Renormalization and Geometry in One-Dimensional and Complex Dynamics

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Mathematics Subject Classification (1991): 58F03, 58F23, 30D05

To my Mother and Father

To Bin and Jeffrey

Preface

This monograph summarizes my research in dynamical systems during the past eight years. Included too are many facts, techniques, and results which I have learned from others and which have greatly enhanced my own work.

In September 1985, I arrived in the United States to pursue a doctoral degree in mathematics. One and a half years later, I wrote to Dennis Sullivan and asked if I could study under his supervision. I became a Ph.D. student of Sullivan and so began this research. During my years as his student, Sullivan gave classes and ran seminars on Tuesday and Thursday at the Graduate Center of the City University of New York, where he held an Einstein Chair. These sessions sometimes lasted all day.

My first problem in this area of research was suggested by Sullivan in one of his classes. The problem was first, to understand the asymptotic geometry of Cantor sets generated by a family of dynamical systems involving a singular point and second, to investigate the geometric property of conjugacy between two such dynamical systems. This work is included in Chapter Two of the present monograph as an application of the Koebe distortion principle. One recognized program is to "fill in" the dictionary between the theory of one-dimensional dynamical systems and the theory of Kleinian groups. Seeking to advance this program and to generalize my first research in this area, I studied the space of geometrically finite one-dimensional maps and the classification of these maps up to conjugacy by quasisymmetric homeomorphisms and up to conjugacy by diffeomorphisms. This work is described in Chapter Three. Part of it comes from my Ph.D. thesis. Frederick Gardiner, Charles Tresser, and Richard Sacksteder gave me their help during this study and Sullivan provided his own insightful suggestions.

In 1987, Peter Veerman gave me a paper of Robert MacKay concerning Denjoy's theorem for circle diffeomorphisms. In this paper, MacKay applies the renormalization method to an old theorem. I became interested in this approach. It is described in Chapter One as an introduction to renormalization theory and as an application of the Denjoy distortion principle. That same year, Welington de Melo and Sebastian van Strien visited the Einstein Chair and presented their results on one-dimensional dynamical systems. Grzegorz Świątek also paid a visit and presented his results on critical circle mappings. Interestingly, they had independently developed a technique to estimate the distortion of a one-dimensional map having a critical point. This technique was later generalized to a larger class of one-dimensional maps by Sullivan; it is called the Koebe distortion principle because of its similarity to Koebe's distortion theorem in one complex variable discovered some eighty years ago. Chapter Two contains several versions of the distortion principle.

A universal rule governs the transition from simple motion to chaos in a one-parameter family of dynamical systems with a unique quadratic critical point. Mitchell Feigenbaum discovered this in the 1970s. The rule can be explained by means of a family of one-dimensional dynamical systems like those generated by quadratic polynomials. Feigenbaum calculated period doubling bifurcations for such a family and showed that the limit of these period doubling bifurcations is a chaotic dynamical system in the family. Furthermore, the appearance of the chaotic dynamical system follows a universal pattern which is described by the so-called Feigenbaum universal number. Oscar Lanford III gave the first proof of this discovery with some computer help. For the chaotic dynamical system, the interesting object is its attractor. The attractor is uncountable, perfect, and totally disconnected: a Cantor set. Feigenbaum, and independently, Pierre Coullet and Charles Tresser, discovered in the 1970s that the geometry of this Cantor set is universal, meaning that it does not depend on the specific family being studied. This discovery is similar to Mostow's rigidity theorem, which says that in the class of closed hyperbolic three-manifolds, topology determines geometry. During my years as a Ph.D. student, some work of Sullivan led to an important mathematical understanding of this discovery. Chapter Four contains part of this work based on my class notes.

During my time as his Ph.D. student, Sullivan showed me how to deform a Feigenbaum-like map. I began to think about this topic and also to study the spectrum of the period doubling operator. Meanwhile, Takehiko Morita visited the Einstein Chair. I told him what I was working on and he showed me a general strategy to study the spectrum of a transfer operator in thermodynamical formalism. I applied this strategy to the study of the tangent map of the period doubling operator by connecting it with a transfer operator. This led eventually to a conceptual proof of the existence of the Feigenbaum universal number in a joint paper with Morita and Sullivan. This is the origin of Chapter Six. During this study, conversations with David Ruelle and Henri Epstein helped me to better understand the spectrum of the period doubling operator and other related topics. In the summer of 1993, Viviane Baladi lectured on thermodynamical formalism at a workshop held in Hillerød, Denmark. After her lectures, I asked her about generalizing some of the results in her lectures to the Zygmund continuous vector space. She showed me some calculations that suggested this possibility. In the summer of 1994, I visited the Forschungsinstitut für Mathematik at the Eidgenössische Technische Hochschule in Zürich to work with Baladi and Lanford on this problem. The fruit of this work is described in a paper written by Baladi, Lanford, and myself. A special case of our result is presented in Chapter Six for the purpose of studying the spectrum of the renormalization operator.

After completing my Ph.D. study in May 1990, I went to the Institute for Mathematical Sciences at Stony Brook in September 1990. There I was influenced by John Milnor's investigations in complex dynamics; I began to work on some problems in this field.

Yakov Sinai constructed Markov partitions for Anosov dynamical systems in the 1960s. Rufus Bowen generalized this method to Axiom A dynamical systems. This method became very important in the study of hyperbolic dynamical systems. During the academic year of 1991, Mitsuhiro Shishikura visited the Institute for Mathematical Sciences at Stony Brook and presented his work in complex dynamics. In his lectures, he introduced me to the result of Jean-Christophe Yoccoz on the local connectivity of the Julia set of a non-renormalizable quadratic polynomial and to the technique called Yoccoz puzzles. This technique was first used by Bodil Branner and John Hubbard in their study of certain cubic polynomials and was successfully used by Yoccoz in his study of non-renormalizable quadratic polynomials. The technique is a little different from that of the Markov partitions but is motivated by the same philosophy. By learning this technique, I was able to apply it, along with my knowledge of infinitely renormalizable maps, to the study of infinitely renormalizable quadratic polynomials. I proved that some conditions on an infinitely renormalizable quadratic polynomial are sufficient to ensure that its Julia set is locally connected. This is described in Chapter Five.

The first time I applied Yoccoz puzzles in my research was in the study of bounded and bounded nearby geometry of certain infinitely renormalizable folding maps and in the quasisymmetric classification of these maps. In this study, I combined the technique of Yoccoz puzzles with Markov partitions and with my previous work on geometrically finite one-dimensional maps. This research is described in Chapter Four.

After completing my contract with the State University of New York at Stony Brook, I started work, in September 1992, at Queens College of the City University of New York. I began to regularly attend Sullivan's seminars at the Graduate Center of the City University of New York. Jun Hu told me there that Sullivan had completed his work about the *a priori* complex bounds for the Feigenbaum quadratic polynomial. This caught my attention because the *a priori* complex bounds is a sufficient condition that the Julia set of a real infinitely renormalizable quadratic polynomial is locally connected. After Hu explained to me the idea of Sullivan's proof of the *a priori* complex bounds for the Feigenbaum quadratic polynomial, I went on to write a joint paper with him about the local connectivity of the Julia set of the Feigenbaum quadratic polynomial. Later I realized that unless I had a complete proof of Sullivan's result, my understanding of the local connectivity of the Julia set of the Feigenbaum polynomial was incomplete. I began a serious study of Sullivan's result, which appears in Chapter Five. During this study, several conversations with Sullivan, along with the thesis of Edson de Faria, provided a lot help. During my research into infinitely renormalizable quadratic polynomials, communication with Curt McMullen via e-mail was very helpful. Several statements were made more precise because of his comments. Moreover, they led me to combine several of my papers in this direction into one self-contained paper, which is the origin of Chapter Five.

After I explained to Sullivan my research about the local connectivity of the Mandelbrot set at certain infinitely renormalizable points, he suggested that I might be able to reduce the computational aspect of many proofs by addressing the argument from a topological perspective. This is also presented in Chapter Five. In this study, many conversations with Tan Lei helped me to better understand the topological structure of the Mandelbrot set.

The survey articles written by E. B. Vul, Ya. G. Sinai, and K. M. Khanin [VSK], by J. Milnor [MI2,MI3], and by J. Hubbard [HUB], the book written by P. Collet and J.-P. Eckmann [COE] and the book edited by P. Cvitanović [CVI] provided valuable guidance, not only at the beginning, but also throughout the whole period of my research.

An invitation from the Advanced Series in Nonlinear Dynamics gave me a chance to work on this research monograph. Its editor, Robert MacKay, not only suggested this monograph but also encouraged me to complete it. Frank Isaacs gave me advice on English and on mathematical presentation.

In addition to the names of colleagues mentioned above, Benjamin Biele-

feld, Elise Cawley, Hsinta Frank Cheng, Guizhen Cui, Jack Diamond, Jozef Dodziuk, Lisa Goldberg, Sen Hu, Huyi Hu, Weihua Jiang, Jeremy Kahn, Linda Keen, Ravi Kulkarni, Genadi Levin, Shantao Liao, Arthur Lopes, Feng Luo, Jiaqi Luo, Mikhail Lyubich, Michael Maller, Jürgen Moser, Waldemar Pałuba, Alberto Pinto, Feliks Przytycki, David Rand, Michael Shub, Meiyu Su, Scott Sutherland, Folkert Tangerman, David Tischler, He Wu, Zhihong (Jeff) Xia, Shing-Tung Yau, Lai-Sang Young have each given valuable counsel and help. During the publication of my results, the referees provided invaluable comments. I offer thanks to all.

During the past eight years, while I visited the following institutes, my research has been supported by the Institut des Hautes Études Scientifiques in France, by the Nonlinear Systems Laboratory in the Mathematics Institute at the University of Warwick in England, by the Forschungsinstitut für Mathematik at the Eidgenössische Technische Hochschule in Switzerland, by the Max-Planck-Institut für Mathematik in Germany, and by the Mathematical Sciences Research Institute in the United States. Valuable research time was granted by the Department of Mathematics at Queens College of the City University of New York. This research was also funded in part by the National Science Foundation, by the Professional Staff Congress-City University of New York Research Award Program, by the City University of New York Collaborative Incentive Research Grant Program, and by the New York State Graduate Research Initiative Grant Program. I am deeply grateful for their support.

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September 1995

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