Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
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Let $\mathcal{QS}$ be the space of all (orientation-preserving) quasisymmetric circle homeomorphisms. Let $\mathcal{Q}$ be those in $\mathcal{QS}$ fixing 1. Let $\mathcal{F}$ be the space of all uniformly quasisymmetric circle endomorphisms of degree 2 fixing 1. (The following can be done for all degrees $d > 2$ with a little bit modification). Suppose $q(z) = z^2$. Let $\alpha : \mathcal{Q} \to \mathcal{F}$ be defined as

$$\alpha(h) = h \circ q \circ h^{-1}.$$

Theorem 1. *The map α is bijective.*

Proof. From Theorems 2 in Lecture 7 and Theorem 1 and Corollary 1 in Lecture 9, we have that $\alpha : \mathcal{Q} \to \mathcal{F}$ is a onto map. We only need to prove that α is injective. Suppose $\alpha(h_1) = \alpha(h_2)$ for $h_1, h_2 \in \mathcal{Q}$. Since 1 is the only fixed point of $\alpha(h_1)$, $\alpha(h_2)$, and q, $h_1(1) = h_2(1) = 1$. Since

$$h_1^{-1}(q^{-1}(1) = h_2^{-1}(q^{-1}(1)),$$

we have that $h_1(-1) = h_2(-1)$. Inductively, since

$$h_1^{-1}(q^{-n}(1) = h_2^{-1}(q^{-n}(1)),$$

we have that $h_1(e^{2\pi i(k/2^n)}) = h_2(e^{2\pi i(k/2^n)})$ for all $0 \leq k < 2^n$. Since the set of all numbers $\{k/2^n | 0 \leq k < 2^n, n = 1, 2, \cdots\}$ is dense in T, we have that $h_1 = h_2$. So α is injective. □

Remark 1. *The bounded nearby geometry and the quasisymmetric property for a conjugacy have been also studied for one-dimensional maps with critical points. Refer to (download site: http://qcpages.qc.cuny.edu/˜yjiang/HomePageYJ/Download/1993GeomFin.pdf) "Geometry of geometrically finite one-dimensional maps". Comm. in Math. Phys., 156 (1993), no. 3, 639-647. Or the book "Renormalization and Geometry in One-Dimensional and Complex Dynamics". Advanced Series in Nonlinear Dynamics, Vol. 10 (1996) World Scientific Publishing Co. Pte. Ltd., River Edge, NJ, xvi+309 pp. ISBN 981-02-2326-9.*

Dual symbolic space and scaling model for $\mathcal{Q}$ and $\mathcal{F}$.

Suppose f is a circle endomorphism of degree 2. As we have seen, we have a sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$. Here η_n contains 2^n intervals, each interval has a unique label $w_n = i_0 i_1 \cdots i_{n-1}$ which we denote as I_{w_n}, where $i_k \in \{0, 1\}$, such that

$$f(I_{w_n}) = I_{\sigma(w_n)}.$$

Now we would like to relabel these intervals. For $w_n = i_0 i_1 \cdots i_{n-1}$, let

$$w_n^* = j_{n-1} \cdots j_1 j_0,$$

where $j_{n-1} = i_0$, $\cdots$, $j_0 = i_{n-1}$. Define the dual shift map

$$\sigma^*(w_n^*) = j_{n-1} \cdots j_1.$$

Then we have the following

$$I_{w_n^*} \subset I_{w_n^*}.$$

We call

$$\Sigma^* = \{w^* = \cdots j_{n-1} \cdots j_0\} = \prod_{-\infty}^{0} \{0, 1\}$$

with the product topology the dual symbolic space. Then the dual shift map is

$$\sigma^* : w^* = \cdots j_{n-1} \cdots j_1 j_0 \to \sigma^*(w^*) = \cdots j_{n-1} \cdots j_1.$$

Then we call (Σ^*, σ^*) the dual symbolic dynamical system. The dual cylinder for a given $w_n^* = j_{n-1} \cdots j_0$ is

$$[w_n^*] = \{w^* = \cdots j'_{m-1} \cdots j'_n j_{n-1} \cdots j_0 \mid j'_k \in \{0, 1\}, k \geq n\}.$$

Then all these dual cylinders form a topological basis for Σ^*.

Now we define another operator in Σ^* called the adding machine $a(w^*)$ as follows: If $w^* = \cdots j_{n-1} \cdots j_1 j_0$ and $j_0 = 0$, then $a(w^*) = \cdots j_{n-1} \cdots j_1 (j_0 + 1)$ and if $j_0 = 1$, then $j_0 + 1 = 0$ and then consider $j_1 + 1$, so on. For each w_n^*, we can also define the adding machine a. We have that for any w_n^*, $I_{w_n^*}$ and $I_{a(w_n^*)}$ are two adjacent intervals. (Note that if $w_n^* = \underbrace{1 \cdots 1}_{n}$, then we define $a(w_n^*) = \underbrace{0 \cdots 0}_{n}$).

For each w_n^*, we define two scalings, one is for bounded nearby geometry and one is for bounded geometry:

$$bng(w_n^*) = \Big| \log \frac{|I_{w_n^*}|}{|I_{a(w_n^*)}|} \Big|$$

and

$$bg(w_n^*) = \frac{|I_{w_n^*}|}{|I_{\sigma^*(w_n^*)}|}.$$

Thus we have two sets of scalings

$$BNG = \{bng(w_n^*) \mid w_n^* = j_{n-1} \cdots j_0,\ j_k \in \{0,1\}\}$$

and

$$BG = \{bg(w_n^*) \mid w_n^* = j_{n-1} \cdots j_0,\ j_k \in \{0,1\}\}.$$

These two sets can be determined to each others.

Exerice 1. *Express scalings in BG in terms of scalings in* BNG *and verse versa.*

Therefore, the sequence of nested Markov partitions $\eta = \{\eta_n\}_{n=0}^{\infty}$ has bounded nearby geometry if and only if there is a constant $C > 0$ such that

$$bng(w_n^*) \leq C,\ \forall w_n^*.$$

And it has bounded geometry if and only if there is a constant $C > 0$ such that

$$bg(w_n^*) \geq C,\ \forall w_n^*.$$

It is also clear that

$$bg(w_n 0) + bg(w_n 1) = 1 \tag{1}$$

which we call it **the summation condition**. Now let us consider the space of scalings (means positive numbers)

$$\mathcal{S} = \{(S(w_n^*))\}$$

satisfies the summation condition (1) and the induced bounded nearby geometry $(bng(w_n^*))$ is in l^{∞}. Then for $f \in \mathcal{F}$, we have that

$$\gamma(f) = (bg(w_n^*)) \in \mathcal{S}.$$

This induces a map

$$\beta(h) = \gamma \circ \alpha \in \mathcal{S}$$

and

Theorem 2. *The maps* $\beta : \mathcal{Q} \to \mathcal{S}$ *and* $\gamma : \mathcal{F} \to \mathcal{S}$ *are both bijective.*

Universal Teichmüller space and the space of uniformly quasisymmetric circle endomorphisms and space of scalings

Remember that $\mathcal{QS}$ is the space of all quasisymmetric homeomorphisms of T. Let $\mathcal{M}$ be the space of all Möbius transformations preserving the unit disk Δ. Then

$$\mathcal{M} = \{M(z) = e^{2\pi\theta i}\frac{z-a}{1-\overline{a}z} \mid |a|<1\}$$

The quotient space

$$\mathcal{UT} = \mathcal{QS}/\mathcal{M}$$

is called the universal Teichmüller space. It is a complex Banach manifold (we will discuss this later, see Remark 2).

Since each element in $\mathcal{M}$ has one complex parameter $a \in \Delta$ and one real parameter $0 \leq \theta < 1$, for any pair of triple points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ arranged in the counter-clockwise order on T, there is a one and only one element $M \in \mathcal{M}$ such that

$$M(z_1) = w_1, \quad M(z_2) = w_2, \quad M(z_3) = w_3.$$

Thus the universal Teichmüller space can be thought as the space of all quasisymmetric homeomorphisms fixing three points on T. Let us assume that this three points are 1, i, -1. Then

$$\mathcal{UT} = \{h \in \mathcal{QS} \mid h(1) = 1,\ h(i) = i,\ h(-1) = -1\} = \mathcal{Q}/\mathcal{M}_1$$

where

$$\mathcal{M}_1 = \{M(z) = \frac{1-\overline{a}}{1-a}\frac{z-a}{1-\overline{a}z} \mid |a|<1\}$$

is the space of all Möbius transformations preserving the unit disk Δ fixing 1.

Now consider the space of all uniformly quasisymmetric circle endomorphisms of a fixed degree $d > 1$ conjugated by $\mathcal{M}$, which we denote as $\mathcal{UF}_d$. For the simplicity, we assume $d = 2$ and write $\mathcal{UF} = \mathcal{UF}_2$.

Theorem 3. *The map α induces a bijective map $\varrho = [\alpha] : \mathcal{UT} \to \mathcal{UF}$.*

Proof. Let $q(z) = z^2$. Then it has a unique fixed point 1 and $q^{-1} = \{1, -1\}$ and $q^{-2}(1) = \{1, i, -1, -i\}$. The space $\mathcal{UF}$ can be thought as the space of those uniformly quasisymmetric circle endomorphisms f fixing 1 such that $f^{-1}(1) = \{1, -1\}$ and $f^{-2}(1) = \{1, i, -1, s\}$. It is clear that of $h \in \mathcal{Q}$ fixing 1, -1, i if and only if $f = \alpha(h) \in \mathcal{UF}$. Thus α induces a onto map ϱ from $\mathcal{UT}$ to $\mathcal{UF}$. We need to prove ϱ is one-to-one. Suppose $\varrho(h_1) = \varrho(h_2)$. Similar to the proof of Theorem 1, $h_1 = h_2$. $\square$

Let $\mathcal{US}$ be the space of scalings $(s(w_n^*))$ in $\mathcal{S}$ such that $s(0) = s(1) = s(00) = 1/2$.

Corollary 1. *The maps* γ *and* β *induce bijective maps* $[\gamma] : \mathcal{UF} \to \mathcal{US}$ *and* $[\beta] : \mathcal{UT} \to \mathcal{US}$*.*

Remark 2. *Note that*

$$\mathcal{Q} = \mathcal{UT} \times \Delta$$

where $\Delta = \{a \mid |a| < 1\}$ *is the unit disk in the complex plane. For any* $h \in \mathcal{Q}$*, let* $t = h(i)$ *and* $s = h(-1)$ *and*

$$a = \frac{1 - i\frac{s(t-1)}{(t-s)}}{1 - i\frac{t-1}{t-s}} \in \Delta.$$

Let

$$G_a(z) = \frac{1-\overline{a}}{1-a}\frac{z-a}{1-\overline{a}z}.$$

Then G_a *maps* $1, t, s$ *to* $1, i, -1$*. So* $G_a \circ h$ *fixes* $1, i, -1$ *and is in* $\mathcal{UT}$*. So if we define* $\chi(h) = (G_a \circ h, a)$*. Then it is a bijective map from* $\mathcal{Q}$ *to* $\mathcal{UT} \times \Delta$*. So* $\mathcal{Q}$ *as well as* $\mathcal{F}$ *and* $\mathcal{S}$ *has induced complex Banach manifold structure such that it is a complex Banach manifold. (We will discuss this later.)*