Note from Robert Suzzi Valli on Lecture of Y. Jiang

in April 3^{rd}, 2009, 9:30 am – 11:30 am.

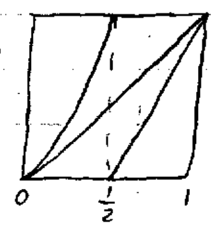
4/3/09

<u>Thm</u>: Spse f and g are uniformly quasisymmetric circle endomorphisms of deg $d>1$. Then they are topologically conjugate by h and h is quasisymmetric. Moreover, f and g both have bounded and bounded nearby geometry.

<u>Thm</u>: A circle endomorphism f of deg $d>1$ has bounded and bounded nearby geometry $\Leftrightarrow$ it is quasisymmetrically conjugate to $q(z)=z^d$.

Questions.

almost expanding circle endomorphism (Spse it's C^2)



$\xrightarrow[\text{by } h]{\text{top conj}} q(z)=z^2$

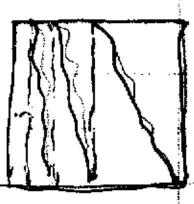
h is <u>not</u> q.s.

$$\boxed{f'(0)=1.\ f'(x)>1,\ \forall x\in(0,1]}$$

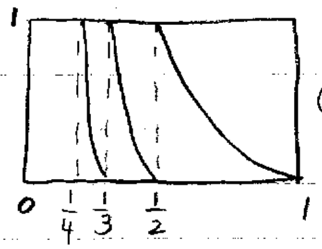
f has no bdd or bdd nearby geom. $\left(\dfrac{|I_{\overbrace{00\cdots0}^{n}}|}{|I_{\underbrace{00\cdots0}_{n-1}}|}\to 1 \ \&\ \dfrac{|I_{\overbrace{00\cdots01}^{n-1}}|}{|I_{\underbrace{00\cdots0}_{n-1}}|}\to 0\right)$

(open problem) Q1: Spse f and g are both almost expanding circle endomorphisms and C^2. Then f and g are top. conj by h. Is h q.s.?

$$G(x)=\frac{1}{x}-\left[\frac{1}{x}\right]:(0,1]\to(0,1] \quad \text{(Gauss map)}$$



f: $\xleftarrow{h}$ G



$\frac{1}{2}<x<1$

$2>\frac{1}{x}>1$

$\left[\frac{1}{x}\right]=1$

$\frac{1}{x}-1$

$x=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}}=[a_1,a_2,\ldots$

$\frac{1}{x}=a_1+\cfrac{1}{a_2+\cdots}$

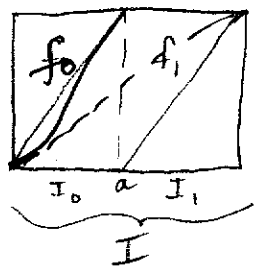
$\left[\frac{1}{x}\right]=a_1$

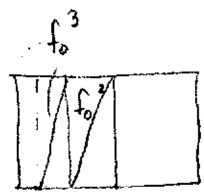
$G(x)=\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}=[a_2 a_3 \ldots]$

$G(x)$ is just shift map in symbolic space of infinite letters.

$$|G'(x)|=\left|-\frac{1}{x^2}\right|>1,\ x\in\left(\frac{1}{n+1},\frac{1}{n}\right) \quad n=1,\ldots$$

<u>Q2</u>: Is h q.s.?





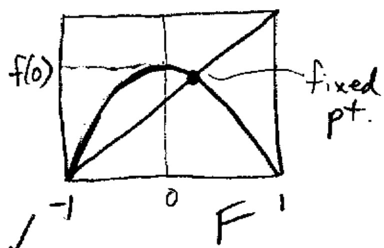
$$I_{0_n} = f_0^{-1}(I_0)$$

$0_n = \overbrace{00\cdots0}^{n}$

$$\hat{f}(x) = f^n : I_{0_n} \to I$$
$$|\hat{f}'(x)| \geq \lambda' > 1, \quad \forall x \in I_{0_n}$$

} this relates Q1 to Q2 ($\hat{f}$ like $G(x)$ - Gauss map)

(bigger problem) **Q3**:



"folding map"

unimodal map

$$f(x) = \mu x(1-x)$$

$$f(x) = F(-|x|^\alpha), \quad \alpha > 1$$

$F : [-1,1] \to [-1, f(0)]$ is a C^3-diffeomorphism + negative Schwarzian derivative

<u>Schwarzian derivative</u>

$$S(F)(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2 < 0$$

$$\Downarrow$$

$$S(F^{-1}) > 0$$

<u>Q3</u>: Spse f and g are both unimodal and are top. conj. by h, i.e. $f \circ h = h \circ g$. Is h q.s.?

<u>specific case</u>: $f(x) = F(-|x|^{2n})$ ← cx. analytic.

$f(z) = F(-z^{2n})$ ← analytic

extend to complex map

(*) $x \mapsto |x|^2$ "x^2"

can extend this map to: $z \mapsto z^2$

- can't do: $x \mapsto |x|^3$

can't extend to: $z \mapsto z^3$.

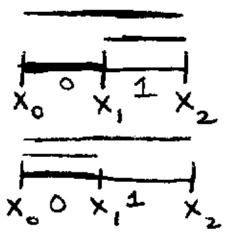
★

Test Ques 2:

<u>Thm</u>: $I = [0,1]$. Spse $f : [0,1] \xrightarrow{\text{cont.}} [0,1]$ has a periodic point of period 3. Then, $\forall m \geq 1$, f has a periodic point of period m.

(<u>Li-York</u>: period 3 $\Rightarrow$ chaos)

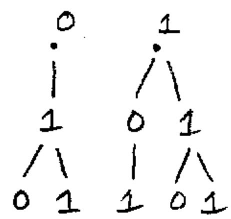
$x_0 \neq x_1 \neq x_2$
$f(x_0) = x_1$
$f(x_1) = x_2$
$f(x_2) = x_0$



$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \Sigma_A = \{\omega = i_0 i_1 \dots \mid a_{i_k i_{k+1}} =$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

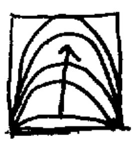
⊛ use mean value thm.



$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd \cdots \rhd 2^n \cdot 3 \rhd 2^n \cdot 5 \rhd \cdots \rhd 2^n \rhd 2^{n-1} \rhd \cdots \rhd 2 \rhd 1$

<u>Thm</u> (Sarkowski): Spse $f : I \to I$ is a cont. map and has a periodic point of period m. Then f has a periodic point of period n for every $m \rhd n$.

$g_\mu(x)$:



$a \leq \mu \leq b$

$\{\tilde{\mu}_n\} \to \tilde{\mu}_\infty$

Feigenbaum constant

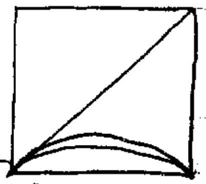
$$\tilde{\delta}_n = \frac{\tilde{\mu}_n - \tilde{\mu}_{n-1}}{\tilde{\mu}_n - \tilde{\mu}_{n+1}} \longrightarrow \delta = \cancel{4.669\ldots} \quad (\text{for } \alpha = 2)$$
$4.669\ldots$

$$\delta_n = \frac{\mu_n - \mu_{n-1}}{\mu_n + \mu_{n+1}}$$

<u>Conj</u>: δ_α is universal and independent of any family as long as α is fixed.

$\alpha = 2$ is OK

(3)



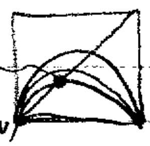
deriv 1 when $\mu = \frac{1}{2}$

For $0 < \mu \le \frac{1}{2}$, $f_\mu^n(x) \to -1$ as $n \to \infty$

$$f_\mu(x) = -\mu x^2 + (\mu - 1)$$
$$0 < \mu \le 2$$

$$f_\mu(-1) = -1$$
$$f_\mu(1) = -\mu$$
$$f_\mu'(0) = 0$$

for $\mu > \frac{1}{2}$



$\mu_n \nearrow \mu_\infty$

$\mu_{n-1} < \mu < \mu_n$

f_μ has a periodic pt. P_μ of period 2^n

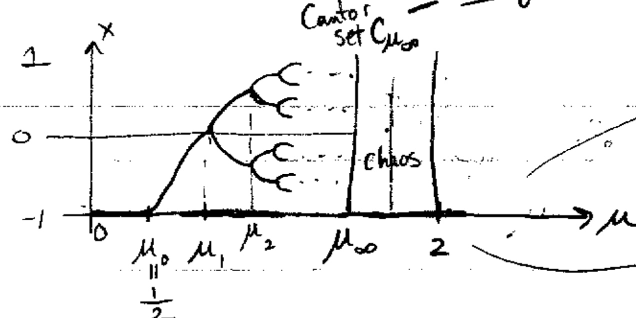
s.t. $f_\mu^n(x) \to \{P_\mu, \ldots, f^{2^n-1}(P_\mu)\} = \mathcal{O}(P_\mu)$ — orbit of P_μ.

$(\mu > \frac{1}{2})$ as $n \to \infty$, for almost every x.

$f_{\mu_\infty}^n(x) \to ?$ as $n \to \infty$

Cantor set C_{μ_∞} — strange attractor

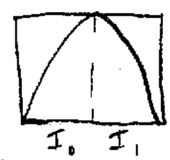
bifurcation



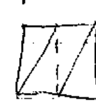
$\mu > 2$

horseshoe map

At $\mu = 2$

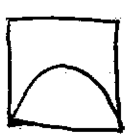


(works like circle expanding map)



Note: Spse $g \xrightarrow[\text{by } h]{\text{top.conj}} f_{\mu_\infty}^n(x) \Rightarrow h$ is q.s.

$\forall f$



+ some smoothing condition

$\exists \mu$ s.t. $f \overset{\text{top.conj}}{\sim} f_\mu$ on a dynamical interesting set