Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
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Purpose of this course

The purpose of this course is designed to study geometric structures of invariant sets of discrete dynamical systems in real one-dimension, complex one-dimension, and real two-dimension. We would like to bring quasiconformal mapping theory into this study. We would also like to study some geometric structure of some space of dynamical systems with a same topological type. In this study, we would like to bring Teichmüller theory into the study. Some key words that we will constantly deal with are

- fixed point and periodic point
- invariant set and Julia set
- invariant measure and Gibbs measure
- transfer operator and Hilbert metric
- Markov partition and symbolic coding
- quasiconformal mapping and quasisymmetric mapping
- holomorphic motion and Teichmüller space
- quasiconformal rigidity and smooth rigidity and rigidity

Main goal in the study of dynamical systems: Make a prediction for the future from past known data under certain rules for a typical starting point.

Setting:

- A topological space X called "phase space", whose elements are called points;
- "time", which could be integer time n or continuous time t;
- a map $f : X \to X$ or a flow ϕ^t on X;

What we want to know is the limits of $\{f^n(x)\}_{n=0}^{\infty}$ for a typical x in X under iterations or the limits of the flow line $\phi^t(x)$ for a typical x when the time t goes to infinity for a typical point x.

We call the group $\{f^n\}_{n=-\infty}^{\infty}$ if f is invertible or the semi-group $\{f^n\}_{n=0}^{\infty}$ if f is non-invertible a discrete dynamical system. We call the flow ϕ^t a continuous dynamical system.

Definition 1. A continuous map $F(t,x) = \phi^t(x) : \mathbb{R} \times X \to X$ is called a flow if for every $t \in \mathbb{R}$, $\phi^t : X \to X$ is a homeomorphism and if it satisfies

$$\phi^{t+s}(x) = \phi^t(\phi^s(x))$$

for all $t, s \in \mathbb{R}$ and $x \in X$.

If X is a smooth manifold and $F(t,x)$ is a smooth map, then for any $x \in X$, $\{\phi^t(x)\}_{t\in\mathbb{R}}$ is a smooth curve in X. Let

$$V(x) = \frac{\partial F(t,x)}{\partial t}\Big|_{t=0} = \frac{d\phi^t(x)}{dt}\Big|_{t=0}.$$

Then V defines a continuous vector field on the tangent bundle on $TX = \cup_{x\in X} T_x X$. Thus the flow $\phi^t(x)$ is just the solution of the ordinary differential equation

$$\frac{dy}{dt} = V(x) \quad \text{with the initial condition } y(0) = x.$$

Actually solutions of an ordinary differential equation on a complete smooth manifold for a vector field with certain smooth regularity form a flow. In many situations, we can use a discrete dynamical system to study a continuous dynamical system. This is due to a smart observation from Poincaré.

Poincaré map: Suppose $\phi^t(x)$ is a continuous flow on an m-dimensional smooth manifold. Suppose there are a point $x_0 \in X$ and a time $t_0 > 0$ such that $\phi^{t_0}(x_0) = x_0$. Then $\{\phi^t(x_0)\}_{t\in\mathbb{R}}$ is called a closed orbit, where the smallest such $t_0 > 0$ is called the period. Suppose $V(x_0) \neq 0$. Since M is locally $\mathbb{R}^m$, we can find an $(m-1)$-dimensional submanifold N of X transversal to $V(x_0)$ at x_0, that is,

$$T_{x_0}X = T_{x_0}N \oplus \{tV(x_0)\}.$$

Since $\phi^{t_0}(x_0) = x_0$ and since $F(t,x) = \phi^t(x)$ is continuous, we have a neighborhood U about x_0 in N such that for every $x \in N$, there is the smallest $t(x) > 0$ close to t_0 such that $\phi^{t(x)}(x) \in N$. Define

$$f(x) = \phi^{t(x)}(x) : U \to N.$$

Then f defines a discrete dynamical system with $f(x_0) = x_0$. So the study of the closed orbit $\{\phi^t(x_0)\}_{t\in\mathbb{R}}$ for the continuous dynamical system ϕ^t is equivalent to the study of the fixed point of the discrete dynamical system $\{f^n\}_{n=0}^{\infty}$. Here f is called a section map, or Pincaré map, or first return map.

Suspension: For a diffeomorphism $f : X \to X$ of a smooth manifold X, we can construct a suspension flow on the suspension manifold X_f. Here the suspension manifold is obtained by gluing $(f(x), 0)$ and $(x, 1)$ on $X \times [0, 1]$. The suspension flow ϕ^t is integral curves of the "vertical" vector field $\partial/\partial t$ on X_f.

Example 1. *Let $X = [0, 1]$ and $f(x) = 1 - x : X \to X$. Then X_f is the Möbius strip. The suspension flow ϕ^t has a period one closed orbit which does not separate M_f and one period two closed orbit which is the boundary of M_f and infinitely many period two closed orbits, each of them cuts M_f into two parts: one is topologically equivalent to M_f and the other is topologically equivalent to $[0, 1] \times S^1$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle.*

Example 2. *Let $X = S^1$ and $f(x) = -z : X \to X$. Then X_f is topologically equivalent to the two torus $T^2 = S^1 \times S^1$.*

In this course, we will more concentrate on discrete dynamical systems.

Dynamics of a discrete dynamical system with contraction property:

Suppose (X, d) is a metric space where $d = d(\cdot, \cdot)$ is the metric. A map $f : X \to X$ is called contracting if there exists $0 < \lambda < 1$ such that

$$d(fx, fy) \leq \lambda d(x, y), \quad \forall x, y \in X,$$

where λ is called a contracting constant. A point $x \in X$ is called a fixed point if $f(x) = x$.

Theorem 1 (Contacting Mapping Principle)**.** *If (X, d) is a complete metric space and if $f : X \to X$ is contracting with the contracting constant λ, then f has a unique fixed point x_0 in X and $f^n(x)$ tends to x_0 exponentially when n goes to infinity for every x in X.*

Proof. For $x, y \in X$ and $n > 0$, we have

$$d(f^n(x), f^n(y)) \leq \lambda d(f^{n-1}(x), f^{n-1}(y)) \leq \cdots \leq \lambda^n d(x, y)$$

for some $0 < \lambda < 1$. This implies that $\{f^n(x)\}_{n=0}^{\infty}$ have the same asymptotic behavior for all $x \in X$ as $n \to \infty$.

Now consider a sequence $\{f^n(x)\}_{n=0}^{\infty}$. For any $m > n > 0$,

$$d(f^m(x), f^n(x)) \leq \sum_{k=0}^{n-m-1} d(f^{n+k+1}(x), f^{n+k}(x))$$

$$\leq \sum_{k=0}^{n-m-1} \lambda^{n+k} d(f(x), x) \leq \frac{\lambda^n}{1-\lambda} d(f(x), x) \to 0, \quad \text{as} \quad n \to \infty.$$

Thus $\{f^n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence and since X is complete, it has a limiting point $x_0 \in X$. Because

$$f(x_0) = f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = x_0,$$

x_0 is a fixed point of f.

If y_0 is another fixed point, since

$$d(y_0, x_0) = d(f(y_0), f(x_0)) \leq \lambda d(x_0, y_0),$$

we have that $d(y_0, x_0) = 0$. So $y_0 = x_0$. This is the uniqueness. We complete the proof. □

The following result says that the unique fixed point x_0 of a contracting map f is preserved by C^0-perturbation and the fixed point changes continuously under C^0-topology.

Corollary 1. *Take f as that in the above theorem. For any $\epsilon > 0$, there is a $\delta \in (0, 1-\lambda)$ such that for any map $g : X \to X$ with*

1) $d(f(x), g(x)) < \delta$ for all $x \in X$ and

2) $d(g(x), g(y)) \leq (\lambda + \delta) d(x, y)$ for all $x, y \in X$.

the fixed point y_0 of g satisfies $d(y_0, x_0) < \epsilon$.

Proof. Take $\delta = \epsilon(1-\lambda)/(1+\epsilon)$. Since $g^n(x_0) \to y_0$, we have

$$d(x_0, y_0) \leq \sum_{k=0}^{\infty} d(g^k(x_0), g^{k+1}(x_0)) \leq \sum_{k=0}^{\infty} (\lambda+\delta)^k d(x_0, g(x_0))$$

$$\leq \frac{\delta}{1-\lambda-\delta} = \epsilon.$$

□

Now we can formulate the above theorem and the corollary into a modern version. Suppose $C(X)$ be the space of all continuous maps from X into X. A map $f \in C(X)$ is called Lipschitz if there is a constant $L > 0$ such that

$$d(f(x), f(y)) \leq L d(x, y), \quad \forall\, x, y \in X.$$

The smallest number $L > 0$ is called the Lipschitz constant and denoted as L_f. Then

$$L_f = \sup_{x \neq y \in X} \frac{d(f(x), f(y))}{d(x, y)}.$$

Let $LC(X)$ be the space of all Lipschitz map in $C(X)$. We can define a Lipschitz metric d_L on $LC(X)$ as

$$d_L(f, g) = \sup_{x \in X} d(f(x), g(x)) + |L_f - L_g|.$$

It is clear that (1) $d_L(f, g) = d_L(g, f)$, (2) $d_L(f, g) = 0$ if and only if $f = g$, and (3)

$$\begin{aligned} d_L(f, g) &= \sup_{x \in X} d(f(x), g(x)) + |L_f - L_g| \\ &\leq \sup_{x \in X} d(f(x), h(x)) + \sup_{x \in X} d(h(x), g(x)) + |L_f - L_h + L_h - L_g| \\ &\leq [\sup_{x \in X} d(f(x), h(x)) + |L_f - L_h|] + [(\sup_{x \in X} d(h(x), g(x)) + |L_h - L_g|] \\ &= d_L(f, h) + d_L(h, g). \end{aligned}$$

Therefore, d_L is a metric on $LC(X)$. (Note that it may happen that $d_L(f, g) = \infty$ for some $f, g \in LC(X)$, but one can modify it by defining

$$\tilde{d}_L(f, g) = \max\{d_L(f, g), 1\}$$

in this case. Since we only care about the case when $d_L(f, g)$ small, so we still use the original form of d_L.) All contracting maps from X to X form a subspace

$$LC_1(X) = \{f \in LC(X) \mid L_f < 1\}.$$

The metric is d_L restricted on $LC_1(X)$.

Theorem 2 (Modern Version of Contacting Mapping Principle)**.** *If (X, d) is a complete metric space, then for every $f \in LC_1(X)$, there is a unique $x(f) \in X$ such that $f(x(f)) = x(f)$. Moreover,*

$$F : (LC_1(X), d_L) \to (X, d); \quad F(f) = x(f)$$

is a continuous operator from the metric space $(LC_1(X), d_L)$ to the metric space (X, d).