}. $\square$

In the following, we always assume that the degree d of f is greater than 1.

Theorem 27. *The Julia set J and the Fatou set F are both completely invariant, i.e., $f(J) = f^{-1}(J) = J$ and $f(F) = f^{-1}(F) = F$. Moreover, $J_{f^n} = J_f$ and $F_{f^n} = F_f$.*

Proof. The proof is from the definition directly. $\square$

Example 12. *Let $f(z) = z^n$ where $n > 1$ is an integer. Then the Julia set J is the unit circle and the Fatou set $F = \Delta \cup \overline{\Delta}^c$, where*

$$\Delta = \{z \in \widehat{\mathbb{C}} \mid |z| < 1\} \quad \text{and} \quad \overline{\Delta}^c = \{z \in \widehat{\mathbb{C}} \mid |z| > 1\}$$

are basins of 0 and ∞, respectively.

Example 13. *Let $f(z) = z^2 - 2$. Take $z = h(w) = w + 1/w$. It maps Δ and $\overline{\Delta}^c$ onto $\widehat{\mathbb{C}} \setminus [-2, 2]$ and the unit circle S^1 onto $[-2, 2]$. Since $h^{-1} \circ f \circ h(w) = w^2$, the Julia set and the Fatou set of f are $J = [-2, 2]$ and $F = \widehat{\mathbb{C}} \setminus [-2, 2]$, which is the basin of ∞.*

The following example is little complicate but fun to know.

Example 14. *Let $f(z) = \lambda z + z^2$ with $|\lambda| = 1$. Suppose there is a domain V_0 containing 0 such that there is a conformal map $\phi : V_0 \to \Delta$ making $\phi \circ f \circ \phi^{-1}(z) = \lambda z$ on Δ. Let B_∞ be the basin of ∞ for f. Then $B_\infty, V_0 \subset F$ and $V_0 \cap B_\infty = \emptyset$ since $f^n(V_0) \not\to \infty$ as n goes to infinity. Let U_0 and U_∞ be the components of the Fatou set F containing V_0 and B_∞. Then we have that $U_\infty = B_\infty$ and $\{f^n|U_0\}_{n=1}^\infty$ is uniformly bounded. The domain U_0 is simply connected. The reason is the following. For any curve γ in U_0, let D_γ be the bounded domain bounded by γ. If U_0 is not simply connected, then there is a such γ such that D_γ contains some point from Julia set. Then $\cup_{n=0}^\infty f^n(D_\gamma) \supseteq \mathbb{C} \setminus \{\text{one point}\}$. But $D_\gamma \cap U_\infty = \emptyset$. This is a contradiction. The simply connected domain U_0 is the maximal domain on which ϕ can be extended such that $\phi \circ f \circ \phi^{-1}(z) = \lambda w$. Here U_0 is called the Siegel disk of f. Let $\tilde{U}_0 = \cup_{n=0}^\infty f^{-n}(U_0)$, It is clear now that $F = \tilde{U}_0 \cup U_\infty$ and $J = \partial \tilde{U}_0 = \partial B_\infty$.*

Let $z \in J$. Suppose U is a neighborhood of z. By Montel's theorem, $E_z = \widehat{\mathbb{C}} \setminus \cup_{n=0}^\infty f^n(U)$ contains at most two points.

Theorem 28. *The set E_z is independent of z. So we can denote it as E. If $E = \{a\}$ contains only one point, we can use a Möbius transformation to move a to infinity, then f is a polynomial. If $E = \{a, b\}$ contains two points, we can use a Möbius transformation to move a to infinity and b to zero, then $f(z) = Cz^d$ or Cz^{-d}. In any cases, E is a subset of F.*

Proof. By the definition, $f^{-1}(E_z) \subseteq E_z$. If $E_z = \{a\}$ contains only one point, then $R(a) = a$. Assume $a = \infty$. Since $f^{-1}(\infty) = \infty$, f has no other poles. So f is a polynomial. Clearly, E_z is independent of z. If $E = \{a, b\}$, then either (1) $f(a) = a$ and $f(b) = b$ or (2) $f(a) = b$ and $f(b) = a$. Assume $a = 0$ and $b = \infty$. In the first case, f is a polynomial and $f(0) = 0$ and $f^{-1}(0) = \{0\}$. So $f(z) = Cz^d$. In the second case, similarly we can get, $f(z) = Cz^{-d}$. □

The set E is called the exceptional set of f. It follows immediately if $z \notin E$, then

$$\overline{\cup_{n=0}^{\infty} f^{-n}(z)} \supset J.$$

This implies the following two theorems.

Theorem 29. *The backward iterates $\cup_{n=0}^{\infty} f^{-n}(z)$ of any point $z \in \mathcal{J}$ is dense in J.*

Theorem 30. *Any non-empty completely invariant set of J is dense in J.*

Since the union of the boundaries of all components of the Fatou set F is closed and completely invariant, it is the Julia set.

Theorem 31. *The Julia set J is a perfect set, that is, J contains no isolated points.*

Proof. Take $z \in J$ and U a neighborhood about z. Suppose z is not a periodic point of f. Then there is a point z_1 such that $f(z_1) = z$ and $f^n(z) \neq z_1$ for all n. Since $z_1 \in J$,

$$\overline{\cup_{n=0}^{\infty} f^{-n}(z_1)} = J.$$

Therefore, there is a point $\xi \in U$ such that $f^n(\xi) = z_1$. Thus $\xi \in J \cap U$ but $\xi \neq z$.

If z is periodic, let $m > 0$ be the minimal integer such that $f^m(z) = z$. If z is the only point in $f^{-m}(z)$, then z is super-attracting. This contradicts to that $z \in J$. So there is a point $z_1 \neq z = f^m(z)$ in $f^{-m}(z)$. Furthermore, $f^j(z) \neq z_1$, $1 \leq j < m$. (Otherwise, $f^j(z) = f^{j+m}(z) = f^m(z_1) = z$, contradicts to the minimal property of m.) Similarly, there is a point $\xi \in U$ such that $f^n(\xi) = z_1$ since

$$\overline{\cup_{n=0}^{\infty} f^{-n}(z_1)} = \mathcal{J}.$$

This $\xi \in \mathcal{J} \cap U$ but $\xi \neq z$. □

Theorem 32. *If J has an interior point, then $J = \widehat{\mathbb{C}}$.*

Proof. Suppose U is a domain contained in J. Then $f^n(U) \subset J$. Since $\cup_{n=0}^{\infty} f^n(U) = \widehat{\mathbb{C}} \setminus E$ and J is closed set, $J = \overline{\cup_{n=0}^{\infty} f^n(U)} = \widehat{\mathbb{C}}$. □

Theorem 33. *Let Γ be the set of all repelling periodic points of f. Then $\overline{\Gamma} = J$.*

Proof. Suppose U is a domain such that $U \cap J \neq \emptyset$ and such that $U \cap \Gamma = \emptyset$. Assume U contains no poles and no critical points of f. Let f_1 and f_2 be two inverse branches of $f|U$. Since there is no solution of $f^m(z) = z$ in U,

$$g_n(z) = \frac{f^n(z) - f_1(z)}{f^n(z) - f_2(z)} \frac{z - f_2(z)}{z - f_1(z)}$$

omits the values $\{0, 1, \infty\}$. By Montel's theorem $\{g_n\}$ is a normal family and hence so is $\{f^n|U\}$. This is a contradiction. Therefore the set of all periodic points of f is dense in J. Since there are only finite number of attracting, super-attracting, and neutral periodic points (see the next section). So $\overline{\Gamma} = J$. □

Concluding our discussion above, we have that the Julia set is non-empty, compact, perfect and is the closure of $\cup_{n=0}^{\infty} f^{-n}(z)$ for any point z in the Julia set. Moreover, either it is the whole Riemann sphere $\widehat{\mathbb{C}}$ or has no interior point and the set of all repelling periodic points is dense in the Julia set.

Recall that a point z is called a periodic point of period n of f if $f^n(z) = z$ and $f^i(z) \neq z$ for $i = 1, \cdots, n-1$. In the previous lectures, we classified a periodic point into attractive, repelling, super-attractive, rational neutral, and irrational neutral. Directly from the definition, we know that an attractive or super-attracting periodic point z is a Fatou point. An repelling periodic point z is in the Julia set.

Next we will see when a neutral periodic point is in the Fatou set or in the Julia set. By the definition, a neutral periodic point of period n is a point $p \in \widehat{\mathbb{C}}$ such that $f^n(p) = p$ and $|(f^n)'(p)| = 1$. Then we divide it into rationally neutral if $(f^n)'(p) = e^{2\pi i p/q}$ with integers $p \geq 0$ and $q \geq 1$, $(p, q) = 1$, and irrationally neutral if $(f^n)'(p) = e^{2\pi i\theta}$ with an irrational number $0 < \theta < 1$.

In the irrational case, we divide it into two cases, Siegel and Cremer, as follows. We call p is a Siegel periodic point if there is a domain $U \subset \widehat{\mathbb{C}}$ about p and a conformal map $\phi : U \to \Delta$ such that $\phi(p) = 0$ and $\phi \circ f^n \phi^{-1}(z) = e^{2\pi i\theta} z$. The maximal such domain U is called the Siegel disk at p. Otherwise, p is called a Cremer periodic point. Without loss of generality, we assume $p = 0$ and $f(0) = 0$. Let $\lambda = f'(0)$ with

$\lambda = e^{2\pi i\theta}$ with an irrational number $0 < \theta < 1$. Consider the Taylor expansion

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$$

in a domain U about 0. To decide 0 is a Siegel fixed point or Cremer fixed point we need to see if the Schröder equation

$$f(h(z)) = h(\lambda z), \quad h(0) = 0, \;\; h'(0) = 1,$$

has a solution or not in some Δ_r, $r > 0$. The reason is that

Lemma 8. *Any analytic solution h of the above Schröder equation is conformal, i.e., h is one-to-one.*

Proof. Suppose $h(z_1) = h(z_2)$ for $z_1, z_2 \in \Delta_r$. Then $h(\lambda^n z_1) = h(\lambda^n z_2)$ for all n. Since θ is irrational, $\{n\theta \pmod 1\}$ is dense on $[0,1]$. So $\{\lambda^n\}$ is dense on the unit circle S^1. By continuity, $h(z_1 e^{2\pi i\tau}) = h(z_2 e^{2\pi i\tau})$ for all τ. Since $h(z_1\cdot)$ and $h(z_2\cdot)$ are analytic, we have $h(z_1 z) = h(z_2 z)$ for all $z \in \Delta_r$. From $h'(0) = 1$, we have $z_1 = z_2$. $\square$

The characterization of the Schröder equation has a solution is that

Theorem 34. *The Schröder equation has a solution if and only if $\{f^n|K\}$ is uniformly bounded for any compact set K in some neighborhood domain U of* 0

Proof. If the Schröder equation has a solution $h : \Delta \to U$, then $f^n(z) = h(\lambda^n h^{-1}(z))$. It is clearly that $\{f^n|K\}$ is uniformly bounded for any compact set K in U.

On the other hand if there is a constant $M > 0$ such that $|f^n(z)| \le M$ for all $n \ge 0$ and all z in some open domain U' of 0. Define

$$\phi_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} \lambda^{-k} f^k(z).$$

Then $\{\phi_n\}$ is a uniformly bounded sequence of analytic functions on U'. So it contains a convergent subsequence. Since

$$\phi_n \circ f = \lambda\phi_n + O(\frac{1}{n}),$$

any such a limit ϕ satisfies

$$\phi \circ f = \lambda\phi.$$

Because $\lambda = f'(0)$, we have $\phi_n'(0) = 1$ and $\phi'(0) = 1$. So we can have a Δ_r for some $0 < r < 1$ such that $h = \phi^{-1} : \Delta_r \to U = h(\Delta_r)$ is a solution of the Schröder equation. $\square$

Since the linear map $z \mapsto \lambda z$ has no periodic point in any small neighborhood about 0, we have that

Theorem 35. *If f has a periodic point in any small neighborhood about 0, then f can not be linearizable*

Applying this theorem, one can construct an example of f such that 0 is a Cremer fixed point of f.

Theorem 36. *There exists a $\lambda = e^{2\pi i\theta}$, $\theta \in [0,1] \setminus \mathbb{Q}$ such that the Schröder equation has no solution for any polynomial f of degree d.*

Proof. Let $f(z) = \lambda z + \cdots + a_d z^d$, $a_d \neq 0$. Suppose there is a disk Δ_r such that f is conjugating to λz by a conformal map $h : \Delta_r \to U$. Consider the d^n fixed points of f^n, that is, all roots of

$$f^n(z) - z = z^{d^n} + \cdots + (\lambda^n - 1)z = 0.$$

One of these root is 0. Let $z_1, \cdots, z_{d^n-1}$ be all the rest roots. Since $f^n(z) = h(\lambda^n h^{-1}(z))$ has only fixed point 0 in Δ_r, $z_i \notin \Delta_r$ for all $0 \leq i \leq d^n - 1$. Thus

$$r^{d^n} \leq \prod_{i=1}^{d^n-1} |z_i| = |1 - \lambda^n|.$$

We now consider a λ such that the last inequality is impossible for all n.

Take a sequence of positive integers

$$q_1 < q_2 < \cdots .$$

Let $\theta = \sum_{k=1}^{\infty} 2^{-q_k}$ and $\lambda = e^{2\pi i\theta}$. Then

$$|1 - \lambda^{2^{q_k}}| \sim 2^{q_k - q_{k+1}}.$$

The last inequality implies that there is a universal constant $C > 0$ such that

$$r^{d^{2^{q_k}}} \leq C 2^{q_k - q_{k+1}}.$$

So

$$d^{2^{q_k}} \log_2 r \leq \log_2 C + q_k - q_{k+1}.$$

Equivalently,

$$q_{k+1} \leq \log_2 C + q_k - d^{2^{q_k}} \log_2 r = d^{2^{q_k}} \left(\frac{\log_2 C + q_k}{d^{2^{q_k}}} - \log_2 r \right) \leq C' d^{2^{q_k}}.$$

If we take $\{q_k\}$ growing fast like $k d^{2^{q_k}}$, then the Schröder equation has no solution. □

Remark 10. *An example in the above theorem is given in 1917 by G. A. Pfeifer. The work was continuous by H. Cremer who proved in 1938 that* $|\lambda| = 1$ *and* $\liminf_{n\to\infty} |1-\lambda^n| = 0$*, then there is an analytic function* $f(w) = \lambda z + a_2 z^2 + \cdots$ *such that Schröder equation has no solution. Refer to*

- *J. Milnor, Dynamics in One Complex Variable, Introductory Lectures. Vieweg, 2nd Edition, 2000.*

Concluding from the above discussion, we have that a Siegel periodic point is in the Fatou set and a Cremer periodic point is in the Julia set. We have seen an example of rational neutral fixed point which is a Cremer fixed point. An example of a Siegel fixed point can be obtained from the following theorem, which we will prove in a later lecture.

A real number $0 < \theta < 1$ is called Diophantine if there are constants $C > 0$ and $0 < \mu < \infty$ such that

$$|\theta - \frac{p}{q}| \geq \frac{C}{q^\mu}$$

for all p and $q \neq 0$. This condition is equivalent to

$$|\lambda^n - 1| \geq C n^{1-\mu}$$

for all $n \geq 1$. For a fixed $\mu > 2$, $|\theta - \frac{p}{q}| \geq C/q^\mu$ holds for almost all real irrational number $\theta \in [0,1]$. Indeed if E is the set of irrational number $\theta \in [0,1]$ such that $|\theta - p/q| < q^{-\mu}$ infinitely many often, then

$$m(E) \leq \sum_{q=n}^{\infty} 2q^{-\mu+1} = O(n^{2-\mu}) \to 0.$$

In particular, almost all irrational numbers in $[0,1]$ are Diophantine.

Theorem 37 (Siegel, 1942)**.** *If* $0 < \theta < 1$ *is Diophantine and if* $f(z) = \lambda z + a_2 z^2 + \ldots$ *is a convergent power series in a neighborhood* U *of* 0 *such that* $\lambda = e^{2\pi i\theta}$*, then there is a solution to the Schröder equation, that is,* f *can be conjugated near* 0 *to the linear map* $z \mapsto \lambda z$*.*

Now let us come to the rational neutral case. Suppose $0 \in U$ is a rational neutral fixed point of a holomorphic map $f : U \to \mathbb{C}$. Let $\lambda = f'(0)$ with $\lambda = e^{2\pi i \frac{p}{q}}$, $p \geq 0$ and $q \geq 1$ are integers. Suppose p and q are relatively prime if $p \neq 0$. Then $f(z) = \lambda z$ or its Taylor expansion of f at 0 is

$$f(z) = \lambda z + a_{n+1} z^{n+1} + \cdots$$

where $a_{n+1} \neq 0$ and $n \geq 1$. In the first case f is a linear map. The most interesting case is the second case. Let us discuss it in three cases:

(1) $\lambda = 1$ and $n = 1$;
(2) $\lambda = 1$ and $n \geq 2$;
(3) $\lambda \neq 1$.

In the case (1), conjugating by $z \mapsto a_2 z$, we may assume that $a_2 = 1$. Then $f(z) = z + z^2 + \cdots$. First we move 0 to ∞ by the conjugacy $w = \alpha(z) = -1/z$, i.e., considering

$$g(w) = \alpha_0 \circ f \circ \alpha_0^{-1}(w) = w + 1 + \frac{b}{w} + O(\frac{1}{w^2}), \quad w \in V = \alpha(U).$$

There is a large constant $C_0 > 0$ such that

$$\Re g^n(w) > \Re w + \frac{n}{2}, \quad \Re w > C_0, \ \ n \geq 1.$$

(In general $n/2 < |g^n(w)| \leq |w| + 2n$ for $\Re w > C_0$.) Take

$$\phi_n(w) = g^n(w) - n - b \log n, \quad \Re w > C_0, \ n \geq 1.$$

Since

$$g^{n+1}(w) = g^n(w) + 1 + \frac{b}{g^n(w)} + O(\frac{1}{n^2}), \quad \Re w > C_0, \ n \geq 1,$$

$$\phi_{n+1}(w) - \phi_n(w) = b\big(\log n - \log(n+1)\big) + \frac{1}{g^n(w)} + O(\frac{1}{n^2}) = O(\frac{1}{n}), \quad \Re w > C_0, \ n \geq 1.$$

So

$$\begin{aligned} |\phi_n(w) - w| &\leq |\phi_1(w) - w| + \sum_{k=1}^{n-1} |\phi_{k+1}(w) - \phi_k(w)| \\ &= O(\log n), \quad \Re w > C_0, \ n \geq 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi_{n+1}(w) - \phi_n(w) &= b\big(\log n - \log(n+1)\big) + g^{n+1}(w) - g^n(w) - 1 \\ &= -\frac{b}{n} + \frac{b}{g^n(w)} + O(\frac{1}{n^2}) \\ &= b\Big(\frac{1}{n + b\log n + \phi_n(w)} - \frac{1}{n}\Big) + O(\frac{1}{n^2}) \\ &= \frac{b}{n^2} O(|b \log n + \phi_n(w)|) + O(\frac{1}{n^2}) \\ &= O(\frac{\log n}{n^2}), \quad \Re w > C_0, \ \ n \geq 1. \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} |\phi_{n+1}(w) - \phi_n(w)| < \infty, \quad \Re w > C_0.$$

This implies that $\phi_n(w)$ converges to $\phi(w)$ as n goes to infinity. This convergence is uniformly on any compact set in $\Re w > C_0$. So ϕ is conformal in $\Re w > C_0$. Since ϕ_n is conformal and $\phi_n(g(w)) = \phi_{n+1}(w) + 1 + b\log(1+\frac{1}{n})$, we have

$$\phi(g(w)) = \phi(w) + 1.$$

We can extend ϕ analytically to any domain B^+ such that $g(B^+) \subseteq B^+$ and $g^n(w) \to \infty$ as $n \to \infty$ for $w \in B^+$. The map $\psi_+ = \phi\alpha$ then conjugates f to $w \mapsto w+1$, i.e.,

$$\psi(f(z)) = \psi(z) + 1, \quad z \in D^+ = \alpha^{-1}(B^+).$$

Similarly we have a conformal map ϕ defined on any domain B^- such that $g^{-1}(B^-) \subseteq B^-$ and $g^{-n}(w) \to \infty$ for $w \in B^-$. The map ϕ conjugates g^{-1} to $w \mapsto w-1$, i.e.,

$$\phi(g^{-1}(w)) = \phi(w) - 1, \quad w \in B^-.$$

The map $\psi_- = \phi\alpha$ then conjugates f^{-1} to $w \mapsto w-1$, i.e.,

$$\psi_-(f^{-1}(z)) = \psi_-(z) - 1, \quad z \in D^- = \alpha^{-1}(B^-).$$

Now let us look at the dynamics of f around 0. Suppose U is a bounded domain. Consider the domain $V = \alpha(U) \cup \{\infty\}$ about ∞. Let a be a real number such that a and $-a$ are in V and such that curve $r_1 : x = -y^2 + a$ and $r_2 : x = y^2 - a$ are all contained in V. Let $B^+ \subset V$ be the domain bounded by r_1 (the right domain cut by r_1) and let $B^- \subset V_1$ be the domain bounded by r_2 (the left domain cut by r_2). Let $T(w) = w + 1$ be the shift map. Then it is clear that $T(B^+) \subset B^+$ and that $T^{-1}(B^-) \subset B^-$. Let $B = B^+ \cup B^-$. Then it is a domain about ∞ in $\widehat{\mathbb{C}}$. Let $D^+ = \alpha^{-1}(B^+)$ and $D^- = \alpha_-^{-1}(B^-)$. They are domains in U bounded by analytic curves tangent at 0. Moreover, $f(D^+) \subset D^+$ and $f^{-1}(D^-) \subset D^-$. Here D^+ is called an *attractive petal* of f at 0 and D^- is called a *repelling petal* of f at 0. Maps ψ_+ and ψ_- define an analytic map ψ on $D = D^+ \cup D^-$ such that

$$\psi(f(z)) = \psi(z) + 1, \quad z \in D.$$

Let K' be the set of all horizontal lines contained in $\overline{B}$ and let $K = \psi^{-1}(K')$. Then K is a compact set and $f(K) = f^{-1}(K) = K$ since $T(K') = T^{-1}(K') = K'$. Conclude the above discussion, we have that

Theorem 38. *There are two domains D^+ and D^- and a compact set K satisfying that*

(1) *D^+ is bounded by an analytic curve tangent at 0 such that $f(\overline{D^+}) \subset D^+ \cup \{0\}$;*

(2) D^- *is bounded by an analytic curve tangent at* 0 *such that* $f^{-1}(\overline{D^-}) \subset D^- \cup \{0\}$;
(3) 0 *is an interior point of* $D = D^+ \cup D^-$;
(4) $K \subset \overline{D}$ *and* $\partial K \cap \partial D \neq \emptyset$;
(5) $f(K) = f^{-1}(K) = K$.
(6) K *consists of two closed domains* K_+ *and* K_- *touching at* 0, *each of them is foliated by completely* f*-invariant analytic curves tangent at* 0, *that is,* $K = K_+ \cup K_-$ *and* $K_+ \cap K_- = \{0\}$ *and* $K_\pm = \cup_t l_{t,\pm}$ *where* $\{l_{t,\pm}\}$ *are analytic curves tangent at* 0 *and* $f(l_{t,\pm}) = f^{-1}(l_{t,\pm}) = l_{t,\pm}$.

In the case (2), conjugating by $z \mapsto bz$ where $b^n = 1/a_n$, we may assume that $a_n = 1$. Then $f(z) = z + z^{n+1} + \cdots$. Let

$$S_j = \{z = re^{2\pi i\theta} \mid \frac{j}{n} \leq \theta < \frac{j+1}{n}\}, \quad j = 0, 1, \cdots, n-1.$$

For each $0 \leq j < n$, considering the map $\beta_j(z) = z^n : S_j \to \mathbb{C}$. Then

$$\tilde{f}(z) = \beta_j \circ f \circ \beta_j^{-1}(z) = (z^{\frac{1}{n}} + z^{\frac{n+1}{n}} + \cdots)^n = z + nz^2 + o(|z|^2)$$

From Case (1), $\tilde{f}$ has an attracting petal $\tilde{D}^+$ bounded by an analytic curve tangent with the positive real line at 0, a repelling petal $\tilde{D}^-$ bounded by an analytic curve tangent with the negative real line at 0, and a totally invariant compact set $\tilde{K}$ foliated by analytic curves tangent with the real line at 0. Let $D_j^+ = \beta_j^{-1}(\tilde{D}^+)$, $D_j^- = \beta_j^{-1}(\tilde{D}^-)$, and $K_j = \beta_j^{-1}(\tilde{K})$. Then D_j^+, $j = 0, \cdots n-1$, are attracting petals of f at 0 and D_j^-, $j = 0, \cdots n-1$, are repelling petals of f at 0, and $K = \cup_{j=0}^{n-1} K_j$ is a completely f-invariant set. Therefore, we have that

Theorem 39. *In any neighborhood* U *of* 0 *where* f *is defined,* f *has*

(1) n *attracting petals* D_j^+, $j = 0, \cdots, n-1$, *such that* $f(\overline{D_j^+}) \subset D_j^+ \cup \{0\}$;
(2) n *repelling petals* D_j^-, $j = 0, \cdots, n-1$, *such that* $f^{-1}(\overline{D_j^-}) \subset D_i^- \cup \{0\}$; *and*
(3) *a completely* f*-invariant compact set* K, *i.e.,* $f(K) = f^{-1}(K) = K$, *consisting of* n *petals* K_j, $j = 0, \cdots, n-1$.

Moreover, 0 *is an interior point of* $D = \cup_{j=0}^{n-1} D_j^+ \cup D_j^-$, *each petal* K_j *is foliated by completely* f*-invariant analytic curves, this means that* $K_j = \cup_t l_{t,j}$ *where* $l_{t,j}$ *are analytic curves and* $f(l_{t,j}) = f^{-1}(l_{t,j}) = l_{t,j}$, *and* $\partial K_j \cap \partial D \neq \emptyset$.

At last, we discuss the case (3). In this case, let $\lambda = e^{2\pi i p/q}$, where $0 < p < q$ and $(p, q) = 1$. Then $f^q(z) = z$ or $f^q(z) = z + a_{n+1}z^{n+1} + \cdots$ with $a_{n+1} \neq 0$ and $n \geq 1$.

Theorem 40. *Suppose $f^q(z) = z + a_{n+1}z^{n+1} + \cdots$ with $a_{n+1} \neq 0$ and $n \geq 1$. In any neighborhood U of 0 where f is defined, f has*

(1) *$n = kq$ attracting petals $D^+_{j,l}$, $j = 0, \cdots, q-1$, $l = 1, \cdots, k$, such that*
$$f(\overline{D^+_{j,l}}) \subset D^+_{j+p \pmod q, l} \cup \{0\};$$

(2) *$n = kq$ repelling petals $D^-_{j,k}$, $j = 0, \cdots, q-1$, $l = 1, \cdots, k$, such that*
$$f^{-1}(\overline{D^-_{j,l}}) \subset D^-_{j+p \pmod q, l} \cup \{0\};$$
and

(3) *a completely f-invariant compact set K, i.e., $f(K) = f^{-1}(K) = K$, consisting of $n = kq$ petals K_j, $j = 0, \cdots, q-1$, $l = 1, \cdots k$.*

Moreover, 0 is an interior point of $D = \cup_{l=1}^k \cup_{i=0}^{q-1} D^+_{j,l} \cup D^-_{j,l}$, each petal $K_{j,l}$ is foliated by analytic curves, this means that $K_{j,l} = \cup_t l_{t,j,l}$ where $l_{t,j,l}$ are analytic curves and $f(l_{t,j,l}) = f^{-1}(l_{t',j,l}) = l_{t,j+p \pmod q, l}$, and $\partial K_{j,l} \cap \partial D \neq \emptyset$.

Corollary 4. *If f has a rational neutral fixed point, then either $f^n = id$ for some $n > 0$ or f can not be linearizable (even topologically).*

Concluding from our discussion in this section, we have that

Theorem 41. *Suppose f is a rational map with the degree $d > 1$. Then the Julia set J of a rational map f contains all repelling periodic points of f and all rational neutral periodic points and all irrational Cremer neutral periodic points. The Fatou set contains all attractive and super-attractive periodic points and all irrational Siegel neutral periodic points.*

Now let us have a complete picture about the local structures around neutral fixed points (for both rational and irrational neutral fixed points).

- R. Pérez-Marco, Fixed points and circle maps. Acta Math., **179** (1997), 243-297.

Theorem 42. *Suppose f is holomorphic and univalent on a domain V. Suppose $0 \in V$ is a fixed point of f, $f(z) = \lambda z + a_2 z^2 + \cdots$, $|\lambda| = 1$. For any domain $0 \in U \subset \overline{U} \subset V$, there is a set K such that*

(1) *K is compact and connected;*

(2) *K is full, i.e., $\mathbb{C} \setminus K$ is connected;*
(3) *$0 \in K$;*
(4) *$K \cap \partial U \neq \emptyset$;*
(5) *$f(K) = f^{-1}(K) = K$;*
(6) *f is linearizable at 0 if and only if there is a U such that K has 0 as an interior point.*

Corollary 5. *There is a unique maximal set $K_{\max}$ in U with the property: $w \in K$ implies that $f^n(w) \in K$ for all $n \geq 0$.*

Proof of Theorem 42 (6.) If f is linearizable at a small neighborhood U, then there is a conformal map $\psi : U \to \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ such that $\phi(f(z)) = \lambda\psi(z)$ for $z \in U$. So $\phi^{-1}(\Delta) \subset U$ is an invariant set of f. Let K be the maximal f-invariant set in $\overline{U}$ containing $\phi^{-1}(\Delta)$. Then K has 0 as an interior point.

If there is a neighborhood U about 0 such that 0 is an interior point of K. Let W be the connected component of K containing 0. Then K is a simply connected domain such that $f(W) = W$. The map f on W is univalent. Let $\phi : W \to \Delta$ be the Riemann mapping, $\phi(0) = 0$. Then

$$\tilde{f} = \phi \circ f \circ \phi^{-1} : \Delta \to \Delta, \quad \tilde{f}(0) = 0,$$

is an isomorphism. The Schwarz lemma (Lemma 3) implies

$$\tilde{f}(z) = e^{2\pi i\theta} z.$$

□

Suppose U is a bounded domain in $\mathbb{C}$. Let S be the set of all compact sets K in $\overline{U}$. Let

$$N_\epsilon(K) = \{z \in \mathbb{C} \mid \text{there is a } w \in K \text{ such that } d(w, z) < \epsilon\}.$$

be the ϵ-neighborhood of K. Let

$$d(K_1, K_2) = \inf\{\epsilon \mid K_1 \subset N_\epsilon(K_2),\ K_2 \subset N_\epsilon(K_1)\}.$$

Then $d(\cdot, \cdot)$ defines a topology in S, which is called the Hausdorff topology in S.

Lemma 9. *The Hausdorff topology is compact.*

Proof. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence in S. For each $n > 0$, let $A_n = \{x_{nm}\}_{m=1}^{\infty}$ be a dense subset in K_n. The for each fixed $m > 0$, $\{x_{nm}\}_{n=1}^{\infty}$ has a convergent subsequence in $\overline{U}$. Therefore, we can find a subsequence $\{n_i\}$ of integers such that $x_{n_i m}$ is convergent for every m with limit points x_m in $\overline{U}$. Let K be the closure of $\{x_m\}_{m=1}^{\infty}$. Then K is a compact set in S such that $d(K_n, K) \to 0$ as $n \to \infty$. □

Lemma 10. *Suppose $\{f_n\}$ is a sequence of analytic maps defined on U_0 and f is an analytic map defined on U_0. Let $U \subset \overline{U} \subset U_0$. Suppose $\{K_n\}$ is a sequence of compact sets in $\overline{U}$ and K is a compact set in $\overline{U}$. If $f_n \to f$ uniformly in $\overline{U}$ and $K_n \to K$ in the Hausdorff topology, then $f_n(K_n) \to f(K)$ in the Hausdorff topology.*

Proof the Rest of Theorem 42. Suppose

$$f(z) = \lambda z + a_2 z + \cdots, \quad \lambda = e^{2\pi i\theta}, \quad \theta \in [0,1] \setminus \mathbb{Q}$$

Take $\theta_n \in \mathbb{Q}$ such that $\theta_n \to \theta$ as $n \to \infty$. Define

$$f_n(z) = e^{2\pi i(\theta_n - \theta)} f = e^{2\pi i\theta_n} z + a_2 z^2 + \cdots.$$

Then 0 is a rational neutral fixed point of f. Following Theorem 40, for each n, there is a compact and connected set $K_n \subset \overline{U}$ such that K_n is completely f_n-invariant, i.e., $f_n(K_n) = f_n^{-1}(K_n) = K_n$, $K_n \cap \partial U \neq \emptyset$. Since the Hausdorff topology is compact, there is a subsequence $\{K_{n_i}\}$ such that $K_{n_i} \to K$ in the Hausdorff topology as $i \to \infty$. But it is clear that $f_{n_i} \to f$ uniformly in $\overline{U}$ as $i \to \infty$. So $K_{n_i} = f_{n_i}(K_{n_i}) \to f(K) = K$ in the Huasdorff topology as $i \to \infty$. Since K_{n_i} is compact and connected, K is compact and connected. Since $0 \in K_{n_i}$ and $K_{n_i} \cap \partial U \neq \emptyset$, $0 \in K$ and $K \cap \partial U \neq \emptyset$. If K is not full, then $\mathbb{C} \setminus K$ has more that one components. Let $B = A_1 \cup \cdots \cup A_n$ be the union of all bounded components and let A_∞ be the unbounded component. Then $f(K \cup B) = K \cup B$. So $K \cup B$ is full and completely f-invariant. We complete the proof. $\square$

Now let us see some applications of Theorem 42. Suppose f is a holomorphic function defined on a domain U_0 containing 0. Suppose 0 is a neutral fixed point of f, i.e., $f(0) = 0$ and $\lambda = f'(0)$ with $|\lambda| = 1$. Let $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$ and $\Delta = \Delta_1$. Suppose $\overline{\Delta}_r \subset U_0$. Let K_r be the maximal completely f-invariant set in $\overline{\Delta}_r$. Let $\phi : \widehat{\mathbb{C}} \setminus K_r \to \Delta$ be the Riemann mapping with $\phi(\infty) = 0$. Let $A = \phi(U \setminus K_r) \subset \Delta$ be an annulus. Then $\tilde{f} = \phi \circ f \circ \phi^{-1} : A \to \Delta$ is a holomorphic map. By the Schwarz reflection principle, we can define a holomorphic map

$$\tilde{F} : A \cup \alpha A \cup S^1 \to \tilde{F}(A \cup \overline{A} \cup S^1)$$

where $\alpha(z) = 1/\overline{z}$. Then $g = \tilde{F}|S^1 : S^1 \to S^1$ is an analytic diffeomorphism.

Theorem 43. *Suppose $\lambda = e^{2\pi i\theta}$, $\theta \in [0,1)$. Then the rotation number $\rho(g) = \theta$.*

Before to prove this theorem, we discuss the Carathéodory- topology on domains. Let $\{U_n\}$ be a sequence of domains in $\mathbb{C}$ and U is another domain in $\mathbb{C}$. Suppose all U_n have a common interior point w_0. We say U_n tends to U in kernal with respect to w_0 if

- **(i):** $U = \{w_0\}$ or U is a domain containing w_0 and for any $z \in U$, there is a neighborhood V about z and $N > 0$ such that $V \subset U_n$ for all $n > N$,
- **(ii):** for any $z \in \partial U$, there is a point $z_n \in \partial U_n$ such that z_n tends to z as n goes to infinity.

Suppose $f_n : \Delta \to U_n$ is univalent such that $f_n(0) = w_0$ and $f_n'(0) > 0$ for every $n > 0$. Suppose $f : \Delta \to U$ is univalent such that $f(0) = w_0$ and $f'(0) > 0$. If f_n tends to f uniformly on compact sets as n goes to infinity, then U_n tends to U in kernal as n goes to infinity. Conversely, if U_n tends to U in kernal as n goes to infinity, then f_n tends to f uniformly on compact sets as n goes to infinity. Note that if f_n tends to f uniformly on compact sets as n goes to infinity, then f_n^{-1} tends to f^{-1} uniformly on compact sets as n goes to infinity. Suppose $\{K_n\}$ is a sequence of compact sets in $\mathbb{C}$. If K_n tends to K in the Hausdorff topology as n goes to infinity, then $\mathbb{C} \setminus K_n$ tends to $\mathbb{C} \setminus K$ in kernal as n goes to infinity. The Carathéodory topology on domains may not describe as how close of domains in distance. This can be seen by the following example. Let $U_n = \mathbb{C} \setminus \big((-\infty, -1/n] \cup [1/n, \infty)\big)$. Let U be the upper-half plane. Let $w_0 \in U$. Then one can see that U_n tends to U in kernal with respect to w_0 as n goes to infinity. Suppose $f_n : \Delta \to U_n$ is the Riemann map such that $f_n(0) = w_0$ and $f_n'(0) > 0$. Let $f : \Delta \to U$ be the Riemann map such that $f(0) = w_0$ and $f'(0) > 0$. Then f_n tends to f uniformly on compact sets. Note that $J_n = f_n^{-1}(\text{lower-half plane})$ tends to a point in S^1 as n goes to infinity. We use the Carathéodory topology in the proof of Theorem 43.

Proof of Theorem 43. Suppose $f(z) = \lambda z + a_n z^{n+1} + \cdots$, $n \geq 1$. If all $a_n = 0$, then it is clear. Suppose $a_n \neq 0$ for the smallest $n \geq 1$. If $\theta \in \mathbb{Q}$, then f has a completely f-invariant set K_r in Δ_r made by m petals ending at 0. Let γ be a curve in $\Delta_r \setminus K_r$ connecting 0 and a boundary point of U. If $\theta = 0$, then we can find γ such that $f(\gamma) = \gamma$. Let $\tilde{\gamma} = \phi(\gamma) \cup \alpha(\phi(\gamma))$. Then $\tilde{f}(\tilde{\gamma}) = \tilde{\gamma}$. That means that g has a fixed point in S^1. Therefore, $\rho(g) = 0$. If $\theta = p/q$ such that $p \neq 0$ and $(p, q) = 1$, then we can find such a curve γ such that $f^{nq}(\gamma) = \gamma$. Let $\gamma_i = f^i(\gamma)$ for $i = 0, 1, \cdots, nq-1$. Then $f(\gamma_i) = \gamma_{i+p \pmod q}$. Thus g has a periodic point s in S^1 such that for $s_i = g^i(s)$, $i = 0, 1, \cdots, nq-1$, $g(s_i) = s_{i+p \pmod q}$. Therefore, $\rho(g) = p/q$.

Now suppose $\theta \in [0,1] \setminus \mathbb{Q}$. Let $\{\theta_n\}$ be a sequence of rational numbers such that $\theta_n \to \theta$ as $n \to \infty$. Let $f_n = e^{2\pi i(\theta_n - \theta)} f$. Then f_n is defined on U_0 and has an invariant set $K_{r,n}$ in Δ_r. Let g_n be the corresponding circle diffeomorphism. Then $\rho(g_n) = \theta_n$. Suppose $\phi_n : \mathbb{C} \setminus K_{r,n} \to \Delta$ is the Riemann map with $\phi_n(\infty) = 0$ and $\phi_n'(\infty) > 0$. Suppose $\tilde{f}_n = \phi_n f_n \phi_n^{-1}$ be the corresponding map defined on annulus $A_n = \phi_n(U \setminus K_{r,n})$. Let $S_{\pm\tau} = \{z \in \mathbb{C} \mid |z| = 1 \pm \tau\}$ where τ is a number close to 0. Since $K_{n,r}$ tends to K_r in the Hausdorff topology as n goes to infinity, $\tilde{f}_n | S_{\pm\tau} \to \tilde{f}$ as $n \to \infty$. The maximal principle implies that $g_n | S^1 \to g | S^1$ as $n \to \infty$. So $\rho(g_n) \to \rho(g)$ as $n \to \infty$. $\square$

Let $\mathcal{G} = \{f(z) = \lambda z + O(z^2) \mid |\lambda| = 1\}$ be the set of all holomorphic germs at 0. Let $\mathcal{A}$ be the set of λ, $|\lambda| = 1$, such that $f(z) = \lambda z + O(z^2)$ is linearizable. Then $\mathcal{A}^c$ is the set of λ, $|\lambda| = 1$, such that there exists a germ $f(z) = \lambda z + O(z^2)$ such that f is not linearizable. Let $\mathcal{S} = \{g : S^1 \to S^1\}$ be the set of all holomorphic diffeomorphisms of the circle. Let $\mathcal{B}$ be the set of rotation numbers ρ such that for any $g \in \mathcal{S}$ with $\rho(g) = \rho$ is holomorphically conjugate to the rigid rotation with rotation number ρ. Then $\mathcal{B}^c$ is the set of rotation numbers ρ such that there exists $g \in \mathcal{S}$ such that g is not holomorphically conjugate to the rigid rotation with rotation number ρ. Then one can check that $\mathcal{A} = \exp(2\pi i \mathcal{B})$.

Suppose f is univalent on U_0 and suppose 0 is an irrational neutral fixed point of f. A domain $0 \in S \subset U$ is called a Siegel disk of f if there is a conformal map $\phi : S \to \Delta_1$ such that $\phi \circ f \circ \phi^{-1}(z) = \lambda z$ for $z \in \Delta$.

Theorem 44. *Suppose S is a Siegel disk of f and $\overline{S} \subset U_0$. Then there is no periodic point of f on ∂S.*

Proof. Suppose f has a periodic orbit $C = \{x_1, \cdots, x_n\}$ on ∂S. Since S is a Siegel disk of f, there is a conformal map $\phi : S \to \Delta$ such that $\phi \circ f \circ \phi^{-1}(z) = \lambda z$. Let $r_n = \{z \in \mathbb{C} \mid |z| = 1 - 1/n\}$ is a circle in Δ. Let $R_n = \phi^{-1}(r_n)$ be a curve in S. Let U_n be the domain bounded by R_n and K_n be the maximal f-invariant set in U_n. Then K_n tends to $K = S$ in the Hausdorff topology as n goes to infinity. Let $V_n = \widehat{\mathbb{C}} \setminus K_n$ and $V = \widehat{\mathbb{C}} \setminus S$. Let $\phi_n : V_n \to \Delta$ be the Riemann map with $\phi_n(\infty) = 0$ and $\phi_n'(\infty) > 0$ and let $\phi : V \to \Delta$ be the Riemann map with $\phi(\infty) = 0$ and $\phi'(\infty) > 0$. Denote $\tilde{f}_n = \phi_n \circ f \circ \phi_n^{-1}$ and $\tilde{f} = \phi \circ f \circ \phi^{-1}$. Let $C_n = \phi_n(C)$. Then $\tilde{f}_n$ (or $\tilde{f}$) can be extended to a circle diffeomorphism g_n (or g) of S^1 (by applying the Schwarz reflection principle). Since $g_n(C_n) = C_n$, g_n has a periodic orbit. Since the number of C_n is less than or equal to n, so there is a subsequence C_{n_k} tends to $\tilde{C} = \phi(C) \subset S^1$ in the Hausdorff topology. The number of $\tilde{C}$ is less than or equal to n too. Since $g(\tilde{C}) = \tilde{C}$, g has a periodic orbit in S^1. This contradicts with the fact that the rotation number $\rho(g)$ is irrational. $\square$

The following lemma used to be useful in the proof of the non-wandering domain theorem (Theorem 48) for rational maps. Since we will use infinitesimal deformations to prove Theorem 48 so we will not use this lemma in the proof of Theorem 48. But it is still fan to know it.

Lemma 11 (Snail Lemma)**.** *Suppose $f \not\equiv id$ is an analytic map defined on a domain U. Suppose $0 \in U$ is a fixed point of f such that $|f'(0)| = 1$. If there is a curve γ in U ending at 0 and a point in ∂U such that $f(\gamma) \subset \gamma$, then $f'(0) = 1$.*

Proof. The map f can have only finite many fixed points in γ because f is not identity. By cutting γ short, we can assume that f has no fixed point in γ except for 0. Under this assumption, f is monotone on γ. This means that there is a natural order on γ from 0 to the other end of γ, for $z_1 < z_2$ in γ, $f(z_1) < f(z_2)$ (decreasing) or $f(z_1) > f(z_2)$ (increasing).

Since $|f'(0)| = 1$, there is a domain U', $\overline{U'} \subset U$, such that $f|U'$ is univalent. Assume γ connects 0 and a point z in $\partial U'$. Let K be the maximal completely f-invariant set in $\overline{U'}$. Then $K \cap \gamma = \{0\}$ since $f^{-1}(\gamma) \neq \gamma$ if f is decreasing and $f(\gamma) \neq \gamma$ if f is increasing. (Note that $f^{-1}(z) \notin \overline{U'}$ if f is decreasing and $f(z) \notin \overline{U'}$ if f is increasing.) Let $\phi : \widehat{\mathbb{C}} \setminus K \to \Delta$ be the Riemann map with $\phi(\infty) = 0$ and $\phi'(\infty) > 0$. Then $\phi(U \setminus K)$ is an annulus in Δ and $\eta = \phi(\gamma)$ is a curve connecting $\phi(\partial U')$ and S^1. Let $g = \phi \circ f \circ \phi^{-1}$. By the Schwarz reflection principle, g can be extended to an analytic diffeomorphism of S^1. We still denote this diffeomorphism as g. Since $g(\eta) \subset \eta$, g has a fixed point on S^1. This implies that the rotation number $\rho(g) = 0$. Therefore, $f'(0) = 1$. $\square$

Suppose $f(z) = \lambda z + a_2 z^2 + \cdots$, $\lambda = e^{2\pi i\theta}$, is a germ at 0. Then it is clear that λ is an analytic invariant, i.e., if h is a conformal map defined on a neighborhood about 0, then $h \circ f \circ h^{-1}(z) = \lambda z + b_2 z^2 + \cdots$. Moreover, we have that

Theorem 45. *The multiplier λ is a topological invariant. More precisely, if $f_1(z) = \lambda_1 z + a_2 z^2 + \cdots$, $\lambda_1 = e^{2\pi i\theta_1}$ and $f_2(z) = \lambda_2 z + b_2 z^2 +$*

$\cdots$, $\lambda_2 = e^{2\pi i\theta_2}$, *are two germs at* 0 *and if* h *is a homeomorphism defined on a neighborhood about* 0 *such that* $f_1 = h \circ f_2 \circ h^{-1}$, *then* $\lambda_1 = \lambda_2$.

Proof. Suppose f_1 and f_2 are both univalent on a domain $0 \in U_0$ and h is defined on U_0. Let $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$ be a disk such that $\overline{\Delta}_r \subset U_0$. Let K_1 and K_2 be the maximal completely f_1 and f_2 invariant sets in $\overline{\Delta}_r$. Let $\phi_1 : \widehat{\mathbb{C}} \setminus K_1 \to \Delta$ and let $\phi_2 : \widehat{\mathbb{C}} \setminus K_2 \to \Delta$ be the Riemann maps with $\phi_1(\infty) = \phi_2(\infty) = 0$ and $\phi_1(\infty) > 0$ and $\phi_2(\infty) > 0$. Then $A_1 = \phi_1(U \setminus K_1)$ and $A_2 = \phi_2(U \setminus K_2)$ are annuli in Δ. Define

$$\tilde{f}_1 = \phi_1 \circ f_1 \circ \phi_1^{-1} : A_1 \to \Delta, \quad \text{and} \quad \tilde{f}_2 = \phi_2 \circ f_2 \circ \phi_2^{-1} : A_2 \to \Delta.$$

By the Schwarz reflection principle, both $\tilde{f}_1$ and $\tilde{f}_2$ can be extended to analytic diffeomorphisms of S^1. We use g_1 and g_2 to denote these diffeomorphisms. Let $H = \phi_2 \circ h \circ \phi_1^{-1} : A_1 \to A_2$. Then H can be extended to a homeomorphism of S^1. We still denote this homeomorphism as H. Then $g_1 = H \circ g_2 \circ H^{-1}$ on S^1. This implies that the rotation numbers $\rho(g_1) = \rho(g_2)$. So $\lambda_1 = \lambda_2$. $\square$

Next we are going to have a study of the Fatou set F. Suppose f is a rational map of degree $d > 1$. Suppose $O = \{z_i = f^i(z)\}_{i=0}^{n-1}$ is an attractive or super-attractive or rational neutral periodic point of period n. Define

$$B = B_O = \{z \in \mathbb{C} \mid f^n(z) \to O, \text{ as } n \to \infty\}$$

the basin of O. Then $B \subset F$. On the other hand, for any domain U such that $U \cap \partial B \neq \emptyset$, then $f^n(z) \to O$ as $n \to \infty$ for $z \in U \cap B$ but $f^n(z) \not\to O$ as $n \to \infty$ for $z \in (\widehat{\mathbb{C}} \setminus B) \cap U$. So $\{f^n|U\}_{n=0}^{\infty}$ is not normal. Therefore, the boundary of the basin of any attractive or super-attractive periodic orbit of f is contained in the Julia set. Since $f(B) = f^{-1}(B) = B$, $\partial B \subset J$ is completely invariant and closed. So $J = \partial B$ from Theorem 30. It is clear that $B_O \subset F$.

Denote $B_{i,O}^*$ be the component containing z_i. Then $B_O^* = \cup_{i=0}^{n-1} B_{i,O}^*$ is called the immediate basin of O.

Theorem 46. *If* O *is attractive or super-attractive, then the immediate basin* B_O^* *contains at least one critical point of* f.

Proof. If O is super-attractive, then $(f^n)'(z_0) = 0$. So one of the points in O is a critical point. Suppose O is attractive. Since $f^n(z_0) = z_0$ and $0 < |\lambda| = |(f^n)'(z_0)| < 1$, there is a neighborhood U_1 about z_0 such that $f^n|U_1 : U_1 \to U_0 \subset U_1$ is homeomorphic. Let $g : U_0 \to U_1$ be its inverse.

Then $|g'(z_0)| = 1/|\lambda| > 1$. Take U_1 simply connected. If U_1 contains no critical point from f^n, then g can be extended holomorphically to U_1, we still denote this extension as g and let $U_2 = g(U_1)$, which is still simply connected. Inductively, suppose we have defined U_n such that U_n is simply connected and $g : U_{n-1} \to U_n$ is an analytic diffeomorphism. If U_n contains no critical point from f^n, then g can be extended to U_n as an analytic diffeomorphism and we can define $U_{n+1} = g(U_n)$. Thus, if $B^*(0, O)$ contains no critical point from f^n, then we can define a sequence of analytic maps $g^n : U_0 \to U_n$ for all $n > 0$. This is a normal family because it omits the Julia set of f which contains more than three points. So there is a subsequence $g^{n_i} : U_0 \to U_{n_i}$ converging uniformly to a holomorphic map g_0 defined on U_0. The Cauchy formula implies that $(g^{n_i})'(z_0)$ closes to $g_0'(z_0)$ for i large. But since $|g'(z_0)| > 1$, $|g^n(z_0)| \to \infty$ as $n \to \infty$. This is impossible. Therefore, there is n such that U_n contains a critical point from f^n. So B_O^* contains a critical point from f. □

Theorem 47. *If O is rational neutral, then the immediate basin $\mathcal{B}_O^*$ contains at least one critical point of f.*

Proof. Without loss of generality, we may assume $f(z_0) = z_0$ and $f'(z_0) = 1$. Then the immediate basin B_O^* contains an attracting petal S at z_0. Let ϕ be the conformal map on S such that $\phi(f(z)) = \phi(z)+1$. Since $f(B_O^*) = B_O^*$, ϕ can be extended to B_O^*, where we still have $\phi((f(z)) = \phi(z) + 1$. Since $\phi(S)$ contains a half-plane, $\phi(B_O^*) \supset \mathbb{C}$. If ϕ has no critical point, then there is a branch ψ of ϕ^{-1} such that $\psi : \mathbb{C} \to B_O^*$ is an entire function. Since $B_O^* \cap J = \emptyset$, ψ can be only a constant function. It is impossible. So ϕ has a critical point c in B_O^*, i.e., $\phi'(c) = 0$. Since $f^n(c) \to z_0$ as $n \to \infty$, we have an integer $m > 0$ such that $f^m(c)$ is in the attracting petal where ϕ is conformal. Since $\phi(f^m(c)) = \phi(c) + m$, $\phi'(f^m(c))(f^m)'(c) = \phi'(c) = 0$. So $(f^m)'(c) = 0$. Therefore, f has a critical point in B_O^*. □

Corollary 6. *Suppose the degree of f is d. Then the number of attractive and super-attractive periodic orbits plus the number of rational neutral periodic orbits is at most $2d - 2$.*

Proof. A degree d rational function f has at most $2d-2$ critical points. □

Remark 11. *The total number of attractive period orbits and neutral periodic orbits (including rational and irrational) is less than or equal*

$2d-2$. *For a polynomial of degree $d>1$, the total number of attractive and super-attractive period orbits and neutral periodic orbits (including rational and irrational) in the complex plane is less than or equal $d-1$. Refer to*

- *M. Shishikura, On the quasiconformal surgery of rational functions. Ann. Sci. Ècole Norm. Sup.,* **20**, *1987, 1-29.*

Suppose $R = P/Q$ is a rational function whose degree $d > 1$.

Definition 20 (Hyperbolicity by Critical Points)**.** *We say that $R(z)$ is hyperbolic if every critical orbit R tends to an attractive or super-attractive periodic orbit.*

Recall that we used to define the hyperbolicity as

Definition 21 (Hyperbolicity by Euclidean Metric)**.** *We say that $R(z)$ is hyperbolic if there are constants $C > 0$ and $\lambda > 1$ such that $|(R^n)'(z)| \geq C\lambda^n$ for all $z \in J$ and all $n \geq 1$.*

Then the above two definitions are equivalent follows by hyperbolic geometry and the non-wandering domain theorem and the classification of Fatou components. Let us have these now.

Let F be the Fatou set for R. A connect component of F is called a *Fatou component* of R. If U is a Fatou component, then $R(U)$ is also a Fatou component and $R^{-1}(U)$ consists of at most d Fatou components. If $R(U) = U$, we call U a *fixed Fatou component*. If $R^n(U) = U$ for some $n \geq 1$, we call U a *periodic Fatou component*. The minimal n is called the period. If $R^m(U)$ is periodic but U is not periodic, we call U a *preperiodic Fatou component*. If all $\{R^n(U)\}$ are distinct, we call U a *wandering domain*.

Theorem 48 (Sullivan)**.** *A rational map has no wandering domains.*

We will prove it in Lecture 8. An important conclusion of this theorem is a complete classification of Fatou components.

Theorem 49. *Let U be a Fatou domain. Then there is a positive integer m such that $R^m(U)$ is periodic. Suppose U itself is periodic of period n, then U can be only following cases:*

(1): *Attracting domain: U contains an attracting fixed point z of R^n and any point $w \in U$ tends to z under iterates of R^n.*

(2): *Super-attracting domain: U contains a super-attracting fixed point z of R^n and any point $w \in U$ tends to z under iterates of R^n.*

(3): *Parabolic domain: ∂U contains a rational neutral fixed point z of R^n with multiplier 1 and any point $w \in U$ tends to z under iterates of R^n.*

(4): *Siegel disk: U is simply connected and contains an irrational neutral fixed point z of R^n such that there is a conformal map $\phi : U \to D_1$ with $\phi(z) = 0$ and $\phi'(z) > 0$, $\phi \circ R^n \circ \phi^{-1}(w) = \lambda w$ for $w \in U$.*

(5): *Herman ring: U is an annulus such that there is a conformal map $\phi : U \to D_1 \backslash \overline{D_r}$, $0 < r < 1$, such that $\phi \circ R^n \circ \phi^{-1}(w) = \lambda w$ (or $\phi \circ R^n \circ \phi^{-1}(w) = r(\lambda w)$ where r is an inversion) for $w \in U$.*

Let C be the (finite) set of all critical points of R. Let $CO = \cup_{i=0}^{\infty} R^i(C)$ be the set of all critical orbits of R. The set $\overline{CO}$ is important because it is the complement all branches of R^{-n}, $n \geq 1$, are locally defined and analytic. As we have proved every attracting domain or super-attracting domain or parabolic domain contains a critical point. A Herman ring or Siegel disk contains no critical point. But we have that

Theorem 50. *If U is a Siegel disk or Herman ring, then ∂U is contained in $\overline{CO}$.*

Proof. Let U be a Siegel disk or Herman ring. Suppose the period of U is n. Let $g = R^n$. Then $g(U) = U$. Suppose $\overline{CO}$ does not contain ∂U. Let D be an open disk disjoint from $\overline{CO}$ which meets ∂U. Since U is a Siegel disk or Herman ring, we can pick D such that D is disjoint from some open invariant non-empty set V of U. Define f_n to be any branch of g^{-1} on D. Since all the f_n omit V, they form a normal family on D. Since g is injective on D, there are other components of $g^{-1}(U)$. Since $\cup_{n=0}^{\infty} R^{-n}(z)$ is dense in the Julia set J for any $z \in J$, there is an integer $m > 0$ such that some component W of $g^{-m}(U)$ distinct from U meets D. If $w \in W \cap D$, then $f_j(w)$ and $f_k(w)$ belong to different components of the Fatou set F for $j \neq k$, or else they belong to a periodic component, which could not be iterated eventually to W then U. Hence $f_k(w)$ tends to J for $w \in D \cap W$, and since J has no interior, any limit of the normal family $\{f_k\}$ is constant on $D \cap W$. On the other hand, since f_k are rotations of U, any limit of $\{f_k\}$ is non-constant on $W \cap D$. This is a contradiction. □

We call R a critical finite rational function if CO is finite. In this case, every point $c \in CO$ is periodic or preperiodic, that is, $R^m(c)$ is periodic for some $m > 0$. If c is periodic, it must be an super-attracting periodic point. If R is critical finite and has no super-attracting periodic points, then every $c \in CO$ is preperiodic. Thus we call such a rational map preperiodic.

Theorem 51. *If R is preperiodic, $J = \widehat{\mathbb{C}}$.*

Proof. If R has Siegel disks or Herman rings, then $\overline{CO}$ will be infinite from the previous theorem. So R has no Siegel disk and Herman ring. Also R has no attracting, super-attracting, and parabolic domain because, otherwise, either R has a super-attracting periodic point or CO is inifinity. From the no wandering domain theorem, the Fatou set F is empty. So the Julia set $J = \widehat{\mathbb{C}}$. $\square$

Following this theorem, we have an example of a rational function whose Julia set is the whole Riemann sphere.

Example 15 (Lattés Map)**.** *Let*

$$R(w) = 1 - \frac{2}{w^2}.$$

Then the Julia set $J = \widehat{\mathbb{C}}$.

Proof. The critical points of R are 0 and ∞. One can check that

$$R(0) = \infty, \quad R(\infty) = 1, \quad R(1) = -1, \quad R(-1) = -1,$$

and -1 is an expanding fixed point of R. So R is preperiodic. $\square$

A Fatou component U is called completely invariant if $R(U) = R^{-1}(U) = U$. For example, if P is a polynomial, then the basin B_∞ of the infinity is a fixed Fatou component and completely invariant. For $P(w) = w^2 - 1$, $P(0) = -1$ and $P(-1) = 0$, the immediate basin $B^*(0)$ of 0 is a periodic Fatou component of period 2.

Theorem 52. *If U is a completely invariant Fatou component, then $J = \partial U$ and every other Fatou component is simply connected. There are at most two completely invariant Fatou components.*

Proof. If U is completely invariant, then $\partial U = R^{-1}(\partial U) = R(\partial U) \subseteq J$. So $\partial U = J$.

Consider $V = \widehat{\mathbb{C}} \backslash \overline{U}$. Since $R^n|V$ omits U, $\{R^n|V\}$ is normal and $V \subset F$. Since U is connected, every component of V is simply connected.

If U further is also simply connected, then $R : U \to U$ is a degree d map and it must contain $d - 1$ critical points of R. Since R can has only $2d - 2$ critical points, R can has at most two completely invariant Fatou components. $\square$

Theorem 53. *The number of Fatou components of a rational function can be* 0*,* 1*,* 2*, or* ∞ *and all this number occurs.*

Proof. Suppose F has only finitely many Fatou component and let U_0 be one of them. Consider a chain of inverse images

$$R(U_{-1}) = U_0, R(U_{-2}) = U_{-1}, \cdots, R(U_{-n}) = U_{-n+1}.$$

There is an integer $n > 0$ such that $R(U_{-n}) = U_{-k+1} = R(U_{-k})$ for $0 \leq k < n$. Then $U_0 = R^n(U_{-n}) = R^n(U_{-k}) = R^{n-k}(U_0)$. Thus each Fatou component U_0 is periodic. Since there are only finite number of Fatou components, there is an integer $N > 0$ such that $R^N(U) = U$ for every Fatou component. Hence each component is completely invariant for R^N. There are at most two such Fatou components. For the Lattés example which we will lecture later, it has 0 Fatou component; for $P(w) = w^2 - 2$, it has one Fatou component; for $P(w) = w^2$, it has two Fatou component. Most of rational maps have ∞ Fatou components. □

Now we prove that

Theorem 54. *Definition 21 and Definition 20 are equivalent.*

Proof. Suppose Definition 21 holds. Then there is a neighborhood U contains J such that $|(R^m)'(z)| > 1$ for all $z \in \overline{U}$. This implies that $\overline{U} \subset R^m(U)$ and all critical point is outside of U. Let $V = \widehat{\mathbb{C}} \setminus \overline{U}$. Then $V \subset F$ and $\overline{R^n(V)} \subset V$. From Theorem 48, every component W of V is eventually periodic. So we suppose $\overline{R^m(W)} \subset W$. Since W is a hyperbolic Riemann surface with the hyperbolic metric $d_{H,W}$, we have a real number $0 < \tau < 1$ such that

$$d_{H,W}(R^m(z), R^m(w)) \leq \tau d_{H,W}(z, w), \quad z, w \in W_1 = R^m(W).$$

From the contracting mapping theorem, R^m has a fixed point $p \in W$. Then p is a super-attracting or attracting fixed point of R^m and all critical points in components of V which are eventually mapped to W are convergent to p under iteration of R. This implies Definition 20.

Now suppose Definition 20 holds. Let $p_1, \cdots, p_k$ are all attracting and super-attracting periodic points. Let $B_*(p_i)$ be the immediate basin of p_i. Let $B = \cup_{i=1}^k B_*(p_i)$. Then we have an integer $m > 0$ such that $B_0 = R^{-m}(B)$ contains all critical points. Thus $\overline{CO}$ is a closed set disjoint with J. Thus we have a neighborhood $U \supset J$ such that $R(U) \supset \overline{U}$ and $R(U) \cap \overline{CO} = \emptyset$. Every component of U is a hyperbolic Riemann surface. Thus R strictly expands the hyperbolic metric $d_{H,U}$, that is, $d_{H,U}(R(z), R(w)) \geq \lambda d_{H,U}(z, w)$ for some $\lambda > 1$

and all $z, w \in U$. The hyperbolic metric $d_{H,U}$ and the Eucildean metric are equivalent on J (by assume $\infty \notin J$). This implies Definition 21. $\square$

Finally, we mention a theorem which is useful in the study of complex dynamics.

Theorem 55 (Denjoy-Wolf)**.** *Let $f : \Delta \to \Delta$ be an analytic function. Suppose f is not an elliptic Möbius transformation. Then there is a point $\alpha \in \overline{\Delta}$ such that for any $z \in \Delta$, $f^n(z) \to \alpha$ as $n \to \infty$.*

Proof. Suppose f is a Möbius transformation. Then either M has one fixed point or two fixed points. If f has one fixed point a. Then let $M(z) = 1/(z-a)$. then $g(z) = M \circ f \circ M^{-1}(z)$ fixes ∞. Thus $g(z) = z + b$, $b \neq 0$. Thus $f^n(z) \to a$ as $n \to \infty$. If f has two fixed points $a \neq b$, then $a, b \in \partial\Delta$. Let $M(z) = (z-a)/(z-b)$. Then $g(z) = M \circ f \circ M^{-1}(z)$ fixes 0 and ∞. Thus $g(z) = \alpha z$. If $|\alpha| = 1$, then f is elliptic, so by the assumption, $|\alpha| < 1$. Thus $f^n(z) \to a$ as $n \to \infty$.

Now suppose f is not a Möbius transformation. It contracts the hyperbolic metric d_H, that is, $d_H(f(z), f(w)) < d_H(z, w)$ for $z, w \in \Delta$.

Consider $f_\epsilon(z) = (1-\epsilon)f(z)$. The image $f_\epsilon(\Delta)$ is contained in a compact set of Δ. Thus it contracts d_H strictly on $f_\epsilon(\Delta)$, that is, there is a real number $0 < \tau_\epsilon < 1$ such that

$$d_H(f_\epsilon(z), f_\epsilon(w)) \leq \tau_\epsilon d_H(z, w), \quad z, w \in f_\epsilon(\Delta).$$

The contracting mapping theorem says f_ϵ has a unique fixed point $\alpha_\epsilon \in \Delta$. Let D_ϵ be the hyperbolic disk of radius $d_H(0, \alpha_\epsilon)$ centered at α_ϵ. Then we have $f_\epsilon(D_\epsilon) \subset D_\epsilon$.

The we have two cases. The first one is that there is a limit point α of α_ϵ as $\epsilon \to 0^+$ in Δ. In this case, Let D be the corresponding limiting hyperbolic disk of D_ϵ, then $f(D) \subset D$. Since $0 \in \partial D$, we have that $f^n(0) \to \alpha$ as $n \to \infty$. This implies that α is the unique limit point of α_ϵ as $\epsilon \to 0^+$ and $f(\alpha) = \alpha$. Now we can use either the Schwarz lemma or hyperbolic geometry to prove that for any $z \in \Delta$, $f^n(z) \to \alpha$ as $n \to \infty$.

The other case is that all limit points of α_ϵ are on $\partial\Delta$. Any limiting hyperbolic disk D of D_ϵ has 0 on its boundary and is tangent with $\partial\Delta$ at a point α. This implies that $f(D) \subset D$ and $f^n(0) \to \alpha$ as $n \to \infty$. Since $\{f^n(0)\}_{n=1}^{\infty} \subset D$, α is the unique limiting point of α_ϵ. For any $z \in \Delta$, we have an integer $k > 0$ such that $f_\epsilon^k(z) \in D_\epsilon$. This implies that $f^k(z) \in D$. So $f^n(z) \to \alpha$ as $n \to \infty$. $\square$

Lecture 7. Lebesgue density and areas of Julia sets revisited.

Suppose A is a measurable subset of $\mathbb{C}$. For any point $p \in A$. let $\Delta_\epsilon(p) = \{z \in \mathbb{C} \mid |z-p| < \epsilon\}$ be the disk of radius $r > 0$ centered at p. Let $Area(S)$ mean the Lebesgue measure of a measurable set $S \subset \mathbb{C}$. We say p is a density point of A if the following limit exists and

$$\lim_{\epsilon \to 0} \frac{Area(A \cap \Delta_\epsilon(p))}{Area(\Delta_\epsilon(p))} = 1.$$

Theorem 56 (Lebesgue Density Theorem)**.** *If $Area A > 0$, then almost all points $p \in A$ are Lebesgue density points.*

Using the Koebe distortion and the Lebesgue density theorem and property of the Julia set, we have another proof of Corollary 1.

Theorem 57. *The Julia set J of a hyperbolic rational map f has zero Lebesgue measure, that is, $Area(J) = 0$.*

Proof. Since a hyperbolic rational map has an attractive or superattractive periodic point, its Fatou set is not empty. So its Julia set is not the whole Riemann sphere $\widehat{\mathbb{C}}$. Let C be the set of all critical points of f and let $P = \overline{\cup_{n=1}^{\infty} f^n(C)}$ be the post-critical set of f. Then $P \cap J = \emptyset$. Let $\epsilon_0 > 0$ be a real number such that the disk $\Delta_{\epsilon_0}(p) \cap P = \emptyset$ for any $p \in J$. Let $\epsilon = \epsilon_0/4$. Then we claim that we have a number $a > 0$ such that $Area(\Delta_\epsilon(p) \setminus J) \geq a$ for any $p \in J$. Let us prove it by contradiction. Suppose we have a sequence $p_n \in J$ such that $Area(\Delta_\epsilon(p_n) \setminus J) < 1/n$ for all $n > 1$. Since J is compact, we have convergent subsequence, which we still denote as p_n, such that $p_n \to p \in J$. Then $Area(\Delta_\epsilon(p) \setminus J) = 0$. This implies that J is dense in $\Delta_\epsilon(p)$. Thus $\Delta_\epsilon(p) \subset J$. But if J has an interior point, then $J = \widehat{\mathbb{C}}$ which is impossible.

Now take arbitrary point $p \in J$. Let $p_n = f^n(p)$. Consider the inverse branch g_n of f^n mapping p_n to p. Then g_n can be analytically extended to $\Delta_{\epsilon_0}(p_n)$ as a conformal map $g_n : \Delta_{\epsilon_0}(p_n) \to g_n(\Delta_{\epsilon_0}(p_n))$. The Koebe's 1/4-theorem (Theorem 22) implies that $g_n(\Delta_{\epsilon_0}(p_n))$ contains a round disk $g_n'(p_n) \cdot \Delta_\epsilon(p)$ which is a disk of radius $\epsilon_n = |g_n'(p_n)|\epsilon$ centered at p. Since $f|J$ is expanding and $g_n'(p_n) = 1/(f^n)'(p)$, we have that $\epsilon_n \to 0$ as $n \to \infty$. Now from the Koebe distortion theorem (Theorem 23), we have a constant $C > 0$ such that

$$C^{-1}|g_n'(p_n)||z-p| \leq |g_n(z) - g_n(p_n)| \leq C|g_n'(p_n)||z-p|$$

and
$$C^{-1} \leq \frac{|g_n'(\xi)|}{|g_n'(\xi)|} < C$$
for all $n \geq 1$, $p \in J$, $\xi, \eta \in \Delta_\epsilon(p_n)$. This implies that
$$\frac{Area(\Delta_{\epsilon_n}(p) \setminus J)}{Area(\Delta_{\epsilon_n}(p))} \geq C \frac{\int_{\Delta_\epsilon(p)\setminus J} |g_n'(\xi)| d\xi}{\int_{\Delta_\epsilon(p)} |g_n'(\eta)| d\eta}.$$
Furthermore,
$$\frac{Area(\Delta_{\epsilon_n}(p) \setminus J)}{Area(\Delta_{\epsilon_n}(p))} \geq C^2 \frac{Area(\Delta_\epsilon(p) \setminus J)}{Area(\Delta_\epsilon(p))} \geq C^2 a = b > 0.$$
So we have that
$$\frac{Area(J \cap \Delta_{\epsilon_n}(p))}{Area(\Delta_{\epsilon_n}(p)} < 1 - b.$$
Thus p is not a Lebesgue density point. This implies that $Area(J) = 0$. □

Similarly, replacing the Koebe distortion by the näive distortion (Lemma 1), we have a proof of Theorem 5 by using the Lebesgue density theorem.

Lecture 8. Sullivan's no wandering domain theorem.

In this lecture, we will prove Theorem 48.

Let $Rat(d)$ be the space of all degree $d > 1$ rational maps. Then it can be identified with an open set of the $2d+1$ projective space $P\mathbb{C}^{2d+1} = \mathbb{C}^{2d+2}/\{\text{complex lines}\}$, that is, we can correspond a rational map

$$g(z) = \frac{a_d z^d + \cdots a_0}{b_d z^d + \cdots + b_0} \in Rat(d)$$

to

$$\iota(g) = (a_d, a_{d-1}, \ldots, a_0, b_d, b_{d-1}, \ldots, b_0)/\{\text{complex lines}\} \in P\mathbb{C}^{2d+1}.$$

Thus, for g in $Rat(d)$, the tangent space $T_f Rat(d) = \mathbb{C}^{2d+1}$. Recall that the definition of $T_g Rat(d)$ is the space of equivalent classes $[g_t]$ of all C^1 paths g_t in $Rat(d)$ passing g at $t = 0$, where two paths g_t and $\widetilde{g}_t$ are equivalent if

$$\frac{dg_t}{dt}\Big|_{t=0} = \frac{d\widetilde{g}_t}{dt}\Big|_{t=0}.$$

Let $T\widehat{\mathbb{C}}$ be the continuous tangent bundle over $\widehat{\mathbb{C}}$. By the definition, the continuous tangent bundle $T\widehat{\mathbb{C}}$ is the space of all equivalent classes $[h_t]$ of C^1 paths h_t in the space of all homeomorphisms $Hoem(\widehat{\mathbb{C}})$ passing the identity, where two paths h_t and $\widetilde{h}_t$ are equivalent if

$$v = \frac{dh_t}{dt}\Big|_{t=0} = \frac{d\widetilde{h}_t}{dt}\Big|_{t=0}.$$

Here v can be viewed as a continuous vector field on $\widehat{\mathbb{C}}$. We normalize all paths fixing 0 and 1 and ∞. Then the only holomorphic vector field is the zero vector field.

Let f be the given rational map in Theorem 48. For any $v = [h_t] \in T\widehat{\mathbb{C}}$, we have a C^1 path

$$f_t = h_t^{-1} \circ f \circ h_t$$

of degree d branched coverings of $\widehat{\mathbb{C}}$. The tangent vector, that is the equivalent class of this C^1 path, is

$$Dv = \frac{df_t}{dt}\Big|_{t=0} = f'(z)v(z) - v(f(z)).$$

Thus we can define a linear operator D from $T\widehat{\mathbb{C}}$ to the tangent space of the space of all degree d branched coverings at f. Note that $T_f Rat(d)$ is a subspace of the later tangent space. Define

$$H = D^{-1}(T_f Rat(d)).$$

It is a subspace of $T\widehat{\mathbb{C}}$. Thus we have a linear operator, which we still denote as D, from H onto $T_f Rat(d)$. Let

$$K = Ker(D) = \{v \in H \mid Dv = 0\}$$

be the kernel of D. Then we have the induced operator

$$\widetilde{D} : QH = H/K \to T_f Rat(d)$$

from D. It is surjective (1-1 and onto). Since $T_f Rat(d)$ is a $2d+1$ dimensional vector space, *the key point in the proof of Theorem 48 is to show that QH is an infinite dimensional vector field space if f has a wandering domain.* This will give us a contradiction.

An infinite dimensional vector space V_Δ on the open unit disk Δ can easily constructed as the space of all vector fields

$$v_\Delta(z) = \begin{cases} \sum_{k=1}^{n} a_k \overline{z}^k, & |z| \le \frac{1}{2} \\ \sum_{k=1}^{n} \frac{a_k}{4^k} z^{-k}, & \frac{1}{2} \le |z| < 1 \end{cases}$$

where a_k are complex numbers. In this vector space, only holomorphic vector field is the zero vector field. Moreover, since v_Δ has an analytic extension in a neighborhood of $\partial\Delta$, $v_\Delta|\partial\Delta = 0$ if and only if $v_\Delta \equiv 0$ on Δ.

Now let us give a characterization of an element $v \in H$ which is equivalent to say that Dv is holomorphic. From the complex analysis, Dv is holomorphic if and only if

$$\overline{\partial} Dv(z) = f'(z)\overline{\partial} v - \overline{\partial} v(f(z))\overline{f'(z)} = 0.$$

Let $\mu = \overline{\partial} v$. Then $v \in H$ if and only if μ is f-invariant, that means

$$\mu(z) = \mu(f(z))\frac{\overline{f'(z)}}{f'(z)}, \quad a.e. \ z \in \widehat{\mathbb{C}}. \tag{2}$$

The key point in this calculation is that we only need $\mu(z)$ is defined almost everywhere on $\widehat{\mathbb{C}}$ and is measurable and in $L^\infty(\mathbb{C})$, that is, $\|\mu\|_\infty < \infty$. (As long as v is a Zyhmund continuous function, we have the above calculation. Thus we have the above calculation for v_Δ.)

Conversely, if $\mu(z) \in L^\infty(\mathbb{C})$ has a compact support in $\mathbb{C}$, define

$$v(z) = P\mu(z) = \frac{1}{\pi}\int\int_{\mathbb{C}} \frac{\mu(\zeta)}{z-\zeta} d\xi d\eta, \quad \zeta = \xi + i\eta. \tag{3}$$

Then v is a continuous vector field on $\widehat{\mathbb{C}}$ (actually Zgymund continuous vector field on $\widehat{\mathbb{C}}$) and one can check that $\overline{\partial} v = \mu$. Thus if μ is f-invariant (that is, it satisfies Equation (2)), then $v \in H$.

The following lemma characterizes v when it is in $K = Ker(D)$. Recall J is the Julia set of f.

Lemma 12. *If $v \in K$, then $v|J = 0$.*

Proof. Since $Dv = 0$, if z is a repelling periodic point of period $n > 0$, then

$$(f^n)'(z)v(z) = v(f^n(z)) = v(z).$$

This implies that $v(z) = 0$ since $|(f^n)'(z)| > 1$. This implies that $v|J = 0$ since the set of all repelling periodic points is dense in J and v is continuous. □

We also need the following Lemma to embed V_Δ into QH.

Lemma 13. *If f has a wandering domain, then it must have a simply connected wandering domain.*

Proof. Suppose U is a wandering domain. Let $U_n = f^n(U)$ for $n > 0$. Then $U_n \cap U_m = \emptyset$ for all $n \neq m$. Since f has only finite number of critical points, there is an integer $N \geq 0$ such that all U_n contains no critical points for $n \geq N$. This implies that

$$U_N \xrightarrow{f} U_{N+1} \xrightarrow{f} \cdots \xrightarrow{f} U_n \xrightarrow{f} U_{n+1} \xrightarrow{f} \cdots$$

is a chain of covering maps and $f^n|U_N$ is a normal family. First U_n can not be a punched disk since $\partial U_n \subset J$ and J has no isolated point. Assume that ∞ is in a Fatou component V which is not one of $\{U_n\}_{n=N}^{\infty}$.

Since $\sum_{n=N}^{\infty} Area(U_n) < \infty$, we have $(f^n)'(z) \to 0$ as $n \to \infty$; otherwise, by 1/4-Kobe Theorem, we have infinitely many U_n contains disks of a definite radius, the sum of the areas of these disks is infinite, it is impossible. If U_m is not simply connected for all m. Let γ_0 be a simply closed curve in U_m enclosing some points from the Julia set. Then the diameter of $\gamma_n = f^n(\gamma_0)$ tends to zero as $n \to \infty$. The bounded component D_n of $\mathbb{C} \setminus \gamma_n$ contains no pole of f since $f^n(U_m) \neq V$. Since f is a Lipschitz map, the bounded component D_n of $\mathbb{C} \setminus \gamma_n$ is mapped to the bounded component D_{n+1} of $\mathbb{C} \setminus \gamma_{n+1}$. Thus the diameter of D_n tends to zero as n goes to ∞. But since D_0 contains points in the Julia set, the image $D_n = f^n(D_0)$ should be bigger and bigger. This is a contradiction. The contradiction implies that one of U_n must be simply connected. □

Proof of Theorem 48. The proof is to embed V_Δ into QH if f has a wandering domain.

Suppose U is a simply connected wandering domain. Since $\partial U \subset J$, the limit set $(\partial U)' \neq \emptyset$ (thus there are at least two points on the boundary), we have a Riemann mapping $\phi : U \to \Delta$. Without loss of generality, we assuming ∞ in U such that $1/2 < |\phi(\infty)| < 1$.

For any $v_\Delta \in V_\Delta$, we have a pullback vector field

$$v_U(z) = \phi^* v_\Delta(z) = v_\Delta(\phi(z))$$

on U. Let $\mu_U = \overline{\partial} v_U$. Then $\mu_U(z) = 0$ if $1/2 < |\phi(z)| < 1$. Moreover, μ_U is continuous except on the analytic curve

$$\gamma = \phi^{-1}\Big(\Big\{z \mid |z| = \frac{1}{2}\Big\}\Big).$$

Let $U_0 = U$ and $U_n = f^n(U)$ for all integers $n \geq 1$. Note that $U_n \cap U_{n'} = \emptyset$ for any $n \neq n' \geq 0$. Then let $U_{nm} = f^{-m}(U_n)$ for all $m \geq 0$ and $n \geq 0$. The grand orbit of U is, by definition,

$$\widetilde{U} = \cup_{m=0}^{\infty} \cup_{n=0}^{\infty} U_{nm},$$

which is the union of pair-wise disjoint domains U_{nm}. Pullback v_U again by using all inverse branches g_n of f^n, $n \geq 0$, and f^m, $m \geq 1$, we define a vector field $v_{\widetilde{U}}$ on $\widetilde{U}$, that is, define

$$v_{U_n}(z) = v_U(g_n(z))$$

on U_n for $n \geq 0$ and define

$$v_{U_{mn}}(z) = v_{U_n}(f^m(z))$$

on U_{mn} for all $m \geq 1$ and $n \geq 0$. Then $v_{\widetilde{U}}$ is a continuous vector field on $\widetilde{U}$. Let $\mu_{\widetilde{U}} = \overline{\partial} v_{\widetilde{U}}$. Then $\mu_{\widetilde{U}}$ extends μ_U and is continuous except on $\widetilde{\gamma} = \cup_{m\geq 1} \cup_{n \geq 0} f^{-m}(f^n(\gamma))$, a union of countable number of analytic curves. It is clear that $\widetilde{\gamma}$ has zero Lebesgue measure. Note that $\mu_{\widetilde{U}}$ is f-invariant from our definition due to the fact that all U_{nm} are pairwise disjoint. Now define

$$\mu(z) = \begin{cases} \mu_{\widetilde{U}}(z), & z \in \widetilde{U}; \\ 0, & z \in \widehat{\mathbb{C}} \setminus \widetilde{U}. \end{cases}$$

From our definition, μ is continuous on $\widehat{\mathbb{C}}$ except on $\widetilde{\gamma}$ and has a compact support in $\mathbb{C}$. μ is also f-invariant. Thus we have a continuous vector field $v = P\mu$ on $\widehat{\mathbb{C}}$ from Equation (3) such that $\overline{\partial} v = \mu$. Since $\overline{\partial}(v - v_{\widetilde{U}}) = 0$, a. e. on $\widetilde{U}$, $h = v - v_{\widetilde{U}}$ is a holomorphic vector field on $\widetilde{U}$. Since μ is f-invariant, $v \in H$ and $Dv \in T_f Rat(d)$. This defines a linear map $v = \mathcal{L} v_\Delta$ from V_Δ into H.

If $v = \mathcal{L} v_\Delta \in K$, Lemma 12 implies that $v|J = 0$ and thus $v|\partial U = 0$. Since the limit set $(\partial U)' \neq \emptyset$ and $v = h + v_U$ is analytic on

$\phi^{-1}(\{z \mid 1/2 < |z| < 1\})$, we have $v = 0$ on $\phi^{-1}(\{z \mid 1/2 < |z| < 1\})$. This implies that

$$h(\phi^{-1}(z)) = -v_U(\phi^{-1}(z)) = -v_\Delta(z) = -\sum_{k=1}^{n} \frac{a_k}{4^k} z^{-k}, \quad 1/2 < |z| < 1.$$

So we have that

$$h(\phi^{-1}(z)) = -\sum_{k=1}^{n} \frac{a_k}{4^k} z^{-k}, \quad |z| < 1.$$

This implies that $a_k = 0$ for all $1 \leq k \leq n$. Thus $h \equiv 0$ on U, $v_\Delta \equiv 0$ on Δ, and $v \equiv 0$ on $\widehat{\mathbb{C}}$.

Concluding from the above, $\mathcal{L}$ embeds V_Δ into QH. More precisely,

$$\widetilde{D} \circ \mathcal{L} : V_\Delta \to T_f Rat(d) = \mathbb{C}^{2d+1}$$

is an injective linear operator. But V_Δ is an infinite dimensional vector space. It is impossible. The contradiction completes the proof of Theorem 48. $\square$

Lecture 9. Hausdorff dimension

Let (X,d) be a metric space. If U is a subset of X, we use $\mathrm{diam}(U)$ to denote the diameter of U. Suppose E is a subset of X. We say $\{U_i\}$ is a δ-cover of E if every $\mathrm{diam}(U_i) \leq \delta$ and $E \subset \cup_i U_i$.

For $\delta > 0$ and $s > 0$, we define

$$\mathcal{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} (\mathrm{diam}(U_i))^s$$

where the infimum is over all countable δ-covers $\{U_i\}$ of E. It is clear that for fixed s, $\mathcal{H}^s_\delta(E)$ is increasing with respect to δ. Now let $\delta \to 0$, we define

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

We will show that $\mathcal{H}^s(\cdot)$ defines an *s-Hausdorff (outer) measure* on X. Recall a function $\mu(\cdot) : 2^X \to [0, +\infty]$ is called an outer measure on X if

1) $\mu(\emptyset) = 0$ (Nullity);
2) $\mu(A) \leq \mu(B)$ if $A \subset B \subset X$ (Monotonicity);
3) For any sequence $(A_n)_{n\geq 1}$ in X we have

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

(σ-subadditivity)

Proposition 3. *$\mathcal{H}^s(\cdot)$ is an outer measure.*

Proof. The nullity and the monotonicity of $\mathcal{H}^s$ are obvious. Only the σ-subadditivity needs to be explained. For any $\epsilon > 0$, $\delta > 0$, and $n \geq 1$ one can find a countable δ-cover $\mathcal{R}_n$ for A_n such that

$$|\mathcal{H}^s_\delta(A_n) - \sum_{U \in \mathcal{R}_n} (\mathrm{diam(U)})^{\mathrm{s}}| \leq \frac{\epsilon}{2^{\mathrm{n}}}.$$

Take $\mathcal{R} = \bigcup_n \mathcal{R}_n$ which is a countable δ-cover of $A = \bigcup_n A_n$. Then

$$\mathcal{H}^s_\delta(A) \leq \sum_{U \in \mathcal{R}} (\mathrm{diam}(U))^s = \sum_{n=1}^{\infty} \sum_{U \in \mathcal{R}_n} (\mathrm{diam}(U))^s \leq \sum_{n=1}^{\infty} \mathcal{H}^s_\delta(A_n) + \epsilon.$$

Let $\delta \to 0$ then $\epsilon \to 0$. $\square$

For any set $E \subset X$, $\mathcal{H}^s(E)$ is non-decreasing as a function of s. Furthermore, if $s < t$, then

$$\mathcal{H}^s(E) \geq \delta^{-(t-s)} \mathcal{H}^t(E), \quad \forall \delta > 0,$$

which implies that if $\mathcal{H}^t(E) > 0$, then $\mathcal{H}^s(E) = \infty$. We define the *Hausdorff dimension* of E, denoted by $\dim_H(E)$ or simply by $\dim E$, as follows

$$\dim E = \inf\{s > 0 : \mathcal{H}^s(E) = \infty\} = \sup\{s > 0 : \mathcal{H}^s(E) = 0\}.$$

As a consequence of the preceding proposition we have the following properties of Hausdorff dimension.

Proposition 4. *We have*

(1) $\dim\emptyset = 0$.
(2) *If* $E \subset F$, *then* $\dim E \leq \dim F$.
(3) *For a countable family* $\{E_i\}$ *of subsets of* X *we have*

$$\dim(\bigcup E_i) = \sup_i \dim E_i.$$

Proof. Here (1) and (2) are obvious. We only prove (3). For (3), we only need to prove $\dim(\bigcup E_i) \leq \sup_i \dim E_i$ because the other direction is also obvious. Assume $\dim E_i < s$ for all i. Then $\mathcal{H}^s(E_i) = 0$. By the σ-subadditivity of the $\mathcal{H}^s$, we get $\mathcal{H}^s(\bigcup_i E_i) = 0$ so that $\dim(\bigcup_i E_i) \leq s$. $\square$

Let $\mu(\cdot)$ be an outer measure on X. A subset $M \subset X$ is said to be μ-measurable if for any subset $S \subset X$ we have

$$\mu(S) = \mu(S \cap M) + \mu(S \setminus M).$$

It is well known that the subfamily $\mathcal{M}_\mu \subset 2^X$ consisting of all μ-measurable sets is a σ-algebra and that the restriction $\mu(\cdot) : \mathcal{M}_\mu \to [0, \infty]$ is a measure. Actually $(X, \mathcal{M}_\mu, \mu)$ is a complete measure space. If every Borel set is μ-measurable (i.e. $\mathcal{B}(X) \subset \mathcal{M}_\mu$), we say μ (considered as an outer measure on 2^X or a measure on $\mathcal{M}_\mu$) is a Borel measure. A Borel outer measure μ is said to be (outer) *Borel regular* if for each subset $A \subset X$ there exists a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$. We will see that the Hausdorff measure $\mathcal{H}^s$ is Borel regular.

Proposition 5. $\mathcal{H}^s$ *is a Borel measure.*

Proof. There is a criterion due to Caratheodory for an outer measure μ to be a Borel measure:

$$\mu(A \cup B) \geq \mu(A) + \mu(B) \tag{4}$$

for all subsets $A \subset X, B \subset X$ such that $d(A, B) > 0$, where $d(A, B)$ is the distance between A and B.

Let us first prove the proposition from the criterion. We have only to show the μ-measurability of an open set O, that is,

$$\mu(S) = \mu(S \cap O) + \mu(S \setminus O) \qquad (\forall S \subset X).$$

We only need to prove that

$$\mu(S) \geq \mu(S \cap O) + \mu(S \setminus O) \qquad (\forall S \subset X).$$

Let $F = O^c$. Let $F_k = \{x \in X : d(x, F) \leq 1/k\}$ be the closed $1/k$-neighborhood of F. Write $O_k = O \setminus F_k$. This is an open set contained in O, which has a distance to F at least $1/k$. Thus $d(S \cap O_k, S \setminus O) > 0$ and so

$$\mu(S) \geq \mu(S \cap O_k) + \mu(S \setminus O). \tag{5}$$

Assume that $\mu(S) < \infty$. Otherwise there is nothing to prove. Consider $C_k = S \cap O_k$. The sequence C_k is increasing. Let $R_k = C_k \setminus C_{k-1}$ (with $C_0 = \emptyset$). It is easy to check that $d(R_{k+2}, R_k) \geq \frac{1}{k(k+1)}$. So,

$$\mu(S) \geq \mu\left(\bigcup_{j=1}^m R_{2j-1}\right) \geq \sum_{j=1}^m \mu(R_{2j-1}),$$

$$\mu(S) \geq \mu\left(\bigcup_{j=1}^m R_{2j}\right) \geq \sum_{j=1}^m \mu(R_{2j}).$$

It follows that both series $\sum_{j=1}^\infty \mu(R_{2j-1})$ and $\sum_{j=1}^\infty \mu(R_{2j})$ are convergent. Observe

$$\mu(S \cap O) = \mu\left(\bigcup_{k=1}^\infty C_k\right) = \mu\left(C_m \cup \bigcup_{k=m}^\infty R_k\right) \leq \mu(C_m) + \sum_{k=m}^\infty \mu(R_k).$$

Since $\sum_{k=m}^\infty \mu(R_k) \to 0$ as $m \to \infty$, the above inequality implies

$$\mu(S \cap O) \leq \lim_m \mu(S \cap O_m).$$

Taking limit in (5), we get

$$\mu(S) \geq \mu(S \cap O) + \mu(S \setminus O).$$

We get the μ-measurability of O.

Now let us prove the criterion for $\mathcal{H}^s$. Given two subsets $A \subset X$ and $B \subset X$ such that $d(A, B) > 0$. Take $0 < \delta < \frac{d(A,B)}{2}$. Let $\mathcal{R}$ be an arbitrary δ-cover of $A \cup B$. None of element in $\mathcal{R}$ can meet both A and B. Hence

$$\begin{aligned}\sum_{E \in \mathcal{R}} (\mathrm{diam} E)^s &= \sum_{E \in \mathcal{R}: E \cap A \neq \emptyset} (\mathrm{diam} E)^s + \sum_{E \in \mathcal{R}: E \cap B \neq \emptyset} (\mathrm{diam} E)^s \\ &\geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B).\end{aligned}$$

Taking the infimum over all such covers $\mathcal{R}$ and then taking limit as $\delta \to 0$ we get

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

This is (4) for the Hausdorff measure. We proved the criterion for $\mathcal{H}^s$. This proves the proposition. □

Proposition 6. *$\mathcal{H}^s$ is Borel regular.*

Proof. Since $\text{diam}(\overline{U}) = \text{diam}(U)$, we may assume that the covering sets in the definition of $\mathcal{H}^s$ are all closed. Let $A \subset X$ be a fixed subset. For each $n \geq 1$, there exists a $1/n$-cover of closed sets $\{F_{n,i}\}_{i \geq 1}$ of A such that

$$\sum_i (\text{diam}(F_{n,i}))^s \leq \mathcal{H}^s_{1/n}(A) + \frac{1}{n}.$$

Take the Borel set $B = \bigcap_{n=1}^\infty \bigcup_{i=1}^\infty F_{n,i}$. Then $A \subset B$ and

$$\mathcal{H}^s(A) = \mathcal{H}^s(B).$$

□

In the definition of the Hausdorff dimension of a subset E of a metric space (X, d), we use all δ-covers of E. However, in applications, it is more convenient to use δ-covers of E by balls to calculate the Hausdorff dimension. Here we will prove it is possible. A ball is defined as a subset $B(x, r) = \{y \in X \mid d(y, x) < r\}$, where x is the center and $r > 0$ is the radius.

Let E be an arbitrary subset of X. For $\delta > 0$ and $s > 0$, we define

$$\mathcal{B}^s_\delta(E) = \inf \sum_{i=1}^\infty (\text{diam}(B))^s$$

where the infimum is over all countable δ-covers $\{B_i\}$ of E by balls. It is clear that for fixed s, $\mathcal{B}^s_\delta(E)$ is increasing with respect to δ. Now let $\delta \to 0$ we get the *s-(outer) measure* of E:

$$\mathcal{B}^s(E) = \lim_{\delta \to 0} \mathcal{B}^s_\delta(E).$$

Theorem 58.

$$\dim E = \inf\{s > 0 : \mathcal{B}^s(E) = \infty\} = \sup\{s > 0 : \mathcal{B}^s(E) = 0\}.$$

Proof. For any $\delta > 0$ and $s > 0$, it is clear that that

$$\mathcal{H}^s_\delta(E) \leq \mathcal{B}^s_\delta(E)$$

since δ-covers of E by balls are also δ-covers of E in the definition of $\mathcal{H}^s_\delta(E)$. Suppose $\{U_i\}$ is a δ-cover of E. Let B_i be some ball containing

U_i of radius $\text{diam}(U_i)$. Then $\{B_i\}$ is a δ-cover of E by balls. We have that

$$\sum_i \text{diam}(B_i)^s \leq \sum_i (2\text{diam}(U_i))^s = 2^s \sum_i (\text{diam}(U_i))^s.$$

Take infima,

$$\mathcal{B}^s_{2\delta}(E) \leq \mathcal{H}^s_\delta(E).$$

Letting $\delta \to 0$, it follows that

$$\mathcal{H}^s(E) \leq \mathcal{B}^s(E) \leq 2^s \mathcal{H}^s(E).$$

It proves the theorem. □

One can also prove that the Hausdorff dimension $\text{dim}E$ can be defined by all δ-covers of E by just closed subsets or just by open subsets. When E is a compact set, one can prove that the Hausdorff dimension $\text{dim}E$ can be defined by all finite δ-covers of E (or by just balls, or just closed subsets or just by open subsets).

The following the Frostmann lemma is useful in the calculation of Hausdorff dimension.

Lemma 14. *Let E be a Borel set in $\mathbb{R}^n$. Suppose there is a probability measure μ on E and positive constants $C > 0$ and α such that*

$$\mu(B(x, r) \cap E) \leq Cr^\alpha$$

for all $x \in E$, $r > 0$, where $B(x, r) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$ is the ball of radius r centered at x. Then $\text{dim}E \geq \alpha$.

Proof. Let $s \leq \alpha$ and $\delta > 0$. Let $\{B_i = B(x_i, r_i)\}$ be a δ-cover of E by balls. There is a universal constant $C(n) > 0$ such that after we remove unnecessary balls each point $x \in E$ will only be covered by at most $C(n)$ balls. Then

$$\sum_i (\text{diam}(B_i))^s = \sum_i (2r_i)^s \geq \sum_i \frac{1}{C2^s}\mu(B_i \cap E) = \frac{1}{C2^s}\int_E \Big(\sum_i 1_{B_i}\Big) d\mu$$

$$\geq \frac{1}{C2^s C(n)} \int_E 1_{\cup_i B_i} d\mu = \frac{1}{C2^s C(n)}.$$

This implies that

$$\mathcal{B}^s(E) \geq \frac{1}{C2^s C(n)} > 0$$

and

$$\text{dim}E \geq \alpha.$$

□

As an application of the distortion result in Lecture 3, we calculate the Hausdorff dimension of a Cantor set which is the invariant set of a $C^{1+\alpha}$-hyperbolic dynamical system.

Theorem 59 ($C^{1+\alpha}$-hyperbolic Cantor set, Hausdorff dimension)**.** *Suppose $f : I_0 \cup I_1 \to I$ is a $C^{1+\alpha}$ expanding map for some $0 < \alpha \leq 1$. Then the Hausdorff dimension of the non-escaping set $\Lambda = \Lambda_f$ under f is bigger than 0 and less than 1, that is,*

$$0 < \dim\Lambda < 1.$$

Proof. We use the same nation as that in the proof of Theorem 5. Without loss of generality, we assume that

$$0 < \gamma \leq \frac{1}{|f'(x)|} \leq \mu < 1.$$

Then we have that

$$\gamma^{n+1} \leq |I_{w_n}| \leq \mu^{n+1}$$

for all $w_n = i_0 \dots i_{n-1}$ and all $n > 0$. Thus we can use all partitions $\eta_n = \{I_{w_n}\}$ to calculate the Hausdorff dimension. That is, we consider

$$\beta_n(s) = \sum_{w_n} |I_{w_n}|^s.$$

Since $\beta_n(s) \geq 2^{n+1}\gamma^{s(n+1)}$, we have that $\dim\Lambda \geq \log 2/(-\log\gamma) > 0$.

For the other direction, we use the inequality

$$\frac{a^s + b^s}{2} \leq \Big(\frac{a+b}{2}\Big)^s$$

to get

$$|I_{w_{n-1}0}|^s + |I_{w_{n-1}1}|^s \leq 2^{1-s}(|I_{w_{n-1}0}| + |I_{w_{n-1}1}|)^s \leq 2^{1-s}(1-c)|I_{w_{n-1}}|^s.$$

Thus we have that

$$\beta_n(s) \leq 2^{(n+1)(1-s)}(1-c)^{n+1}$$

and that

$$\dim\Lambda \leq \frac{\log 2 + \log(1-c)}{\log 2}.$$

□

Lecture 10. Hausdorff dimension and pressure function.

Now let come back to study the Hausdorff dimension of an expanding holomorphic dynamical system (or $C^{1+\alpha}$ expanding one dimensional dynamical system).

Suppose Ω is a bounded simply connected region. Suppose $\Omega_1, \ldots, \Omega_d$ are bounded simply connected regions such that

$$\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \ \ \forall\, 1 \leq j \neq j \leq d$$

and

$$\cup_{i=1}^{d} \overline{\Omega}_i \subset \Omega.$$

Let $f : \cup_{i=1}^{d} \overline{\Omega}_i \to \overline{\Omega}$ be a holomorphic map, that means that f is holomorphic on a neighborhood of $\cup_{i=1}^{d} \overline{\Omega}_i$. Suppose $f|\overline{\Omega}_i : \overline{\Omega}_i \to \overline{\Omega}$ is conformal for every $1 \leq i \leq d$. Let

$$J = \cap_{n=0}^{n} f^{-n}(\overline{\Omega})$$

be the non-escaping set of f (we will call it the Julia set later).

(In one-dimensional dynamical systems, we have a similar setting as follows: Let $I = [0, 1]$ and $I_i = [a_i, b_i]$, $i = 1, \ldots, d$, $a_1 = 0 < b_1 < a_2 < b_2 < \cdots < a_d < b_d = 1$. Suppose $f : K_0 = \cup_i^d I_i \to I$ is a $C^{1+\alpha}$ map such that each $f : I_i \to I$ is a homeomorphism. Let

$$J = \cap_{n=1}^{\infty} f^{-n}(I)$$

be the non-escaping set. Suppose f is expanding, that is, $|(f^n)'(x)| \geq C\lambda^n$ for two constants $C > 0$ and $\lambda > 1$ and all $n > 0$ and all x such that $f^i(x) \in K_0$, $0 \leq i \leq n-1$. Then all arguments in the following in this section work for this setting just replacing the Koebe distortion by the naive distortion.)

Then from the hyperbolic geometry we know that $f : J \to J$ is a hyperbolic Markovian map with the initial partition

$$\eta_0 = \{J_0, \ldots, J_d\}$$

where $J_i = J \cap \overline{\Omega}_i$. Then we have a bijective map

$$\pi : \Sigma_d = \prod_{n=0}^{\infty} \{1, \ldots, d\} \to J.$$

Thus J is a Cantor set.

Without loss of generality, we assume that $\Omega = \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $0 < r < 1$ be a such number such that

$$J \subset \Delta_r = \{z \in \mathbb{C} \mid |z| < r\}.$$

Let

$$g_i = (f|\overline{\Omega}_i)^{-1} : \Delta \to \Omega_i, \quad 1 \le i \le d.$$

For any $w_n = i_0 \dots i_{n-1}$ of a sequence of $\{1, \dots, d\}$,let

$$g_{w_n} = g_{i_0} \circ \dots g_{i_{n-1}} : \Delta \to \Omega_{w_n} = g_{w_n}(\Delta).$$

Then g_{w_n} is conformal. From the Koebe distortion theorem (Theorem 23), we have that

$$\Big(\frac{1-r}{1+r}\Big)^4 \le \frac{|g'_{w_n}(z_1)|}{|g'_{w_n}(z_2)|} \le \Big(\frac{1+r}{1-r}\Big)^4, \quad \forall |z_1|, |z_2| \in \Delta_r,$$

and

$$|g'_{w_n}(0)|\frac{|r|}{(1+|r|)^2} \le |g_{w_n}(z) - g_{w_n}(0)| \le |g'_{w_n}(0)|\frac{|r|}{(1-|r|)^2}, \quad \forall z \in \Delta_r.$$

Therefore, we have a constant $C > 0$ such that

$$\frac{|g'_{w_n}(z)|}{C} \le \mathrm{diam}(\Omega_{w_n}) \le C|g'_{w_n}(z)|, \quad \forall z \in \Delta_r. \tag{6}$$

In other words,

$$C^{-1}|(f^{n+1})'(z)|^{-1} \le \mathrm{diam}(\Omega_{w_n}) \le C|(f^{n+1})'(z)|^{-1}, \quad \forall z \in \Omega_{w_n}. \tag{7}$$

Since $g_{w_n} : \Omega \to \Omega_{w_n}$ is a contracting map, from the contracting fixed point theorem, we have a unique fixed point $p_{w_n} \in \Omega_{w_n}$ of g_{w_n}. It is the unique fixed point $p_{w_n} \in \Omega_{w_n}$ of f^{n+1}, that is, $f^{n+1}(p_{w_n}) = p_{w_n}$. Let $\lambda_{w_n} = (f^{n+1})'(p_{w_n})$ be the multiplier of f^{n+1} at p_{w_n}. Then we have that

$$C^{-1}|\lambda_{w_n}|^{-1} \le \mathrm{diam}(\Omega_{w_n}) \le C|\lambda_{w_n}|^{-1}, \quad \forall w_n. \tag{8}$$

(In the $C^{1+\alpha}$ expanding one-dimensional case, we can use the naive distortion (Lemma 1) to replace the Koebe distortion to get the inequalities (6), (7), and (8).)

Now consider the function $\phi_t(z) = -t\log|f'(z)|$. For any $z \in J$, let $S_{n,t}(z) = \sum_{i=0}^n \phi_t(f^i(z))$. The topological pressure for ϕ_t is, by definition,

$$P(\phi_t) = \lim_{n\to\infty}\frac{1}{n}\log\Big(\sum_{w_n} e^{S_{n,t}(p_{w_n})}\Big) = \lim_{n\to\infty}\frac{1}{n}\log\Big(\sum_{w_n}|\lambda_{w_n}|^{-t}\Big)$$

It is clearly that

$$P(\phi_t) = \lim_{n\to\infty}\frac{1}{n}\log\Big(\sum_{z\in\Omega_{w_n}} e^{S_{n,t}(z)}\Big) = \lim_{n\to\infty}\frac{1}{n}\log\Big(\sum_{z\in\Omega_{w_n}}|(f^n)'(z)|^{-t}\Big)$$

because from the Koebe distortion (or the naive distortion in one-dimensional dynamical systems)), we have a constant $C > 0$ such that

$$C^{-1}|\lambda_{w_n}| \leq |(f^{n+1})'(z)| \leq C|\lambda_{w_n}|, \quad \forall z \in \Omega_{w_n}, \quad \forall w_n.$$

Lemma 15. *Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers. Suppose $a_{n+m} \leq a_n + a_m$. Then*

$$\lim_{n\to\infty} \frac{a_n}{n}$$

exists.

Proof. Let $l = \inf\{a_n/n \mid n \geq 1\}$. For any $\epsilon > 0$, there is $n_0 > 0$ such that

$$l \leq \frac{a_{n_0}}{n_0} \leq l + \epsilon/2.$$

Let $n_1 > n_0$ be a number such that $\max\{a_k/n_1, k = 1, \cdots, n_0 - 1\} \leq \epsilon/2$. Then any $n > n_1$, let $n = pn_0 + k$, where $0 \leq k < n_0$. Define $a_0 = 0$. Then

$$l \leq \frac{a_n}{n} \leq \frac{pa_{n_0} + a_k}{pn_0 + k} \leq \frac{a_{n_0}}{n_0} + \frac{a_k}{n} \leq l + \epsilon.$$

This says that

$$\lim_{n\to\infty} \frac{a_n}{n} = l.$$

□

Theorem 60. *The limit*

$$P(\phi_t) = \lim_{n\to\infty} \frac{1}{n} \log\Big(\sum_{w_n} e^{S_{n,t}(p_{w_n})}\Big) = \lim_{n\to\infty} \frac{1}{n} \log\Big(\sum_{w_n} |\lambda_{w_n}|^{-t}\Big)$$

exists.

Proof. Since

$$P(\phi_t) = \lim_{n\to\infty} \frac{1}{n} \log\Big(\sum_{z\in\Omega_{w_n}} e^{S_{n,t}(z)}\Big),$$

let $b_n = \sum_{z\in\Omega_{w_n}} e^{S_{n,t}(z)})$, we have that

$$\begin{aligned} b_{n+p} = \sum_{z\in\Omega_{w_{n+p}}} e^{S_{n+p,t}(z)} &= \sum_{z\in\Omega_{w_{n+p}}} e^{S_{n,t}(z)} e^{S_{p,t}(f^n(z))} \\ &\leq \sum_{z\in\Omega_{w_n}} e^{S_{n,t}(z)} \sum_{w\in\Omega_{w_p}} e^{S_{p,t}(w)} \leq b_n b_p. \end{aligned}$$

Let $a_n = \log b_n$. Then we have that $a_{n+p} \leq a_n + a_p$. The previous lemma implies that the limit exists. □

Remark 12. *The topological pressure can be defined for any continuous function ϕ. In particular, when $\phi \equiv 0$, the $P(0) = h(f)$ is just the growth rate we studied before (or called the topological entropy). As we already know, in our case studied above, $P(0) = \log d$.*

Consider the function $\alpha(t) = P(\phi_t)$. It is differentiable with $\alpha'(t) < 0$ and $\alpha''(t) > 0$. Thus $\alpha(t)$ is strictly decreasing and concave upward function of t. Since $\alpha(0) = h(f) > 0$ and $\alpha(t) \to -\infty$ as $\to \infty$. We have a unique number t_0 such that $\alpha(t_0) = 0$. The following theorem is due to Bowen and Ruelle originally, refer to

- R. Bowen. Hausdorff dimension of quasi-circles. Publ. of IHES, Volume 50 (1979).
- D. Ruelle, Repellers for real analytic maps. Ergodic Theory Dynamical Systems **2** (1982), 99107.

Theorem 61. *The Hausdorff dimension* $\dim J = t_0$.

Combing with Theorem 19, we got

Corollary 7. $0 < \dim J < 2$.

Proof of Theorem 61. The collection of set $\{\Omega_{w_n}\}$ is a δ-cover of J for any large n. From the Koebe distortion, there is a constant $C > 0$ such that

$$C^{-1}|\lambda_{w_n}|^{-1} \leq \mathrm{diam}(\Omega_{w_n}) \leq C|\lambda_{w_n}|^{-1}, \quad \forall n > 0, \forall w_n.$$

Thus

$$C^{-1} = C^{-1}\sum_{w_n} |\lambda_{w_n}|^{-t_0} \leq \sum_{w_n} \Big(\mathrm{diam}(\Omega_{w_n})\Big)^{-t} \leq C\sum_{w_n} |\lambda_{w_n}|^{-t_0} = C.$$

This implies that $\mathrm{diam} J \leq t_0$.

To prove $\mathrm{diam} J \geq t_0$. We need the Gibbs measure for ϕ_{t_0} (see Theorem 62), that is, we have a measure μ on J such that there is constant $D > 0$,

$$D^{-1}|(f^{n+1})'(z)|^{-t_0} = D^{-1}e^{S_{n,t}(z)} \leq \mu(J_{w_n}) \leq De^{S_{n,t}(z)} = D|(f^{n+1})'(z)|^{-t_0}$$

for any w_n and $z \in J_{w_n}$ (since $P(\phi_{t_0}) = 0$).

We also need the Koebe distortion, that is, we have two constants $c_1, c_2 > 0$ such that

$$B(g_{w_n}(0), c_1|g'_{w_n}(0)|) \subset \Omega_{w_n} \subset B(g_{w_n}(0), c_2|g'_{w_n}(0)|).$$

for any $n > 0$, w_n. This implies that there are two constants $d_1, d_2 > 0$ such that

$$B(g_{w_n}(0), d_1|(f^{n+1})'(z)|^{-1}) \subset \Omega_{w_n} \subset B(g_{w_n}(0), d_2|(f^{n+1})'(z)|^{-1})$$

for any $n > 0$, w_n, and $z \in \Omega_{w_n}$.

Suppose $B(a,r) = \{z \in \mathbb{C} \mid |z-a| < r\}$ is any ball such that $B(a,r) \cap J \neq \emptyset$. For any $z \in B(a,r) \cap J$, $B(a,r) \subset B(z,2r)$. There is a unique integer $n_1 > 0$ such that

$$|(f^{n_1+2})'(z)|^{-1} \leq 2rd_1^{-1} \leq |(f^{n_1+1})'(z)|^{-1}.$$

Thus

$$B(a,r) \subset B(g_{w_{n_1}}(0), 2r) \subset B(g_{w_{n_1}}(0), d_1|(f^{n_1+1})'(z)|^{-1}) \subset \Omega_{w_{n_1}}.$$

Let $m = \inf_{z\in J} |f'(z)|^{-1}$. Then

$$r^{t_0} \geq \Big(\frac{d_1 m}{2}\Big)^{t_0} |(f^{n_1+1})'(z)|^{-t_0} \geq D^{-1}\Big(\frac{d_1 m}{2}\Big)^{t_0} \mu(J_{w_{n_1}}) \geq C\mu(B(a,r) \cap J)$$

where

$$C = D^{-1}\Big(\frac{d_1 m}{2}\Big)^{t_0}$$

is a constant. From Lemma 14, we have that

$$\dim J \geq t_0.$$

$\square$

In the proof of Theorem 61, we need the Gibbs theory, let us state it here and we will give a full study later, for example, refer to

- Y. Jiang, Nanjing Lecture Notes In Dynamical Systems. Part One: Transfer Operators in Thermodynamical Formalism. FIM Preprint Series, ETH-Zurich, June 2000.
- Y. Jiang, A proof of existence and simplicity of a maximal eigenvalue for Ruelle-Perron-Frobenius operators. Letters in Math. Phys. 48 (1999), no. 3, 211-219.
- A. Fan and Y. Jiang, On Ruelle-Perron-Frobenius Operators. I. Ruelle Theorem. Comm. in Math. Phys., 223 (2001), 125-141.

Let $A = (a_{ij})_{d\times d}$ be a $0-1$ matrix, that is, $a_{ij} = 0$ or 1. A $(n+1)$-sequence $w_n = i_0 i_1 \ldots i_{k-1} i_k \ldots i_{n-1} i_n$ of $\{1, \ldots, d\}$ is called admissible if $a_{i_{k-1}i_k} = 1$ for all $1 \leq k \leq n$. Let

$$\Sigma_A = \{w = i_0 i_1 \ldots i_{k-1} i_k \ldots\}$$

be the space of all ∞-admissible sequence of $\{1, \ldots, d\}$. For a given $w_n = i_0 i_1 \ldots i_{k-1} i_k \ldots i_{n-1} i_n$,

$$[w_n] = \{w' = j_0 j_1 \ldots j_{k-1} j_k \ldots \mid j_k = i_k,\ 0 \leq k \leq n\}$$

is a cylinder. All cylinders form a topological basis for Σ_A. With this topology, Σ_A is a compact topological space. Let

$$\sigma_A : w = i_0 i_1 \ldots \to \sigma_A(w) = i_1 \ldots$$

be the shift.

Consider the space $C(\Sigma_A)$ of all continuous functions defined on Σ_A. Let

$$|\phi|_\infty = \max_{w\in\Sigma_A} |\phi(w)|$$

be the maximal norm of ϕ. A function $\phi \in C(\Sigma_A)$ is called a Hölder continuous if there are constants $C > 0$ and $0 < \beta < 1$ such that

$$|\phi(w) - \phi(w')| \leq C\beta^n$$

for any $w, w' \in [w_n]$. Let $C^h(\Sigma_A)$ be the space of all Hölder continuous functions.

Theorem 62. *For any $\phi \in C^h(\Sigma_A)$, there is a real number $P = P(\phi)$ and a probability measure $\mu = \mu_\phi$ on Σ_A such that*

$$C^1 \leq \frac{\mu([w_n])}{\exp(-nP + \sum_{i=0}^n \log\phi(\sigma_A^i(w))} \leq C$$

for any cylinder $[w_n]$ and and $w \in [w_n]$, where $C > 0$ is a constant.

Remark 13. *The function ϕ is called a potential. The number P is called the pressure for the potential ϕ. The measure μ is called the Gibbs measure associate to the potential ϕ. Usually, we write*

$$S_n(w) = S_{n,\phi}(w) = \sum_{i=0}^n \log\phi(\sigma_A^i(w)).$$

Using the Ruelle's Perron-Frobenius theorem and the study of transfer operators (which we will fully develop with the Gibbs theory), we knew that

$$P(\phi) : C^h(\Sigma_A) \to \mathbb{R}$$

is a real analytic function. Refer to

- Y. Jiang, Nanjing Lecture Notes In Dynamical Systems. Part One: Transfer Operators in Thermodynamical Formalism. FIM Preprint Series, ETH-Zurich, June 2000.

Now consider a holomorphic family of hyperbolic holomorphic dynamical systems

$$f_c : \cup_{i=1}^d \Omega_i(c) \to \Omega, \ \ c \in \Delta$$

where

(1) all $\Omega_i(c)$ and Ω are simply connected regions in $\mathbb{C}$,
(2) $\overline{\Omega}_i(c) \cap \overline{\Omega}_j(c) =$ for any $1 \leq i, j \leq d$,
(3) $f_c : \Omega_k(c) \to \Omega$ is conformal for every $1 \leq k \leq d$,
(4) $\cup_{i=1}^d \overline{\Omega}_k(c) \subset \Omega$,

(5) for any $z \in J_c$, $f_c(z)$ depends on $c \in \Delta$ holomorphically.

For $c = 0$, $\phi_{0,t}(w) = -t \log |f_0'(\pi(w))|$ defined on Σ_d is a Hölder continuous function. For any w_n, there exists a unique $p_{w_n}(c) \in \Omega_{w_n}(c)$ such that $f_c^{n+1}(p_{w_n}(c)) = p_{w_n}(c)$. The multiplier

$$|\lambda_{w_n}(c)| = |(f_c^{n+1})'(p_{w_n}(c))| > 1.$$

Thus, implicitly function theorem implies that $p_{w_n}(c)$ defines a holomorphic function on Δ. Let Per be the set of all periodic points of f_0. Then

$$h(c, p_{w_n}) = p_{w_n}(c) : \Delta \times Per \to \mathbb{C} \tag{9}$$

is a holomorphic motion.

In general, let E be a subset in the Riemann sphere $\widehat{\mathbb{C}}$, a map $h(c, z) : \Delta \times E \to \widehat{\mathbb{C}}$ is called a holomorphic motion if

- $h(0, z) = z$ for every $z \in E$,
- for each $c \in \Delta$, $h_c(z) = h(c, z)$ defines a bijective map from E to $h_c(E)$,
- for each $z \in E$, $h^z(c) = h(c, z)$ defines a holomorphic map from Δ to $\widehat{\mathbb{C}}$.

The following lemma is first proved by Màñé, Sad, and Sullivan in

- Màñé, R. M., Sad, P. and Sullivan, D. On the dynamics of rational maps. Ann. c. Norm. Sup. 96 (1983), 193217.

Lemma 16 (λ-Lemma)**.** *The map h can be extended to a holomorphic motion $\overline{h}(c, z) : \Delta \times \overline{E} \to \widehat{\mathbb{C}}$.*

Furthermore, we have

Theorem 63 (Slodkowski's Theorem)**.** *The map $\overline{h}$ can be extended to a holomorphic motion $H(c, z) : \Delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

And

Theorem 64 (Bers-Royden's Theorem)**.** *Suppose $H(c, z) : \Delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic motion. Then for each $c \in \Delta$, $H_c(z) = H(c, z) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defines a quasiconformal homeomorphism with whose quasiconformal dilatation $K(H_c) \leq (1 + |c|)/(1 - |c|)$.*

We will lecture these two theorems in details later following papers.

- F. Gardiner, Y. Jiang, and Z. Wang, Holomorphic motions and related topics, Geometry of Riemann Surfaces, London Mathematical Society Lecture Note Series, No. 368, 2010, 166-193.

- Y. Jiang, S. Mitra, and Z. Wang, Liftings of holomorphic maps into Teichmller spaces, Kodai Mathematical Journal, 32 (2009), No. 3, 544-560.
- F. Gardiner and Y. Jiang, Guiding isotopies and Slodkowskis theorem, Preprint.

Theorem 65. *Suppose H is a quasiconformal homeomorphism on a simply connected region Ω with quasiconformal dilatation $K > 1$. Let $J \subset \Omega$ be a compact subset. Then $H|J$ is an α-Hölder continuous function for some $0 < \alpha = \alpha(K) < 1$, that is,*

$$|H(z) - H(w)| \leq C|z - w|^{\alpha}, \quad \forall z, w \in J,$$

where $C = C(K) > 0$ is a constant.

We will also lecture this theorem later, refer to

- L. V. Ahlfors. Lectures on Quasiconformal Mapping,, volume 38 of University Lecture Series. Amer. Math. Soc., 2006.

Applying holomorphic motions, we can extend the holomorphic motion in (9) to a holomorphic motion, which we still denote as

$$h(c, z) : \Delta \times J \to \widehat{\mathbb{C}}$$

since $J = \overline{Per}$. From the definition and the continuity of $h_c(z) = h(c, z)$ on J, we have $h_c : J \to J_c$ and

$$h_c(f_0(z)) = f_c(h_c(z)), \quad z \in J.$$

Since h_c is the restriction of a quasiconformal homeomorphism H_c, so h_c and h_c^{-1} are both α-Hölder continuous. Define

$$\phi_{c,t}(w) = -t \log |f_c'(h_c^{-1}(\pi(w)))|$$

on Σ_d. It is a family of Hölder continuous functions which depends on $c \in \Delta$ real analytically. Thus the pressure function $P(\phi_{c,t})$ is a function depends on c and t both real analytic. The unique solution t_c such that $P(\phi_{c,t_c}) = 0$ is a real analytic function defined on $c \in \Delta$. Since t_c is the Hausdorff dimension $\mathrm{diam} J_c$, we conclude that, originally due to Ruelle, refer to

- D. Ruelle, Repellers for real analytic maps. Ergodic Theory Dynamical Systems **2** (1082), 99107.

Theorem 66. *The Hausdorff dimension $\mathrm{diam} J_c$ is real analytically depends on $c \in \Delta$.*

In the proof of this theorem, I still owes you two big theories, the Gibbs theory and the holomorphic motion theory. We will lecture them later. Using the above theorem, Ruelle showed that

Corollary 8. *Suppose* $q_c(z) = z^2 + c$ *is the quadratic polynomial and suppose* J_c *is its Julia set. The Hausdorff dimension* $\mathrm{diam} J_c$ *is analytic function on the main cardioid* $\mathcal{M}_0$ *and*

$$\mathrm{diam} J_c = 1 + \frac{|c|^2}{4\log 2} + o(|c|^2).$$

Lecture 11. Quasiconformal Mappings and Applications

11.1. Grötzsch argument.

Recall that the Riemann Mapping Theorem (Theorem 10). A simple closed curve in $\mathbb{C}$ is called a Jordan curve if it is an image of the unit circle under a homeomorphism.

Theorem 67 (Carathèodory Theorem). *Suppose Ω is a simply connected domain. Suppose $\partial\Omega$ is a Jordan curve. Then H can be extended to the closure $\overline{\Omega}$ as a homeomorphism from $\overline{\Omega}$ onto $\overline{\Delta}$.*

We say $\partial\Omega$ is locally connected if for any point $z \in \partial\Omega$ and any neighborhood U about z, $U \cap \partial\Omega$ is connected. More generally, we have

Theorem 68 (Carathèodory Theorem). *Suppose Ω is a simply connected domain and suppose $\partial\Omega$ is locally connected. Then the inverse $H^{-1} : \Delta \to \Omega$ can be extended to a continuous map from $\overline{\Delta} \to \overline{\Omega}$.*

Now let us consider the following problem. Suppose $R = [0, a, a + ib, ib]$ is a rectangle with vertices 0, a, $a + ib$, and ib in the complex plane. Let $R' = [0, a', a' + ib', ib']$ be another rectangle with vertices 0, a', $a' + ib'$, and ib' in the complex plane. According to Theorem 10, we have a conformal map $H : R \to R'$ and according to Theorem 67, H can be extended as a homeomorphism $H : R \to R'$. How we add one more condition that $H(0) = 0$, $H(a) = a'$, $H(a + ib) = a' + ib'$, and $H(ib) = ib'$.

Problem 6. *When can we find a conformal map H from R to R' satisfying the above condition?*

Before answering the above problem, let us have some basic calculation in complex analysis.

Let $z = x + iy$ and $\overline{z} = x - iy$ be the coordinate system in $\mathbb{C}$. Then we have the change of coordinate

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

We calculate differentials

$$dz = dx + idy \quad \text{and} \quad d\overline{z} = dx - idy.$$

Since $\partial/\partial x$ and $\partial/\partial y$ are two differential operators, we have other differential operators

$$\frac{\partial}{\partial z}=\frac{1}{2}\Big(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}}=\frac{1}{2}\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big).$$

So for any orientation-preserving C^1 map $f:\Omega\to\mathbb{C}$, we have

$$df(z)=\frac{\partial f(z)}{\partial x}dx+\frac{\partial f(z)}{\partial y}dy=\frac{\partial f(z)}{\partial z}dz+\frac{\partial f(z)}{\partial \overline{z}}d\overline{z}.$$

We use the following notation

$$\partial f(z)=\frac{\partial f(z)}{\partial z} \quad \text{and} \quad \overline{\partial} f(z)=\frac{\partial f(z)}{\partial \overline{z}}.$$

So we have

$$df(z)=\partial f(z)dz+\overline{\partial} f(z)d\overline{z}$$

is a linear map from the tangent plane at $z\in\Omega$ to the tangent plane at $f(z)$. Consider $\overline{f}(z)=\overline{f(z)}$. Then

$$\partial\overline{f}(z)=\overline{\overline{\partial} f(z)} \quad \text{and} \quad \overline{\partial}\overline{f}(z)=\overline{\partial f(z)}.$$

The Jacobian $J(f)(z)$ is

$$J(f)(z)=\left\|\begin{array}{cc} \partial f(z) & \overline{\partial} f(z) \\ \partial\overline{f}(z) & \overline{\partial}\overline{f}(z)\end{array}\right\| = |\partial f(z)|^2-|\overline{\partial} f(z)|^2>0.$$

Consider the unit circle $|dz|=1$ on the tangent plane at z, then the image of the unit circle is $df(z)$ with

$$0<|\partial f(z)|-|\overline{\partial} f(z)|\le |df(z)|\le |\partial f(z)|+|\overline{\partial} f(z)|.$$

Let

$$\mu(z)=\frac{\overline{\partial} f(z)}{\partial f(z)}=re^{2\pi i\theta}.$$

Then we have $0\le|\mu(z)|<1$ and

$$df(z)=(1+\mu(z)\frac{d\overline{z}}{dz})\partial f(z)dz$$

and

$$|\partial f(z)|(1-|\mu(z)|)\le df(z)\le |\partial f(z)|(1+|\mu(z)|).$$

Suppose $\arg\mu=\theta$. Then the maximum can be reached at $\arg dz=\theta/2$ and the minimum can be reached at $\arg dz=(\theta+\pi)/2$. Thus $df(z)$ is an ellipse with the major axis $l(z)=|\partial f(z)|(1+|\mu(z)|)$ and the minor axis $s(z)=|\partial f(z)|(1-|\mu(z)|)$. The ratio $K(f)(z)=l(z)/s(z)$ is called the quasiconformal dilatation of f at z. A conformal map preserves circles. That is $df(z)$ is a circle. In other words, f is conformal at z if and only if

$$K(z) = 1 \Longleftrightarrow \mu(z) = 0 \Longleftrightarrow \overline{\partial} f(z) = 0.$$

Now suppose $f : R \to R'$ is a C^1 diffeomorphism such that $f(0) = 0$, $f(a) = a'$, $f(a+ib) = a'+ib'$, and $f(ib) = ib'$. Let $c_y(t) = t+iy, 0 \leq t \leq a$ be a straight line in R. Then $f(c_y(t))$ is a C^1 curve in R' connecting $f(iy)$ and $f(a+iy)$. Then

$$a' \leq |\int_{c_y} df(z)| \leq \int_{c_y} |df(z)|$$

$$\leq \int_{c_y} (|\partial f(z)| + |\overline{\partial} f(z)|)|dz| = \int_{c_y} J(f)(z)^{1/2} K(f)(z)^{1/2} |dz|.$$

So

$$a'b \leq |\int\int_R df(z)| \leq \int\int_R J(f)(z)^{1/2} K(f)(z)^{1/2} |dz \wedge d\overline{z}|$$

$$\leq (\int\int_R J(f)(z)|dz \wedge d\overline{z}|)^{1/2} (\int\int_R K(f)(z)|dz \wedge d\overline{z}|)^{1/2}$$

$$= (a'b')^{1/2}(ab)^{1/2}(\max_{z\in R} K(f)(z))^{1/2}.$$

This implies that

$$(a'/b') : (a/b) \leq \max_{z\in R} K(f)(z).$$

Thus the answer to Problem 6 is that there is a conformal map from R onto R' preserving the vertices if and only if $a'/b' = a/b$. The number a/b is called the modulus of R is denoted as $\mathrm{mod}(R)$. It is a conformal invariant.

In general, for a C^1 diffeomorphism $f : \Omega \to \mathbb{C}$, let

$$K(f) = \max_{z\in\Omega} K(f)(z).$$

Then we call it a K-quasiconformal diffeomorphism. Furthermore, we have that

Theorem 69. *Suppose R and R' are two rectangles. Then there is a K-quasiconformal C^1-diffeomorphism preserving the vertices $f : R \to R'$ if and only if*

$$\frac{1}{K}\mathrm{mod}(R') \leq \mathrm{mod}(R) \leq K\mathrm{mod}(R').$$

Suppose Ω is a double connected region. Then we have a conformal map $f : \Omega \to A_r = \{z \in \mathbb{C} \mid 1 < |z| < r\}$, where r is unique. The modulus $\mathrm{mod}(\Omega)$ of Ω is defined as

$$\mathrm{mod}(\Omega) = \frac{1}{2\pi} \log r.$$

Then it is a conformal invariant. Furthermore, we also have by the above argument that

Theorem 70. *Suppose Ω and Ω' are two double connected regions. Then there is a K-quasiconformal C^1-diffeomorphism preserving the vertices $f : \Omega \to \Omega'$ if and only if*

$$\frac{1}{K}\mathrm{mod}(\Omega) \leq \mathrm{mod}(\Omega') \leq K\mathrm{mod}(\Omega).$$

11.2. Extremal length.

Suppose Ω is a domain in the complex plane. Let $\rho \geq 0$ be a measurable function on Ω. Then ρ is called a conformal metric on Ω. Suppose $\mathcal{F} = \{\gamma(t)\}$ be a family of piecewise C^1 curves in Ω. The ρ-length is

$$l_\rho(\gamma) = \int_\gamma \rho|dz| = \int_a^b \rho(\gamma(t))\gamma'(t)dt$$

if $t \to \rho(\gamma(t))$ is measurable; otherwise, $l_\rho(\gamma) = \infty$. The ρ-length of $\mathcal{F}$ is

$$L_\rho(\mathcal{F}) = \inf\{l_{\rho(\gamma)} \mid \gamma \in \mathcal{F}\}.$$

The extremal length of $\mathcal{F}$ is, by definition,

$$\lambda(\mathcal{F}) = \sup_\rho \frac{(L_\rho(\mathcal{F}))^2}{A_\rho},$$

where supremum is taken over all conformal metrics $\rho \geq 0$ on Ω such that areas $A_\rho = \int\int_\Omega \rho^2 dxdy < \infty$. If the supremum can be reached by a conformal metric $\rho \geq 0$ on Ω. Then such a ρ is called an extremal conformal metric of $\mathcal{F}$.

Theorem 71. *Suppose Ω and Ω' are two domains. If $f : \Omega \to \Omega'$ is a K-quasiconformal C^1-diffeomorphism, then*

$$\frac{1}{K}\lambda(\mathcal{F}) \leq \lambda(f(\mathcal{F})) \leq K\lambda(\mathcal{F})$$

for any family $\mathcal{F}$ of piecewise C^1 curves in Ω (then $f(\mathcal{F})$ is a family of piecewise C^1 curves in Ω'. In particular, the extremal length is conformal invariant.

Proof. Consider a conformal metric $\rho(z)dz$ on Ω. Then

$$\tilde{\rho}(w) = \frac{\rho(f^{-1}(w))}{|\partial f(f^{-1}(w))| - |\overline{\partial} f(f^{-1}(w))|}$$

defines a conformal metric on Ω'. For $\gamma(t) \in \mathcal{F}$, $\delta(t) = f(\gamma(t)) \in f(\mathcal{F})$. We have that

$$\tilde{\gamma}'(t) = \partial f(\gamma(t))\gamma'(t) + \overline{\partial} f(\gamma(t))\overline{\gamma'(t)}.$$

This implies that $l_\rho(\gamma) \leq l_{\tilde{\rho}}(\tilde{\gamma})$. Thus $L_\rho(\mathcal{F}) \leq L_{\tilde{\rho}}(f(\mathcal{F}))$. On the other hand, for $w = u + iv$ and $z = x + iy$,

$$A_{\tilde{\rho}} = \int\int_{\Omega'} (\tilde{\rho}(w))^2 dudv = \int\int_{\Omega} \Big(\frac{\rho(z)}{|\partial f(z)| - |\overline{\partial} f(z)|} \Big)^2 J(f)(z) dxdy$$
$$= \int\int_{\Omega} (\rho(z))^2 \frac{|\partial f(z)| + |\overline{\partial} f(z)|}{|\partial f(z)| - |\overline{\partial} f(z)|} dxdy \leq K A_\rho.$$

This implies that

$$\lambda_{\tilde{\rho}}(f(\mathcal{F})) \geq \frac{1}{K} \lambda_\rho(\mathcal{F}).$$

The other side inequality can be obtained similarly. $\square$

Let $R = R(0, a, a+ib, ib)$ be the rectangle. Let $\mathcal{F}$ be the family of all piecewise C^1-curves connecting two points on the both vertical sides of R. Let $\rho_0 = 1$. Then $|dz|$ is the Lebesgue measure on R. It is easy to see $L_{\rho_0}(\mathcal{F}) = a$ and $A_{\rho_0} = ab$. Thus

$$\lambda(\mathcal{F}) \geq \frac{a^2}{ab} = \frac{a}{b}.$$

Suppose $\rho \geq 0$ is any conformal metric on R. Suppose $L_\rho(\mathcal{F}) \neq 0$. By multiplying a constant on ρ, we can assume that $L_\rho(\mathcal{F}) = a$. This implies that

$$\int_0^a (\rho(z) - 1) dx \geq 0.$$

Then we get

$$0 \leq \int\int_R (\rho - 1)^2 dxdy = \int\int_R \rho^2 dxdy - 2 \int\int \rho dxdy + \int\int_R 1 dxdy$$
$$\leq \int\int_R \rho^2 dxdy - \int\int 1 dxdy.$$

This says that

$$A_\rho \geq ab.$$

So

$$\frac{(L_\rho(\mathcal{F}))^2}{A_\rho} \leq \frac{a^2}{ab} = \frac{a}{b}.$$

Thus $\lambda(\mathcal{F}) = a/b = \mathrm{mod}(R)$ and the Lebesgue metric is the extremal conformal metric.

Exercise 3. *Let $A_r = \{z \in \mathbb{C} \mid 1 \leq |z| \leq r\}$. Let $\mathcal{F}$ be the family of piecewise C^1 curves connecting two points in the unit circle $|z| = 1$ and the circle $|z| = r$. Find the extremal length $\lambda(\mathcal{F})$ and compare with the modulus $\mathrm{mod}(A_r) = (1/2\pi)\log r$. Find the extremal conformal metric.*

11.3. Quasiconformal mappings.

We now are ready to give a general definition of a quasiconformal homeomorphism. A quadrilteral $B = B(z_1, z_2, z_3, z_4)$ is a (closed) Jordan domain with four distinct points z_1, z_2, z_3, z_4 marked on the boundary of B in the counterclockwise sense. Then two arcs $[z_1, z_2]$ and $[z_3, z_4]$ are called horizontal sides and two arcs $[z_2, z_3]$ and $[z_4, z_2]$ are called vertical sides. Let $\mathcal{F}$ be the family of all C^1 piecewise curves connecting two points on two different vertical sides. The modulus of B is defined as the extremal length of $\mathcal{F}$, that is,

$$\mathrm{mod}(B) = \lambda(\mathcal{F}).$$

We have a conformal map f from B onto a unique $R = R(0, a, a+i, i)$ such that $f(z_1) = 0$, $f(z_2) = a$, $f(z_3) = a + i$, and $f(z_4) = i$. The number $\mathrm{mod}(R) = a$ is uniquely determinant. It is just the modulus $\mathrm{mod}(B)$ of B. That is,

$$\mathrm{mod}(B) = \lambda(\mathcal{F}) = \mathrm{mod}(R).$$

The quadrilateral $B' = B(z_2, z_3, z_4, z_1)$ has the modulus $\mathrm{mod}(B') = 1/\mathrm{mod}(B)$.

Theorem 72. *Suppose $B = B(z_1, z_2, z_3, z_4) = B_1(z_1, w_2, w_3, z_4) \cup B_2(w_2, z_2, z_3, w_4)$ is a quadrilateral which is a union of two quadrilaterals $B_1 = B_1(z_1, w_2, w_3, z_4)$ and $B_2 = B_2(w_2, z_2, z_3, w_4)$. Then*

$$\mathrm{mod}(B) \geq \mathrm{mod}(B_1) + \mathrm{mod}(B_2).$$

The equality holds if and only if there is a conformal map $f : B \to R(0, a, a+i, i)$ such that $f(B_1) = R(0, a_1, a_1+i, i)$ and $f(B_2) = R(0, a_2, a_2+i, i)$, where $a = a_1 + a_2$.

Corollary 9. *Suppose B is a quadrilateral and $n-1$ curves connecting points on its horizontal sides cut it into n sub-quadrilaterals $B_1, \cdots, B_n$. Then*

$$\mathrm{mod}(B) \geq \sum_{i=1}^{n} \mathrm{mod}(B_i).$$

Definition 22. *Suppose Ω is a domain in the complex plane. Suppose $f : \Omega \to f(\Omega)$ is an orientation-preserving homeomorphism. We say f is a K-quasiconformal if*

$$\frac{1}{K}\mathrm{mod}(B) \leq \mathrm{mod}(f(B)) \leq K\mathrm{mod}(B)$$

for any quadrilateral $B \subset \Omega$.

Clearly, f^{-1} is also a K-quasiconformal homeomorphism if f is a K-quasiconformal. If f_1 is K_1-quasiconformal and f_2 is K_2-qausiconformal, then $f_1 \circ f_2$ is K_1K_2-quasiconformal.

Theorem 73. *Suppose $f : \Omega \to \Omega'$ is 1-quasiconformal. Then f is conformal.*

Proof. Let B be a quadrilateral in Ω. Then $B' = f(B)$ is a quadrilateral in Ω' and $a = \mathrm{mod}(B') = \mathrm{mod}(B)$. Let $g_1 : B \to R = R(0, a, a+i, i)$ and $g_2 : B' \to R = R(0, a, a+i, i)$ be two conformal map. Then $\tilde{f} = g_2 \circ f \circ g_1^{-1} : R \to R$ is 1-quasiconformal. For any $z \in R$, let l be the vertical line passing z. Then l cuts R into two rectangles R_1 and R_2 such that $a_1 = \mathrm{mod}(R_1)$ and $a_2 = \mathrm{mod}(R_2)$ and $a = a_1 + a_2$. Since $\tilde{f}$ is 1-quasiconformal, $\mathrm{mod}(\tilde{f}(R_1)) = a_1$ and $\mathrm{mod}(\tilde{f}(R_2)) = a_2$. This implies that

$$\mathrm{mod}(\tilde{R}) = \mathrm{mod}(\tilde{f}(R_1)) + \mathrm{mod}(\tilde{f}(R_2)).$$

This implies that $\tilde{f}(l) = l$. Similarly , if l is a horizontal line in R, we have that $\tilde{f}(l) = l$. This implies that $\tilde{f}(z) = z$ and $f(z) = g_2^{-1} \circ g_1(z)$ is conformal. □

Instead to using quadrilaterals in the definition of a quasiconformal mapping, one can also use topological annuli. A topological annulus A is a (closed) double connected domain whose boundary consists of two Jordan curves. Then we have a conformal mapping mapping $f : A \to A_r = \{z \in \mathbb{C} \mid 1 \leq |z| \leq r\}$ where r is uniquely determinant. The modulus of A is $\mathrm{mod}(A) = (1/2\pi)\log r$. As Exercise 3, $\mathrm{mod}(A) = \lambda(\mathcal{F})$, the extremal length of the the family of piecewise C^1 curves connecting two points in two boundary Jordan curves.

Definition 23. *Suppose Ω is a domain in the complex plane. Suppose $f : \Omega \to f(\Omega)$ is an orientation-preserving homeomorphism. We say f is a K-quasiconformal if*

$$\frac{1}{K}\mathrm{mod}(A) \leq \mathrm{mod}(f(A)) \leq K\mathrm{mod}(A)$$

for any topological annulus $A \subset \Omega$.

Using the extremal length technique, we have that

Corollary 10. *Suppose A is a topological annulus and $A_1, \cdots, A_n \subset A$ are n pairwise disjoint annulus separate two boundary components of A. Then*

$$\mathrm{mod}(A) \geq \sum_{i=1}^{n} \mathrm{mod}(A_i).$$

Suppose $f : \Omega \to f(\Omega)$ is a homeomorphism. We say it is locally K-quasiconformal if for every point $z \in \Omega$, there is a neighborhood U of z such that $f : U \to f(U)$ is K-quasiconformal.

Theorem 74. *The homeomorphism $f : \Omega \to \Omega'$ is K-quasiconformal if and only if it is locally K-quasiconformal.*

Proof. We need only to prove the "if" part. Let $B \subset \Omega$ be a quadrilateral. Since B is compact, we can cut it into small quadrilaterals Q_{ij} such that each Q_{ij} falls in the definition neighborhood of local quasiconformalilty and such that

$$\mathrm{mod}(Q) = \sum_i \mathrm{mod}(Q_i)$$

where $Q_i = \sum_j Q_{ij}$, and

$$\frac{1}{\mathrm{mod}(Q_j)} = \sum_i \frac{1}{\mathrm{mod}(Q_{ij})}$$

Thus

$$\frac{1}{K}\mathrm{mod}(Q_{ij}) \leq \mathrm{mod}(f(Q_{ij})) \leq K\mathrm{mod}(Q_{ij}).$$

Using the property that

$$\mathrm{mod}(f(Q)) \geq \sum_i \mathrm{mod}(f(Q_i))$$

and

$$\frac{1}{\mathrm{mod}(f(Q_i))} \leq \sum_j \frac{1}{\mathrm{mod}(f(Q_{ij}))},$$

we get

$$\frac{1}{K}\mathrm{mod}(Q) \leq \mathrm{mod}(f(Q)) \leq K\mathrm{mod}(Q).$$

∎

Suppose A_1 and A_2 are two doubly connected domains. Suppose $f : A_1 \to A_2$ is a holomorphic degree two covering map. Then we have

$$\mathrm{mod}(A_2) = 2\mathrm{mod}(A_1).$$

The reason is that consider conformal maps $g_1 : A_1 \to A_{r_1} = \{z \in \mathbb{C} \mid 1 \leq |z| \leq r_1\}$ and $g_2 : A_2 \to A_{r_2} = \{z \in \mathbb{C} \mid 1 \leq |z| \leq r_2\}$. Then $\tilde{f} = g_2 \circ f \circ g_1^{-1} : A_{r_1} \to A_{r_2}$ is a holomorphic degree covering map preserving inner circles and outer circles. Now by the Schwarz reflection theorem, $\tilde{f}$ can be extended to a holomorphic degree two map from $\mathbb{C}$ into itself and fixes 0 and preserves the unit circle. This implies that $\tilde{f}(z) = \lambda z^2$ with $|\lambda| = 1$. Thus we have that $r_2 = r_1^2$.

Now consider the hyperbolic disk $(\Delta, d(\cdot, \cdot))$. For any two points $z, w \in \Delta$, let $x = d(z, w)$ be its hyperbolic distance. Now consider the disk $\Delta_{r(x)} = \{z \in \mathbb{C} \mid |z| < r(x)\}$ with the hyperbolic distance $d_{r(x)}(\cdot, \cdot)$ such that $d_{r(x)}(0, 1) = d(z, w)$. We have a conformal map from $\Delta_{r(x)} \setminus [0, 1]$ onto $\Delta \setminus l_{zw}$ where l_{zw} is the hyperbolic geodesic connecting z and w. Here $r(x)$ is uniquely determined by x and real analytic depending on x. It is also strictly decreasing with $r(x) \to \infty$ as $x \to 0^+$ and $r(x) \to 1^+$ as $x \to \infty$. Now consider the doubly connected domain $D_{r(x)} = \Delta_{r(x)} \setminus [0, 1]$. The modulus $m(x) = \mathrm{mod}(D_r)$ is also an analytic strictly increasing function with $m(x) \to \infty$ as $x \to \infty$ and $m(x) \to 0$ as $x \to 1^+$. Let $A_{e^{2\pi m(x)}} = \{z \in \mathbb{C} \mid 1 < |z| < e^{2\pi m(x)}\}$ be the round annulus. Then we have a conformal map $g : A_{e^{2\pi m(x)}} \to D_{r(x)}$ which can be extended as a continuous function on the boundary. The boundary mapping maps the unit circle into the interval $[0, 1]$. The Schwarz reflection theorem implies that we have a degree two branched holomorphic cover $\tilde{g} : \tilde{A} = \{z \in \mathbb{C} \mid -e^{2\pi m(x)} \leq |z| \leq e^{2\pi m(x)}\} \to (\Delta, \{z, w\})$ with two branched points -1 and 1 and two branched values z and w. The modulus $\mathrm{mod}(\tilde{A}) = 2m(x)$.

Theorem 75. *Suppose $f : \Delta \to \Delta$ is a K-quasiconformal homeomorphism. Then we have that*

$$\frac{1}{K} m(d(z, w)) \leq m(d(f(z), f(w)) \leq K m(d(z, w))$$

for any $z, w \in \Delta$.

Proof. Let $\tilde{g}_1 : \tilde{A}_1 \setminus \{-1, 1\} \to \Delta \setminus \{z, w\}$ and $\tilde{g}_2 : \tilde{A}_2 \setminus \{-1, 1\} \to \Delta \setminus \{f(z), f(w)\}$ be two degree two holomorphic covers constructed above. Then the K-quasiconformal map $f : \Delta \setminus \{z, w\} \to \Delta \setminus \{f(z), f(w)\}$ can be lift to a K-quasiconformal homeomorphism $\tilde{f} : \tilde{A}_1 \to \tilde{A}_2$. This implies that

$$\frac{1}{K}\mathrm{mod}(\tilde{A}_1) \le \mathrm{mod}(\tilde{A}_2) \le K\mathrm{mod}(\tilde{A}_1).$$

Thus

$$\frac{1}{K}m(d(z,w)) \le m(d(f(z), f(w)) \le Km(d(z,w)).$$

□

We have the following important corollaries.

Corollary 11. *Suppose $f : \Delta \to \Delta$ is a K-quasiconformal homeomorphism with $f(0) = 0$. For any $0 < r < 1$, there is a constant $C(r) > 0$ such that*

$$|f(z) - f(w)| \le C(r)|z - w|^{1/K}$$

for any $|z|, |w| \le r$.

Remark 14. *Furthermore, by using the Grözsch extremal modulus, we have a more strong result:*

$$|f(z) - f(w)| \le 16|z - w|^{1/K}$$

for any $|z|, |w| < 1$. Refer to

- *Ahlfors, L. V. Lectures on Quasiconformal Mapping, University Lecture Series. 38, Amer. Math. Soc., 2006.*

Corollary 12. *Any K-quasiconformal homoemorphism $f : \Delta \to \Delta$ can be extended to a homeomorphism $\overline{f} : \overline{\Delta} \to \overline{\Delta}$.*

Corollary 13. *The family $\mathcal{K}$ of all K-quasiconformal homeomorphisms $f : \Delta \to \Delta$ with $f(0) = 0$ is a normal family, that is, any sequence in $\mathcal{K}$ has a convergent subsequence on any compact subset of Δ and the limiting function is again in $\mathcal{K}$.*

11.4. Analytic definition and the measurable Riemann mapping theorem.

Suppose I is an interval in the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is absolutely continuous on I if for every positive number $\epsilon > 0$, there is a positive number $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I satisfies

$$\sum_k |y_k - x_k| < \delta,$$

then

$$\sum_k |f(y_k) - f(x_k)| < \epsilon.$$

From the real analysis, the statement that f is absolutely continuous on $I = [a, b]$ is equivalent to the following statements,

i. f has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(x)dx \quad \forall x \in I;$$

ii. there exists a Lebesgue integrable function g on I such that

$$f(x) = f(a) + \int_a^x g(x)dx \quad \forall x \in I.$$

Thus we have the following properties for an absolutely continuous function $f : I = [a, b] \to \mathbb{R}$:

1) f is of bounded variation;
2) f maps every zero Lebesgue measure subset E of I to a zero Lebesgue measure set $f(E)$.

Therefore, we have another equivalent statement: $f : I \to \mathbb{R}$ is absolutely continuous if and only if f is continuous, is of bounded variation, and maps every zero Lebesgue measure subset E of I to a zero Lebesgue measure set $f(E)$.

Definition 24. *Suppose U is a domain in the complex plane. A continuous map $f = u + iv : U \to \mathbb{C}$ is called absolutely continuous on lines (ACL) if u and v when restricted on almost all horizontal lines and almost all vertical lines are absolutely continuous.*

If f is absolutely continuous on lines, then it has partial derivatives f_x and f_y for almost all $z = x + iy \in U$. From Egoroff's theorem, f is differentiable at almost all $z \in U$. Thus we have $f_z(z)$ and $f_{\overline{z}}(z)$ for almost all $z \in U$ such that

$$f(w) - f(z) = f_z(z)(w - z) + f_{\overline{z}}(z)(\overline{w} - \overline{z}) + o(|w - z|).$$

We then define a measurable function

$$\mu(z) = \frac{f_{\overline{z}}(z)}{f_z(z)}, \quad z \in U.$$

Moreover. f_z and $f_{\overline{z}}$ are partial derivatives in the distribution sense, that is,

$$\int\int f_z(z)h(z)dzd\overline{z} = -\int\int f(z)h_z(z)dzd\overline{z}$$

and
$$\int\int f_{\overline{z}}(z)h(z)dzd\overline{z} = -\int\int f(z)h_{\overline{z}}(z)dzd\overline{z}$$
hold for all C^∞ functions with compact support.

Definition 25. *Suppose $f : U \to V$ is an orientation-preserving homeomorphism from the domain U to the domain V. The we say f is K-quasiconformal for some $K > 1$ if*

a) *f is ACL;*
b) $\|\mu\|_\infty \leq k = (K-1)/(K+1)$.

Suppose f is K-qiasiconformal in Definition 25. Then for any quadrilateral $B = B(z_1, z_2, z_3, z_4) \subset U$, we have a conformal map ϕ from B onto the rectangle $R = R(0, a, a+bi, bi)$ and a conformal map ψ from $f(B) = B'(f(z_1), f(z_2), f(z_3), f(z_4))$ to the rectangle $R = R(0, a', a'+b'i, b'i)$. Then $\tilde{f} = \phi \circ f \circ \psi^{-1} : R \to R'$ is a homeomorphism, ACL, maps vertices to vertices. Following Grötzsch argument in Lecture 10.1, we can show that
$$(a'/b') : (a/b) \leq K.$$
This says $m(B') \leq Km(B)$. Thus f is K-quasiconformal in Definition 23. If f is a quasiconfornmal mapping in Definition 23, then it is ACL. So it has partial derivatives f_z and $f_{\overline{z}}$ a.e. and differentiable a.e. This implies $b)$ in Definition 25. For more detailed arguments, refer to

- L. V. Ahlfors, Quasiconformol Mappings. Van Nostrand, 1966
- O. Lehto and K. Virtanen, Quasiconformal Mappings in the Plane. Springer-Verlag, 1973.

Thus Definition 25 and Definition 23 are equivalent.

Checking f is ACL in Definition 23 is important even if f has partial derivatives almost everywhere. For example, let Λ be a Cantor set as we constructed in Lecture 3 as the non-escaping set set for a dynamical system $f : I_0 \cup I_1 \to I$ with zero Lebesgue measure. Then we have $[0,1] = G \cup \cup_{w_k} G_{w_k} \cup \Lambda$. Define $f(x) = 1/2$ on G and
$$\beta(x) = \frac{1}{2^2} + \sum_{j=0}^{k} \frac{i_j}{2^{k+1}} \quad x \in \overline{G}_{w_k}.$$
Then $\beta : [0,1] \to [0,1]$ is a onto map with $\beta'(x) = 0$ for all $x \in \cup_{w_k} G_{w_k}$ but is not absolutely continuous and maps Λ to $[0,1]$. It is called a devil staircase. Now consider the homeomorphism
$$f(z) = z + \beta(\frac{z+\overline{z}}{2})i$$

on the unit square $D = [0,1] \times [0,1]$. It is clear f has partial derivatives

$$f_z = 1$$

and

$$f_{\overline{z}} = 0$$

for almost all $z \in D$. But it is not quasiconformal because, otherwise, it must be conformal (refer to Theoren 73), but clearly it is not conformal. The reason is that it is not ACL.

From Definition 25, for a K-quasiconformal map f on U, we define a measurable function μ_f on U such that $k = \|\mu_f\|_\infty < 1$, where $K = (1+k)/(1-k)$. The measurable function μ_f is called a Beltrami coefficient. In general, let

$$M(\mathbb{C}) = \{\mu \in L^\infty \mid \|\mu\|_\infty < 1\}$$

is the unit open ball of L^∞ functions on $\mathbb{C}$. Consider the Beltrami equation,

$$w_{\overline{z}} = \mu w_z, \quad a.e. \tag{10}$$

where $\mu \in M(\mathbb{C})$ is called a Beltrami coefficient. If w is a quasiconformal homeomorphism, then it is a solution of the Betrami equation with the Betrami coefficient $\mu_w = w_{\overline{z}}/w_z$. A quasiconformal homeomorphism is called normalized if it fixes $0, 1, \infty$. The following theorem plays an important role in Teichmüller theory and theory of Riemann surfaces.

Theorem 76 (Measurable Riemann Mapping Theorem)**.** *For any $\mu \in M(\mathbb{C})$, the Beltrami equation has a quasiconformal solution. And the normalized solution, denoted as w^μ, is unique and holomorphically depends on μ.*

To prove this theorem, we first need to study two operators: Let $\zeta = \xi + \eta i$,

$$T\mu(z) = -\frac{1}{\pi}\int\int_{\mathbb{C}} \frac{\mu(\zeta)}{(\zeta - z)^2} d\xi d\eta = \lim_{\epsilon \to 0^+} -\frac{1}{\pi}\int\int_{|z-\zeta|>\epsilon} \frac{\mu(\zeta)}{(\zeta - z)^2} d\xi d\eta$$

where $\mu \in C_0^2$ (C^2 with compact support in $\mathbb{C}$) and

$$P\mu(z) = -\frac{1}{\pi}\int\int_{\mathbb{C}} \mu(\zeta)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big) d\xi d\eta$$

where $\mu \in L^p(\mathbb{C})$ for $p > 2$.

Lemma 17. *$P\mu$ is continuous and satisfies the uniform Hölder condition with exponent $1-2/p$, that is,*

$$|P\mu(z) - P\mu(w)| \leq K_p\|\mu\|_p|z-w|^{1-2/p}, \quad z, w \in \mathbb{C}$$

where $K_p > 0$ is a constant.

Lemma 18. *For any $\mu \in C_0^2$, $T\mu$ exists and is of class C^1. Furthermore,*

$$(P\mu)_{\overline{z}} = \mu \quad \textit{and} \quad (P\mu)_z = T\mu$$

and

$$\int\int_{\mathbb{C}} |T\mu(\zeta)|^2 d\xi d\eta = \int\int_{\mathbb{C}} |\mu(\zeta)|^2 d\xi d\eta.$$

In the above lemma, $T\mu$ is an isometry on $L^2(\mathbb{C}) \cap C_0^2$. Since C_0^2 is dense in L^2, T can be extended to an isometry on L^2. As we notice that $T\mu$ also acts on L^p for $p > 2$. We have the following important inequality.

Lemma 19 (Zygmund-Calderon)**.** *For any $\mu \in C_0^2$, we have*

$$\|T\mu\|_p \leq C_p\|\mu\|_p$$

for any $p > 1$ and $C_p \to 1$ as $p \to 2$.

This lemma implies that $T\mu$ can be extended as a bounded operator defined on $L^p(\mathbb{C})$ for $p > 1$. Thus we two operators $T\mu$ and $P\mu$ defined on $L^p(\mathbb{C})$ for $p > 2$ and generalize Lemma 18 as follows.

Lemma 20. *For any $\mu \in L^p(\mathbb{C})$, $p > 2$, we have*

$$(P\mu)_{\overline{z}} = \mu \quad \textit{and} \quad (P\mu)_z = T\mu$$

in the distribution sense.

Theorem 77 (Special Case)**.** *For any $\mu \in M(\mathbb{C})$ with compact support, the Beltrami equation (10) has a unique solution such that $f(0) = 0$ and $f_z - 1 \in L^p(\mathbb{C})$.*

Proof. Let us prove the uniqueness first. Suppose f is a solution in the theorem. Then we have $f_{\overline{z}} = \mu f_z$. Since μ has a compact support and $f_z - 1 \in L^p(\mathbb{C})$, $f_{\overline{z}} \in L^p(\mathbb{C})$. Then we have $Pf_{\overline{z}}$ such that $(Pf_{\overline{z}})_{\overline{z}} = f_{\overline{z}}$. Consider the function $F = f - Pf_{\overline{z}}$. We have $F_{\overline{z}} = 0$ a.e.. Therefore, F is analytic from Weyl's lemma. $F' = f_z - (Pf_{\overline{z}})_z = f_z - Tf_{\overline{z}}$. Since $Tf_{\overline{z}} \in L^p(\mathbb{C})$ and $f_z - 1 \in L^p(\mathbb{C})$, we get $F' - 1 \in L^p(\mathbb{C})$. This is possible only if $F' = 1$. Thus $F = z + a$ for some constant a. This implies that $f = z + a + Pf_{\overline{z}}$. But $f(0) = 0$ implies that $f = z + Pf_{\overline{z}}$.

Suppose g is another solution. Then we also have $g = z + Pg_{\overline{z}}$. This implies that

$$f - g = P(f_{\overline{z}} - g_{\overline{z}}) = P(\mu(f_z - g_z)).$$

This implies that

$$f_z = g_z = T(\mu(f_z - g_z)).$$

Lemma 19 implies that

$$\|f_z - g_z\|_p \leq kC_p\|f_z - g_z\|_p$$

where $k = \|\mu\|_\infty < 1$. Let $p \to 2$. Then $\|\mu\|_\infty C_p \to k$, this implies that $f_z = g_z$ and $f_{\overline{z}} = g_{\overline{z}}$ a.e. Thus both $f - g$ and $\overline{f} - \overline{g}$ are analytic. Thus $f - g = const$. This constant must be 0. We get $f = g$.

The proof of the uniqueness also suggests a proof of the existence. Consider the equation

$$h = T(\mu h) + T\mu. \tag{11}$$

We first solve this equation in $L^p(\mathbb{C})$ for a number $p > 2$ such that $\lambda = kC_p < 1$. Let $\mu_0 = \mu$, $\mu_n = T\mu_{n-1}$ for $n \geq 1$. Lemma 19 implies that

$$\|\mu_n\|_p \leq \lambda\|\mu_{n-1p} \leq \cdots \leq \lambda^n\|\mu_0\|_p.$$

Then we have a convergent series

$$h = \sum_{n=1}^{\infty} \mu_n \tag{12}$$

in $L^p(\mathbb{C})$. Then h is a solution of Equation (11). We claim

$$f = z + P(\mu(h+1))$$

is a solution of Equation (10). The reason is that first, $\mu(h+1) \in L^p$, so $P(\mu(h+1))$ is well-defined and continuous function on $\mathbb{C}$ (actually, a Hölder continuous function on $\mathbb{C}$). Secondly,

$$f_{\overline{z}} = \mu(h+1), \quad f_z = T(\mu(h+1)) + 1 = h + 1. \tag{13}$$

This implies that $f_z - 1 \in L^p(\mathbb{C})$. Clearly $f(0) = 0$. $\square$

The solution in the theorem is called a normalized solution of (10). Next we claim that the solution f is a homeomorphism. From this claim that f is a quasiconformal homeomorphism since it satisfies the equation (10).

Now let us prove the claim. First we prove it under assumption tha $\mu \in C_0^1$. Then we have μ_z. We would like first solve the equation

$$\lambda_{\overline{z}} = \mu\lambda_z + \mu_z. \tag{14}$$

Similar to find a solution of the Beltrami equation (10), we first find the solution

$$q = T(\mu q) + T\mu_z$$

by convergent power series

$$q = T\mu_z + T\mu T\mu_z + \ldots + T\mu T\mu T\mu_z + \ldots$$

in $L^p(\mathbb{C})$ by choosing $p > 2$ such that $kC_p < 1$. Set

$$\sigma = P(\mu q + \mu_z) + const$$

such that $\sigma \to 0$ as $z \to \infty$. Thus σ is continuous and

$$\sigma_{\overline{z}} = \mu q + \mu_z, \quad \sigma_z = T(\mu q + \mu_z) = q.$$

Define $\lambda = e^{\sigma}$. Then it is the solution of Equation (14). Let $p = \lambda$ and $q = \mu\lambda$. We have that $p_{\overline{z}} = q_z$ are continuous. Green's Theorem implies that

$$f(z) = \int_0^z p dz + q d\overline{z}$$

defines a C^2 function on $\mathbb{C}$. It is the unique solution of Equation (10). Since $\sigma \to 0$ as $z \to \infty$, $f_z = \lambda \to 1$ as $z \to \infty$. Thus $f(z) \to \infty$ as $z \to \infty$. The Jacobian

$$J(f)(z) = |f_z|^2 - |f_{\overline{z}}|^2 = (1 - |\mu|^2)e^{2\sigma} > 0, \ \ \forall z \in \mathbb{C}.$$

This implies that f is locally 1-1. This combining with $f(z) \to \infty$ as $z \to \infty$, we get that both f and f^{-1} are 1-1 and onto and continuous, therefore, $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism.

From Lemma 19 and (11), we have

$$\|h\|_p \leq kC_p\|h\|_p + C_p\|h\|_p.$$

So

$$\|h\|_p \leq \frac{C_p}{1 - kC_p}\|\mu\|_p. \tag{15}$$

From (6),

$$\|f_{\overline{z}}\|_p \leq \frac{1}{1 - kC_p}\|\mu\|_p.$$

Lemma 17 implies that

$$|f(z) - f(w)| \leq \frac{K_p}{1 - kC_p}\|\mu\|_p|z - w|^{1-2/p} + |z - w|. \tag{16}$$

Applying on f^{-1}, we get

$$|z - w| \leq \frac{K_p}{1 - kC_p}\|\mu\|_p|f(z) - f(w)|^{1-2/p} + |f(z) - f(w)|. \tag{17}$$

Now, for any $\mu \in L^\infty(\mathbb{C})$ with compact support, let $\mu_n \in C_0^1$ such that $\mu_n \to \mu$ a.e. and $\|\mu_n\|_\infty \leq k$. Then the normalized solutions f_n are K-quasiconformal homeomoprhisms for $K = (1+k)/(1-k)$ for all $n > 0$. Then (17) implies that f is 1-1 since $f_n \to f$ and $\|\mu_n\|_p \to \|\mu\|_p$ as $n \to \infty$. Similarly, f^{-1} is also 1-1. Both f and f^{-1} are continuous. So f is K-quasiconformal homeomorphism.

Proof of Theorem 76. If μ has a compact support, we only need to normalize f by using a Möbis transformation. The unique solution fixing 0, 1, ∞ is denoted as f^μ.

If μ has no compact support, we decompose $\mu = \mu_1 + \mu_2$ such that μ_1 has a compact support and $\mu_2 = 0$ in a neighborhood of 0. Then we f^{μ_1}. Define

$$\mu_3(z) = \mu_2(1/z)\frac{z^2}{\overline{z}^2}.$$

Then μ_3 has a compact support. So we have f^{μ_3}. Then $1/f^{\mu_3}(1/z)$ defines a homeomorphism of $\mathbb{C}$ which fixes 0, 1, ∞. We call this homeomorphism as $f^{\mu_2} = 1/f^{\mu_3}(1/z)$. It is the unique solution of the Beltrami equation (10) with the Beltrami coefficient μ_2. Now define

$$\mu_4 = \Big(\big(\frac{\mu - \mu_2}{1 - \mu\overline{\mu}_2} \big) \big(\frac{f_z^{\mu_2}}{\overline{f_z^{\mu_2}}} \big) \Big) \circ (f^{\mu_2})^{-1}.$$

Then μ_4 has a compact support. So we have f^{μ_4}. Now define

$$f^\mu = f^{\mu_4} \circ f^{\mu_2}.$$

We have that f^μ is the unique solution of Beltrami equation (10) with the Beltrami coefficient μ.

From Equation (12), if we consider $t\mu$ for $\|\mu\|_\infty = 1$ and $t \in \Delta$, h is convergent power series of t. Thus $f^{t\mu_2}$ and $f^{t\mu_4}$ are holomorphically dependent on $t \in \Delta$. This implies $f^{t\mu}$ is also holomorphically dependent on $t \in \Delta$. $\square$

Lecture 12. Local dynamics of a holomorphic map, irrational neutral case.

Suppose f is a holomorphic map defined on a domain U containing 0. Suppose $0 \in U$ is an irrational neutral fixed point, that is, $f(0) = 0$ and$f'(0) = e^{2\pi i\theta}$ for $\theta \in [0,1] \setminus \mathbb{Q}$. Let $\lambda = e^{2\pi i\theta}$. Consider the Taylor expansion

$$f(w) = \lambda w + a_2 w^2 + a_3 w^3 + \cdots .$$

If all $a_i = 0$, then $f(w) = \lambda w$ is a linear map. In general, we would like to know that it is linear under an appropriate coordinate. We still use

$$\Delta_r = \{w \in \mathbb{C} \mid |w| < r\}$$

to denote the disk centered 0 with radius $r > 0$. If there is a conformal map $\phi : U' \subset U \to \Delta_r$ such that $\phi \circ f = \lambda\phi$ on U', then we call f is linearizable. By considering $h = \phi^{-1}$, this is equivalent to that the Schröder equation

$$f(h(w)) = h(\lambda w), \quad h(0) = 0, h'(0) = 1,$$

has a solution as we studied before.

A real number $\theta \in [0,1] \setminus \mathbb{Q}$ is called Diophantine if there are constants $C > 0$ and $0 < \mu < \infty$ such that

$$|\theta - \frac{p}{q}| \geq \frac{C}{q^\mu}$$

for all p and $q \neq 0$. This condition is equivalent to

$$|\lambda^n - 1| \geq Cn^{1-\mu}$$

for all $n \geq 1$. For a fixed $\mu > 2$, $|\theta - \frac{p}{q}| \geq C/q^\mu$ holds for almost all real number $\theta \in [0,1] \setminus \mathbb{Q}$. Indeed if E is the set of $\theta \in [0,1] \setminus \mathbb{Q}$ such that $|\theta - p/q| < q^{-\mu}$ infinitely many often, then

$$m(E) \leq \sum_{q=n}^{\infty} 2q^{-\mu+1} = O(n^{2-\mu}) \to 0.$$

In particular, almost all real numbers in $[0,1] \setminus \mathbb{Q}$ are Diophantine. A positive answer for the linearization problem is given in 1942 by Siegel by using the Diophantine condition.

Theorem 78. *If θ is Diophantine and if $f(w) = \lambda w + \cdots$ has a fixed point 0 with multiplier $\lambda = e^{2\pi i\theta}$, then there is a solution to the Schröder equation, that is, f can be conjugated near 0 to the linear map $w \mapsto \lambda w$.*

Proof. Let $C > 0$ and $\mu > 0$ be two constants such that $|\lambda^n - 1| \geq Cn^{1-\mu}$ for all $n \geq 1$. We want to find a conformal solution h defined on a disk Δ_r such that $h(\lambda w) = f(h(w))$. If we define $f(w) = \lambda w + \tilde{f}(w)$ and $h(w) = w + \tilde{h}(w)$. Then the equation can be written as

$$\tilde{h}(\lambda w) - \lambda \tilde{h}(w) = \tilde{f}(h(w)), \quad w \in \Delta_r.$$

Instead to solve $\tilde{h}$ directly, we consider a coordinate change $\psi(w) = w + \tilde{\psi}(w)$ with $\tilde{\psi}(w) = O(w^2)$. Then

$$\psi^{-1} \circ f \circ \psi(w) = \lambda w + \tilde{g}(w).$$

If $\tilde{g} \equiv 0$, then $h = \psi$ is a solution for the Schröder equation. If $\tilde{g} \neq 0$, we want $|\tilde{g}(w)| < |\tilde{f}(w)|$. Suppose

$$\tilde{f}(w) = \sum_{j=2}^{\infty} a_j w^j, \quad w \in \Delta_r, \;\; r < 1.$$

Let

$$\tilde{\psi}(w) = \sum_{j=2}^{\infty} b_j w^j.$$

Solving the equation

$$\tilde{\psi}(\lambda w) - \lambda \tilde{\psi}(w) = \tilde{f}(w),$$

we get $b_j = a_j/(\lambda^j - \lambda)$. By the Diophantine condition, the series

$$\tilde{\psi}(w) = \sum_{j=2}^{\infty} \frac{a_j}{\lambda^j - \lambda} w^j$$

converges and defines an analytic function on Δ_r.

For a $\delta > 0$, let $r > 0$ small so that $|\tilde{f}'(w)| < \delta$ for $w \in \Delta_r$. Now let us estimate $\tilde{g}$ in

$$g(w) = \psi^{-1} \circ f \circ \psi = \lambda w + \tilde{g}(w).$$

Since

$$f^{(j)}(0) = \frac{(j-1)!}{2\pi i} \int_{\partial \Delta_r} \frac{\tilde{f}'(\xi)}{\xi^j} d\xi,$$

$$|a_j| = |f^{(j)}(0)/j!| \leq \frac{\delta}{j r^{j-1}}.$$

Let $\Delta_{r(1-\eta)} = \{w \in \mathbb{C} \mid |w| < r(1-\eta)\}$, where $0 < \eta < 1/5$. For $w \in \Delta_{r(1-\eta)}$,

$$\begin{aligned}
|\tilde{\psi}'(w)| &\leq \sum_{j=2}^{\infty} \frac{ja_j}{|\lambda^{j-1}-1|} r^{j-1}(1-\eta)^{j-1} \\
&\leq C\delta \sum_{j=2}^{\infty} j^{\mu-1}(1-\eta)^{j-1} \\
&\leq C\delta \sum_{j=2}^{\infty} j^{[\mu]}(1-\eta)^{j-1} \\
&\leq C_0\delta \sum_{j=2}^{\infty} j(j-1)\cdots(j-[\mu]+1)(1-\eta)^{j-1} = \frac{C_0\delta}{\eta^{[\mu]+1}}.
\end{aligned}$$

Take $0 < \delta < \eta$ small, we also assume $C_0\delta < \eta^{[\mu]+2}$. Then

$$|\tilde{\psi}'(w)| < \eta, \qquad w \in \Delta_{r(1-\eta)}.$$

So we have

$$\psi(\Delta_{r(1-4\eta)}) \subset \Delta_{r(1-3\eta)}.$$

Furthermore, ψ takes every value in $\Delta_{r(1-2\eta)}$ precisely once because $|\psi(w)| \geq r(1-2\eta)$ for $w \in \partial\Delta_{r(1-\eta)}$, the argument principle, and $\psi(w) = 0$ has only one solution 0 in $D_{r(1-\eta)}$. Now consider $g = \psi^{-1} \circ f \circ \psi$ and $U_i = \Delta_{r(1-i\eta)}$. Since $\psi(U_4) \subset U_3$ and $\delta < \eta$, $f(U_3) \subset U_2$. Moreover, since $\psi^{-1}(U_2) \subset U_3$, $g(U_4) \subset U_1$.

Let us further estimate $\tilde{g}$ on U_4. Since

$$\tilde{g}(w) + \tilde{\psi}(\lambda w + \tilde{g}(w)) = \lambda\tilde{\psi}(w) + \tilde{f}(w + \tilde{\psi}(w)), \quad w \in U_4,$$

$$\tilde{g}(w) = \tilde{\psi}(\lambda w) - \tilde{\psi}(\lambda w + \tilde{g}(w)) + \tilde{f}(w + \tilde{\psi}(w)) - \tilde{f}(w), \quad w \in U_4.$$

Now we conclude our discussion above. Take $f(w) = \lambda w + \tilde{f}(w)$ such that $|\tilde{f}'| \leq \delta < \eta < 1/5$ on $U_0 = \Delta_r$ and such that $C_0\delta < \eta^{[\mu]+2}$. Then we can find a conjugacy $\psi(w) = w + \tilde{\psi}(w)$ such that for

$$g(w) = \psi^{-1} \circ f \circ \psi(w) = \lambda w + \tilde{g}(w),$$

$$|\tilde{g}'(w)| \leq C_0\delta^2 r\eta^{-([\mu]+2)}(1-\eta)^{-1}, \quad w \in U_4.$$

Let us label $\delta_0 = \delta$ and $\eta_0 = \eta$. Define

$$r_{n+1} = r_n(1-5\eta_n), \quad \eta_{n+1} = \frac{\eta_n}{2}, \quad \delta_{n+1} = C_0\delta_n^2\eta_n^{-([\mu]+2)}2^{-([\mu]+2)}.$$

The require condition $C_0\delta_n \leq \eta_n^{[\mu]+2}$ is now easy to verify by the induction because if it holds for n, then

$$C_0\delta_{n+1} = C_0^2\delta_n^2\eta_n^{-([\mu]+2)}2^{-([\mu]+2)} \leq \eta_n^{2([\mu]+2)}\eta_n^{-([\mu]+2)}2^{-([\mu]+2)} = \eta_{n+1}^{[\mu]+2}.$$

So it also holds for $n+1$. Thus we can construct inductively sequences of maps $\{\psi_n\}$ and $\{g_n\}$ with $g_0 = f$ and

$$g_n = \psi_n^{-1} \circ g_{n-1} \circ \psi_n = \psi_n^{-1} \circ \psi_{n-1}^{-1} \circ \cdots \circ \psi_1^{-1} \circ f \circ \psi_1 \circ \cdots \circ \psi_n.$$

Let $\tau = r \prod_{i=1}^{\infty}(1 - 5\eta_n) > 0$. Then we have that

$$\tilde{g}_n'(w) \leq \delta_n r_n/(1 - \eta_n), \quad w \in \Delta_\tau = \{w \in \mathbb{C}; |w| < \tau\}, \;\; n > 0.$$

Thus $\psi_1 \circ \cdots \circ \psi_n$ and g_n tend to a conformal map h and the linear map λw, respectively, as n goes to infinity. So h conjugates f to λw. $\square$

The above Theorems view the linearization problem from the combinatorical point of views. There are many new developments in this direction. For example, a more precise combinatorical condition such that f is linearizable is so called the Brjuno condition as follows. Suppose $\theta \in [0, 1] \setminus \mathbb{Q}$. Let

$$\theta = [a_1, a_2, \cdots,] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

be the continuous fractional expansion of θ. Let

$$\frac{p_n}{q_n} = [a_1, a_2, \cdots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \frac{1}{a_n}}}.$$

Then θ is called a Brjuno number if

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty.$$

The sufficiency of this condition was proved by Brjuno in 1965 for $f(w) = e^{2\pi i\theta}w + w^2 + \cdots$ and necessity condition was established by Yoccoz in 1988.

We will discuss the following topics in the future semesters.

Lecture 13. Holomorphic motions and applications

13.1. λ-Lemma, from Mane-Sad-Sullivan's paper

13.2. Bers-Royden theorem, from Bers-Royden's paper.

13.3. Slodkowski theorem, Chirka's proof from Gardiner-Jiang-Wang's paper.

13.4. Proof of the lifting theorem, an equivalent statement to Slodkowski's theorem, from Jiang-Mitra-Wang's paper

13.5. Canonical holomorphic replacement, a new proof of Slodkowski's theorem, from Gardiner-Jiang's paper

13.6. Infinitesimal holomorphic motions, from Gardiner-Jiang-Wang's paper

13.7. Structural stability theorem, from Mane-Sad-Sullivan's paper

Lecture 14. Gibbs Theory

14.1. Expanding and mixing dynamical systems.

14.2. Maximal eigenvalue of a Ruelle-Perron-Frobenius operators.

14.3. Smooth invariant measures for expanding dynamical systems.

14.4. Chains of Markovian projections.

14.5. Gibbs distributions.

14.6. Hilbert metric, second eigenvalue, and convergence speed.

14.7. Spectra of Ruelle-Perron-Frobenius operators.

14.8. Essential spectrum radius of a Ruelle-Perron-Frobenius operator.

Lecture 15. Characterization of rational maps

15.1. Branched coverings.

15.2. Orbifold structures.

15.3. Rational maps

15.4. The Teichmüller space of a closed subset

15.5. Pull-back operator.

15.6. Multi-curves

15.7. Characterization of post-critically finite rational maps

15.8. Geometrically finite branched coverings

15.9. Characterization of sub-hyperbolic rational maps

15.10. Characterization of some entire and meromorphic functions