Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
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Periodic point:

For a dynamical system, $f : X \to X$, we call a point $x \in X$ a periodic point of period $m > 0$ if $f^m(x) = x$ but $f^i(x) \neq x$ for all $1 \leq i \leq m-1$. We use $Per_m(f)$ to denote the set of all period points of period m. In particular, if $m = 1$, x is called a fixed point. We use $Fix(f)$ to denote the set of all fixed points.

Exerice 1. *Suppose* $f(z) = z^d : S^1 \to S^1$. *Find* $Fix(f^m)$, $Per_m(f)$.

Suppose X is an n-dimensional C^1 manifold and $f : X \to X$ is a C^1 map. Suppose p is a periodic point of period m. Then $f^m(p) = p$. The derivative $Df^m(p) : T_pX \to T_pX$ is a linear operator where T_pX is the tangent space of X at p. Since we only consider local dynamics for f near p, we can introduce local coordinates near p with p as the origin 0. In these coordinates $Df^m(0)$ becomes an $n \times n$ matrix. So suppose U is a neighborhood of 0 in $\mathbb{R}^n$ and f is a C^1 map defined on U ($f : U \to f(U)$ with $f(0) = 0$). Let $\| \cdot \|$ be a fixed norm on $\mathbb{R}^n$. Let

$$B(r) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$$

be the closed ball of radius $r > 0$ and centered 0. For r small, $B(r) \subset U$. Consider $X = B(r)$. Let $C^1(X)$ be the space of all C^1 maps; each is defined on some neighborhood of X. For any $g \in C^1$, we can define the C^0 norm for g as

$$\|g\|_0 = \sup_{x \in X} \|g(x)\|.$$

Let

$$Dg(x) : \mathbb{R}^n \to \mathbb{R}^n$$

be the derivative of g at x. As a linear operator, we can define its C^0 norm as

$$\|Dg(x)\|_0 = \sup_{\|v\|=1} \|Dg(x)v\|.$$

Then C^0 norm for Dg is defined as

$$\|Dg\| = \sup_{x \in X} \|Dg(x)\|_0.$$

The C^1 norm of g is

$$\|g\|_1 = \|g\|_0 + \|Dg\|_0.$$

The C^1 distance between two maps $g, h \in C^1(X)$ is

$$d_1(g,h) = \|g - h\|_1.$$

We say g is close to f in the C^1-topology if their C^1 distance $d_1(g, f)$ is small.

Theorem 1 (Preserving of Periodic Point by Small Perturbation)**.** *If $Df^m(0)$ does not have one as an eigenvalue then every map g sufficiently closed to f in the C^1-topology has a unique periodic point of period m close to 0.*

Proof. Consider $F = f^m - id$. Then its derivative $DF(0) = Df^m(0) - I$. Since 1 is not among the eigenvalues of $DF(0)$, the map $F = f^m - id$ is locally invertible by the Inverse Function Theorem. More precisely, by picking r small, $F : X \to F(X)$ is invertible and its inverse $F^{-1} : F(X) \to X$ is also a C^1 map.

Suppose g is a map close to f in the C^1-topology. We can write $g^m = f^m - \eta$ where η has a small C^1 norm

$$\|\eta\|_1 = ||\eta||_0 + ||D\eta||_0.$$

Now we need to solve the equation,

$$x = g^m(x) = f^m(x) - \eta(x) = F(x) + x - \eta(x).$$

Equivalently, solve the equation

$$F(x) - \eta(x) = 0$$

or the equation

$$x = F^{-1}(\eta(x)).$$

A key argument in the proof is that since the C^0 norm of the derivative DF^{-1} is bounded and the C^1 norm of η is small,

$$F^{-1} \circ \eta : X \to X$$

is a contracting map (note that X is a complete space with the distance $d(x,y) = \|x - y\|$). More precisely, let $L = ||DF^{-1}||_0$ be the C^0-norm of DF^{-1} and let

$$||\eta||_1 = \|\eta\|_0 + ||D\eta||_0 \leq \epsilon$$

be the C^1 norm of η. We get

$$||F^{-1}(\eta(x)) - F^{-1}(\eta(y))|| \leq \epsilon L||x-y||$$

for $x, y \in X$ and, since $F^{-1}(0) = 0$,

$$||F^{-1}(\eta(0))|| = ||F^{-1}(\eta(0)) - F^{-1}(0)|| \leq L||\eta(0)|| \leq \epsilon L,$$

and hence

$$||F^{-1}(\eta(x))|| \leq ||F^{-1}(\eta(x)) - F^{-1}(\eta(0))|| + ||F^{-1}(\eta(0))|| \leq \epsilon L||x|| + \epsilon L.$$

Thus if

$$\epsilon \leq \frac{r}{L(1+r)},$$

then

$$F^{-1} \circ \eta : X \to X$$

is a contracting map. From the Contracting Mapping Principle, $F^{-1} \circ \eta$ has a unique fixed point $x = x(g) \in B(r)$. Thus $g^m(x) = x$. Furthermore, $\|x(g)\| \to 0$ as $d_1(g, f) \to 0$.

By taking $r > 0$ small, we can assume that $f^i(B(r)) \cap B(r) = \emptyset$ for $1 \leq i \leq n-1$. When g is C^1 close to f, this property is still kept, that is, $g^i(B(r)) \cap B(r) = \emptyset$ for $1 \leq i \leq n-1$. So $g^i(x) \neq x$ for $1 \leq i \leq n-1$. This says that x is a periodic point of period m for g. ☐

Dynamics of Linear Maps and Local Dynamics:

Suppose X is an n-dimensional C^1-manifold and suppose $f : X \to X$ is a C^1 map with a periodic point p of period $m > 0$. Then the derivative $Df^m(p)$ is a linear map from $T_pX \to T_pX$ where T_pX is the tangent space of X at p. In a "good" situation, the local dynamics of f near p is determined by this linear map. Here a "good" situation, we will refer to when $Df^m(p)$ is "hyperbolic" as we will explain later. By considering local coordinates at p, we can think $Df^m(p)$ as an $n \times n$ matrix A. Thus the linear map $Df^m(p) : T_pX \to T_pX$ can be thought as

$$y = Ax : \mathbb{R}^n \to \mathbb{R}^n.$$

Let $sp(A)$ be the set of all eigenvalues of A and define

$$r(A) = \max\{|a| \mid a \in sp(A)\}$$

be the spectral radius of A. It is a quantity independent of choice of norms on $\mathbb{R}^n$.

Given any norm $|| \cdot ||$ on $\mathbb{R}^n$, we define the norm of A as

$$||A|| = \sup_{||v||=1} ||Av||.$$

Clearly, $||A|| \geq r(A)$.

Lemma 1. *For any $\delta > 0$ there exists a norm on $\mathbb{R}^n$ such that*

$$||A|| < r(A) + \delta.$$

Proof. First we can find a basis in $\mathbb{R}^n$ such that A has Jordan normal form. That is,

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix}$$

where each block is either a Jordan block

$$A = \begin{pmatrix} \lambda & 1 & & & 0 \\ 0 & \lambda & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & \lambda & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

for a real eigenvalue λ or

$$A = \begin{pmatrix} \rho R_\phi & Id & \cdots & 0 \\ & \rho R_\phi & Id & 0 \\ & & \ddots & \\ 0 & & \cdots & \rho R_\phi \end{pmatrix}$$

for a pair of complex eigenvalues $\lambda = \rho e^{i\phi}$ and $\overline{\lambda} = \rho e^{-i\phi}$, where

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$R_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

Now consider a diagonal change of coordinate

$$\begin{pmatrix} 1 & & & 0 \\ & \delta^{-1} & & \\ & & \ddots & \\ 0 & 0 & \cdots & \delta^{-m+1} \end{pmatrix}$$

for a Jordan block for a real eigenvalue and

$$\begin{pmatrix} Id & & & 0 \\ & \delta^{-1} Id & & \\ & & \ddots & \\ 0 & 0 & \cdots & \delta^{-m+1} Id \end{pmatrix}$$

for a pair of complex eigenvalues. Now for the standard Euclidean norm with respect to this new basis, we have that

$$||A|| = \sup_{||v||=1} ||Av|| \leq r(A) + \delta.$$

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