Lecture 12 5/1/09

10:00 am – 12:00 noon

by Fred Gardiner.

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} \qquad t^2 - x^2 - y^2 - z^2$$

Minkowski space in $\mathbb{R}^4$.

$t^2 - x^2$ in $\mathbb{R}^2$ $\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$, $A = \begin{pmatrix} \frac{1}{\sqrt{1-k^2}} & \frac{k}{\sqrt{1-k^2}} \\ \frac{k}{\sqrt{1-k^2}} & \frac{1}{\sqrt{1-k^2}} \end{pmatrix}$

Title:

Complex structure on Teichmüller space

$A\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix}$ $t^2 - x^2 = (t')^2 - (x')^2$. invariant under A.

In Euclidean: $t^2 + x^2$ $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ invariant for $t^2 + x^2$.

$PSL(2, \mathbb{C}) \cong$ Lorentz gp $= SO^+(1,3)$

$PSL(2, \mathbb{R})$.

$t^2 = x^2 + y^2 + z^2$ is a cone.

$\zeta = u + iv$

$$\begin{pmatrix} u^2+v^2+1 \\ 2u \\ -2v \\ u^2+v^2-1 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad t^2 = 4u^2+4v^2+u^4+v^4+1+2u^2+2v^2+2u^2v^2$$

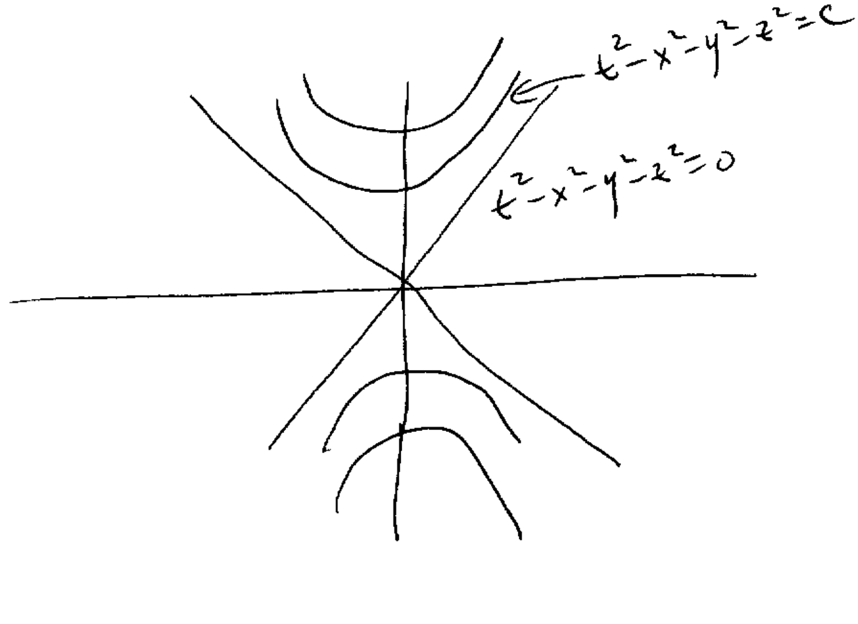
$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = \text{a Hermitian } 2\times 2 \text{ matrix}$$

(i.e. $X^t = \bar{X}$)

$\det = t^2 - x^2 - y^2 - z^2$. $\quad x, y, z, t$ are all real.

$P \in SL(2,\mathbb{C})$, $\det = 1$.

$$X \longrightarrow PXP^*, \quad P^* = \text{conjugate transpose} = \bar{P}^t$$



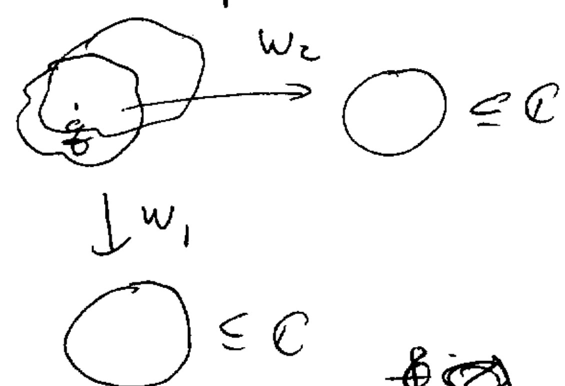
manifold structure

complex manifold structure

$R \in$ Hausdorff space, Riemann surface.

$S = \{ f : \Delta \to R,\ f(0) = p.\ f \text{ is holomorphic} \}$

$f(z) = q \in R$



$w_1 \circ w_2^{-1}$: holo.

$\zeta \in \Delta \xrightarrow{f} R.$

$(*)$ $\inf \left\{ \dfrac{1}{|f'(0)|} \;\middle|\; f \in S \right\}$, f' means fixed local coordinate $\left.\dfrac{d(w \circ f)}{d\zeta}\right|_{\zeta = 0} = f'(0).$

$\|$

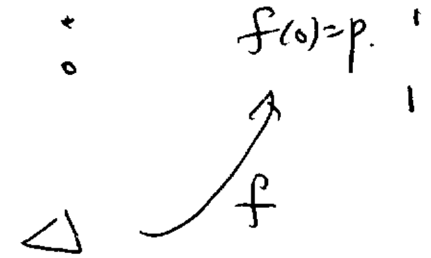
$\rho(p)$. depends on local coordinate w.

$\rho(p)\,|dp|$ is invariant by local coordinate w.

i.e. $\rho(w_1)\,|dw_1| = \rho(w_2)\,|dw_2|.$

The f that realizes the infimum in $(*)$ is the universal covering of R by Δ.

z as coordinate

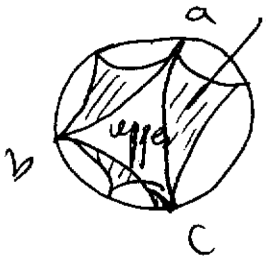


∞ $\quad R = \mathbb{C} \backslash \{0, 1\}$.

one linear map $f_r : \Delta \to \Delta_r(p)$

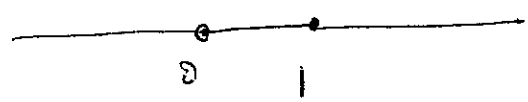
$\Rightarrow f_r'(0) = r$.

lower half plane



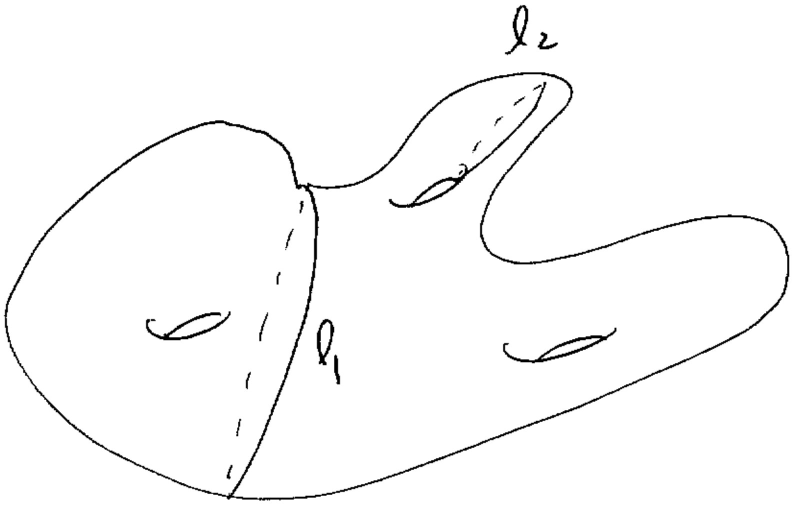
Riemann mapping maps $\Delta \to$ upper half plane with boundary segments $(0,1)$, $(1,\infty)$, $(-\infty, 0)$.

$a, b, c \to 0, 1, \infty$.

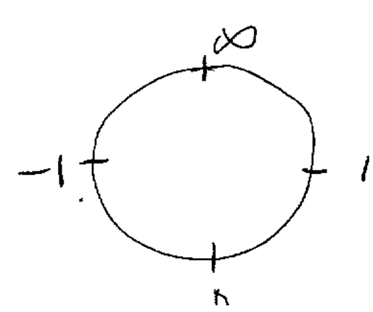


reflect in the triangle $\Rightarrow f : \Delta \to \mathbb{C} \backslash \{0,1\}$ is the universal covering of $\mathbb{C}\backslash\{0,1\}$ by Δ.

In general:

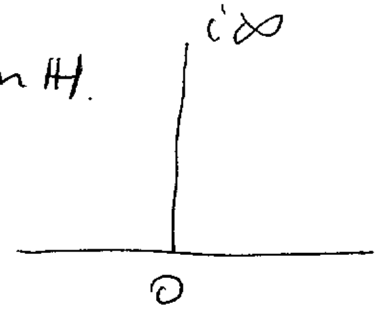


$\cong \Delta / \Gamma$ = covering group $\subset PSL(2, \mathbb{R})$



Pick one closed curve and assume it

$A \in \Gamma$, $Az = \lambda z$, $\lambda > 1$ $(0, +i\infty)$ in $\mathbb{H}$.



$\ln \lambda$, hyperbolic ~~length~~ of l_1

$A'(0) = \lambda$

Tsuji, Fricke-Klein. If R is of finite type, then a finite number of lengths of homotopic classes of closed curve on R determines the covering group.

Γ_1. Γ_2 acting on $\mathbb{H}$.

$w : \mathbb{H} \to \mathbb{H}$ $w \Gamma_1 w^{-1} = \Gamma_2$, i.e. $w \gamma w^{-1} = \chi(\gamma) \in \Gamma_2$
$\forall \gamma \in \Gamma_1$

χ is an isomorphism from Γ_1 on Γ_2.

L a Möbius trans. in determined by its values at 3 pts $\Lambda(\Gamma)$ = limit set of Γ.

Thinking about

W restricted to $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$.

~~$W: R \to R$~~

$m > \Delta$.

$$\begin{array}{ccc} \Delta & \xrightarrow{\tilde{w}} & \Delta \\ \downarrow & & \downarrow \\ R_1 & \xrightarrow{w} & R_2 \end{array} \qquad M^{-1} \le \frac{|\tilde{w}(x+t) - \tilde{w}(x)|}{|\tilde{w}(x) - \tilde{w}(x-t)|} \le M$$

$T_2 = \omega T_1 \omega^{-1}$. $\quad \omega(0)=0, \quad \omega(1)=1. \quad \omega(\infty)=\infty.$

$\omega(z) = z + \varepsilon V(z) + O(\varepsilon) \quad z \in \mathbb{H}.$

$\omega_\varepsilon(x) = x + \varepsilon V(x) + O(\varepsilon).$

$$M \le \overset{\rho(x,t)}{\overset{\shortparallel}{\frac{|\omega(x+t)-\omega(x)|}{|\omega(x)-\omega(x-t)|}}} \le M$$

If $\varepsilon = 0$, $\rho(x,t)=1$. if ε small

$$\frac{1}{1+\delta(\varepsilon)} \le \frac{|\omega_\varepsilon(x+t)-\omega_\varepsilon(x)|}{|\omega_\varepsilon(x)-\omega_\varepsilon(x-t)|} \le 1+\delta(\varepsilon) \qquad \delta(\varepsilon)\to 0 \text{ as } \varepsilon \to 0.$$

$$\left| \frac{V(x+t)-V(x-t)-2V(x)}{t} \right| \le C_\varepsilon \quad (**)$$

$$T_\varepsilon = \omega_\varepsilon T \omega_\varepsilon^{-1}$$

$$\frac{V(\gamma(x))}{\gamma'(x)} - V(x) = \text{quadratic polynomial in } x. \ (***)$$

$$\gamma_\delta(z) = \frac{a_\delta z + b_\delta}{c_\delta z + d_\delta} \qquad \gamma_0(z) = z.$$

$a_\delta, b_\delta, c_\delta, d_\delta$ are differentiable function of δ.

$$\lim_{\delta \to 0} \frac{1}{\delta}\left(\gamma_\delta(z) - z\right) = \frac{1}{\delta}\left(\frac{a_\delta z + b_\delta}{c_\delta z + d_\delta} - z\right)$$

$$= \lim_{\delta \to 0} \frac{1}{\delta} \, \frac{a_\delta z + b_\delta - c_\delta z^2 - d_\delta z}{c_\delta z + d_\delta}. \qquad a_0 = 1, \; d_0 = 1, \; c_0 = 0, \; b_0 = 0.$$

$$= -\dot{c} z^2 + (\dot{a} - \dot{d}) z + \dot{b}.$$

$\Rightarrow$ The V's ~~that~~ satisfy $(**)$ $(***)$ factored by quad polys comprise the tangent space to $\mathrm{Teich}(\Gamma)$.

$Z =$ a real vector space

$I : Z \to Z$. $I^2 = -\text{identity}$

$(a+ib)V \doteq aV + IbV$, $V \in Z$.

$c_1(c_2 V) = (c_1 c_2) V$.

$(a_1 + ib_1)\big((a_2 + ib_2)V\big) = (a_1 + ib_1)(a_2 + ib_2) V$

Bers embedding: Schwarzian derivative

Hilbert transform:

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1) Harmonic conjugates

2) $PV - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{V(t)}{t-x} dt$

3) Same sort of integral to periodic functions
cotangent $(t-x)$. ~~by rep~~ t replace $(t-x)$.

4) Pompeiu formula.
- Cauchy

$$V(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \left(\frac{1}{\zeta - z} + \text{convergence term} \right) \mu(\zeta)\, d\xi\, d\eta.$$

$\mu(\bar{\zeta}) = \mu(\zeta)$

$\mu \in L_\infty$. Hilbert transf is just $I\mu$, ie

$$IV = -\frac{1}{\pi} \iint \left(\frac{1}{\zeta - z} + \text{con. term} \right) I\mu(\zeta)\, d\xi\, d\eta.$$

$$I\mu = \begin{cases} i\mu & \text{in } \mathbb{H}. \\ -i\mu & \text{in } \mathbb{H}^*. \end{cases}$$

$\Rightarrow$ Z under ~~this~~ Hilbert transform is a
Complex Banach manifold.