Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
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Yunping Jiang

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Dynamics of Linear Maps and Local Dynamics (continued):

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ are said to be equivalent if there is a constant $C>0$ such that

$$C^{-1}\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1 \quad \forall v \in \mathbb{R}^n.$$

Since all norms in $\mathbb{R}^n$ are equivalent, we have that

Corollary 1. *Given any norm $||\cdot||$ on $\mathbb{R}^n$, for any $\epsilon>0$, there exists a constant $C=C(\epsilon)>0$ such that*

$$||A^n v|| < C(r(A)+\epsilon)^n ||v||.$$

Definition 1. In general, we call a map $f: X \to X$ contracting if there are two constants $C>0$ and $0<\lambda<1$ such that

$$d(f^n(x), f^n(y)) \leq C\lambda^n d(x,y)$$

for all $n>0$ and $x, y \in X$.

Exerice 1. *Prove the Contracting Mapping Principle (Theorem 1) under this more general definition.*

Corollary 2. *If all eigenvalues of A have absolute values less than one, then $A:\mathbb{R}^n \to \mathbb{R}^n$ is contracting with respect to any norm on $\mathbb{R}^n$. And A has a unique fixed point 0 and $A^m v \to 0$ exponentially as $m \to \infty$ for every $v \in \mathbb{R}^n$.*

Exerice 2. *Suppose A is a 2×2 matrix whose all eigenvalues have absolute values less than one. Discuss the dynamics of $A:\mathbb{R}^2 \to \mathbb{R}$ in different cases.*

Definition 2. A linear map $A:\mathbb{R}^n \to \mathbb{R}^n$ is called hyperbolic if all of its eigenvalues have absolute values different from one.

For every real eigenvalue λ, let

$$E_\lambda = \{v \in \mathbb{R}^n \mid (A - \lambda Id)^k v = 0 \quad \text{for some } k\}$$

be its root space. Similarly, for a pair of complex eigenvalues λ and $\overline{\lambda}$, we can consider $A : \mathbb{C}^n \to \mathbb{C}^n$ as a complex map. Then we can define its root space E_λ and $E_{\overline{\lambda}}$. Let

$$E_{\lambda,\overline{\lambda}} = \mathbb{R}^n \cap (E_\lambda \otimes E_{\overline{\lambda}}).$$

Let

$$E^s = E^s(A) = \bigoplus_{|\lambda|<1} E_\lambda \oplus \bigoplus_{|\lambda|<1} E_{\lambda,\overline{\lambda}}$$

and

$$E^u = E^u(A) = \bigoplus_{|\lambda|>1} E_\lambda \oplus \bigoplus_{|\lambda|>1} E_{\lambda,\overline{\lambda}}$$

and

$$E^0 = E^0(A) = \bigoplus_{\lambda=1,-1} E_\lambda \oplus \bigoplus_{|\lambda|=1} E_{\lambda,\overline{\lambda}}.$$

We call E^s, E^u, and E^0 the stable space, the unstable space, and the central space, for A, respectively. Then

$$\mathbb{R}^n = E^s \oplus E^0 \oplus E^u$$

and

$$A(E^s) \subseteq E^s, \quad A(E^0) \subseteq E^0, \quad \text{and} \quad A(E^u) \subseteq E^u.$$

Moreover, $A^m v \to 0$ for $v \in E^s$ and $A^{-m} v \to 0$ for $v \in E^u$ as $m \to +\infty$. We say that A is hyperbolic is equivalent to say that $E^0 = \{0\}$, that is,

$$\mathbb{R}^n = E^s \oplus E^u.$$

Definition 3. Suppose X is an n-dimensional C^1 manifold and $f : X \to X$ is a C^1 map. Suppose p is a periodic point of period $m > 0$ of f. We call p a hyperbolic periodic point if the derivative $Df^m(p) : T_pX \to T_pX$ is a hyperbolic linear map.

Theorem 1. *(Hartman-Grobman Theorem) Suppose $f : X \to X$ is a C^1 map from an n-dimensional C^1 manifold X to itself. Suppose p is a hyperbolic fixed point of f. Then there are neighborhoods U_1 and $U_2 = f^{-1}(U_1)$ of p in X and neighborhoods V_1 and $V_2 = (Df(p))^{-1}(V_1)$ of 0 in $T_pX = \mathbb{R}^n$ and a homeomorphism $h : U_1 \to V_1$ such that $h(U_1 \cap U_2) = V_1 \cap V_2$ and*

$$h \circ f = Df(p) \circ h$$

on $U_1 \cap U_2$. In other words, $f|(U_1 \cap U_2)$ and $Df(p)|(V_1 \cap V_2)$ are conjugate.

The above theorem says that for a hyperbolic fixed point p of f, the local dynamics of f near p is same as the local dynamics of the linear map $Df(p)$ topologically.

Exerice 3. *Prove the Hartman-Grobman Theorem when all eigenvalues of $Df(p) : T_pX \to T_pX$ have absolute values > 1 by using the Contracting Mapping Principle. Can you find another way to prove it under the same assumption?*

Problem 1. *Under what condition, can we promote h to be C^1. More general, under what condition, can we promote h to be C^2, $\cdots$, C^k, or C^ω if f is a C^2. $\cdots$, C^ω. This discussion should divide into cases: real one-dimension and real higher dimension. Similarly, for a complex map, we should discuss it into cases: complex one-dimension and complex higher dimension.*

In one real and complex dimensional cases, we have some kinds of complete solutions, which we will talk this later. However, in higher dimensional cases, it is a difficult problem.

In order to understand a non-hyperbolic linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, we need consider $A : E^0 \to E^0$. This is divided into three cases: (1) $A : E_1 \to E_1$; (2) $A : E_{-1} \to E_{-1}$; (3) $A : E_{\lambda,\bar{\lambda}} \to E_{\lambda,\bar{\lambda}}$. The case (1) is an identity map when A is restricted on the eigenspace and the case (2) is a reflection when A is restricted on the eigenspace. These two cases are trivial. The case (3) is rotation when it is restricted on the eigenspace, that is, the eigenspace $\mathbb{R}^2$ is foliated into circles and A restricted on each circle is a rotation. Thus we only need to discuss the dynamics of A restricted on the unit circle. Let

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

be the unit circle and, if $\lambda = e^{2\pi i\phi}$, then

$$A(z) = e^{2\pi i\phi} z : S^1 \to S^1.$$

If $\phi = p/q$ with $(p, q) = 1$, then every point in S^1 is a periodic point of period q. If we arrange the orbit $\{A^n z\}_{n=0}^{\infty}$ counter-clockwise on S^1 as $z_0 = z$, z_1, $\cdots$, z_{q-1}, then $Az_i = z_{i+p \pmod{q}}$ for all $0 \leq i \leq q-1$.

More interesting dynamics of a rotation is when ϕ is an irrational number.

Definition 4. Suppose X is a topological space and $f : X \to X$ is an invertible map. We call f topologically transitive if there exists a point $x \in X$ such that its orbit $O(x) = \{f^m(x)\}_{m=-\infty}^{\infty}$ is dense in X. It is called minimal if the orbit $O(x)$ is dense in X for every $x \in X$.

A subset $Y \subset X$ is called invariant if $f(Y) \subset Y$. Then the statement that f is minimal is equivalent to say that f has no invariant closed subset.

Theorem 2. *If ϕ is irrational, then $A : S^1 \to S^1$ is minimal.*

Proof. Let $\overline{O}(x) = \overline{\{f^m(x)\}}_{m=-\infty}^{\infty}$ be the closure of the orbit of any point $x \in S^1$. If it is not dense, the complement $S^1 \setminus \overline{O}(x)$ is an non-empty invariant open set, which is a union of open intervals. Let I be one of the longest intervals. Since the rotation preserves the length, we have that $|I| = |A^m I|$ for all m. But $A^k I \cap A^l = \emptyset$ for any $k \neq l$. This is because that if $A^k I \cap A^l I \neq \emptyset$, then $A^k I = A^l I$ (otherwise $A^k I \cup A^l I$ is a longer interval than I in $S^1 \setminus \overline{O}(x)$, contradiction). This implies that $A^{k-l} : I \to I$. With loss of generality, suppose $k > l$. Then there is a point $x \in S^1$ such that $A^{k-l}x = x$. Suppose $x = e^{2\pi i\theta}$. Then $e^{2\pi i[(k-l)\phi+\theta]} = e^{2\pi i\theta}$. Thus

$$[(k-l)\phi + \theta] = \theta + p$$

for some integer p. Therefore, $\phi = p/(k-l)$ is a rational number. This contradicts to our assumption that ϕ is irrational.

Since $\cup_{n=\infty}^{\infty} A^n I \subset S^1$, we have that

$$\infty = \sum_{n=-\infty}^{\infty} |A^n I| \leq |S^1| < \infty.$$

This is a contradiction. The contradiction says that $S^1 \setminus \overline{O}(x)$ must be non-empty. We have proved the theorem. $\square$