Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
Spring Semester of 2009, Friday, 9:30am-11:30am

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Lecture IV, February 20, 2009

Dynamics of Linear Maps and Local Dynamics (continued): Irrational circle rotations is a basic example for the study of dynamics of topological groups. A topological group G is a group with a topology such that each group multiplication by a fixed element in the group is a continuous map from the topological space into itself and the inverse is also a continuous map from the topological space into itself. For example, the unit circle S^1 is an Abelian topological group.

Let $L_{g_0} : G \to G$ be the group multiplication by g_0

$$L_{g_0} g = g_0 g : G \to G.$$

The orbit of the unit element $e \in G$ is cyclic group $\{g_0^n\}_{n=-\infty}^{\infty}$. From Theorem 2 in Lecture 3, S^1 has not proper infinite closed subgroup.

Theorem 1. *If L_{g_0} is topologically transitive, then it is minimal.*

Proof. Since L_{g_0} is topologically transitive, there is a g such that $O(g) = \{g_0^n g\}_{n=-\infty}^{\infty}$ is dense in G. Now for any $g' \in G$, $g_0^n g' = g_0^n g((g^{-1}g')$. Thus the orbit $O(g') = O(g)(g^{-1}g')$. Since $g^{-1}g' : G \to G$ is continuous, $O(g')$ is dense in G. □

Another example of an Abelian topological groups is a torus

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n} = \{w = (z_1, \cdots, z_n) \mid z_i \in S^1\}$$

which plays a central role in the study of completely integrable Hamiltonian systems. The multiplication

$$w \cdot w' = (z_1 z_1', \cdots, z_n z_n)$$

if $w = (z_1, \cdots, z_n)$ and $w' = (z_1', \cdots, z_n')$. The group multiplication by w_0 is

$$L_{w_0} w = w_0 \cdot w.$$

Theorem 2. *Suppose $w_0 = (e^{2\pi i \phi_1}, \cdots, e^{2\pi i \phi_1})$. The group multiplication L_{w_0} is minimal if and only if the number ϕ_1, $\cdots$, ϕ_n, and* 1 *are rationally independent, that is, if* $\sum_{i=1}^{n} k_i\phi_i = 0 \pmod 1$*, then* $k_1 = k_2 = \cdots = k_n = 0$*.*

Exerice 1. *Prove this theorem. You can refer to Katok and Hassenlblatt's book "Introduction to the Modern Theory of Dynamical Systems", Cambridge University, pages 28-31.*

Maps with extremely complicate dynamics but we now understood quite well:

Suppose (X, d) is a metric space. By the definition, X is called compact if every cover $\{U_\alpha\}_{\alpha\in\Lambda}$ by open sets has a sub-cover $\{U_{\alpha_n}\}_{n=1}^{m}$ consisting of finite number of open sets in this cover. Equivalently, X is compact if every sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$. A closed subset of a compact space is compact. The space X is called totally disconnect if every connected component of X contains only one point. A point $x \in X$ is called a limiting point if for every neighborhood U of x, there is a $y \neq x$ in U. Let X' be the set of all limiting points. Then X is called perfect if $X = X'$. Topologically, a Cantor set is a compact, totally disconnected, and perfect metric space.

Suppose $I = [0, 1]$ is the unit interval and $I_0 = [0, a]$ and $I_1 = [b, 1]$ are two subintervals where $0 < a < b < 1$. A C^1 map f defined on $I_0 \cup I_1$ is said to be degree two if $f|I_i$ from I_i to I is bijective for $i = 0$, 1; f is said to be expanding if there are constants $C > 0$ and $\lambda > 1$ so that $|(f^{\circ n})'(x)| \geq C\lambda^n$ whenever $f^{\circ i}(x)$ are in $I_0 \cup I_1$ for all $i = 0$, 1, …, $n - 1$.

Suppose $f : I_0 \cup I_1 \to I$ is a degree two expanding map. Let $G = (a, b)$ be the complement of $I_0 \cup I_1$ in I. A number x in I is said to be escaping to G if $f^{\circ k}(x)$ is in G for some integer $k \geq 0$ (where $f^{\circ 0}$ is the identity). The set $\Omega \subseteq I$ of escaping points is an open subset of the real line. The complement Λ of Ω in I is called the non-escaping set under f. It is a compact (closed and bounded) subset of the real line $\mathbf{R}$. We also call Λ the maximal invariant set of f in $[0, 1]$.

A linear example

Example 1. *($\frac{1}{3}$-Cantor set) . Suppose* $a = 1/3$ *and* $b = 2/3$*. Define*

$$f(x) = \begin{cases} 3x, & if\ 0 \leq x \leq a; \\ 3x - 2, & if\ b \leq x \leq 1 \end{cases}$$

Then f is a degree two expanding map for which the non-escaping set Λ under f is the famous $\frac{1}{3}$-Cantor set.

Proof. Every point $x \in [0,1]$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{i_n}{3^n}$$

where $i_n \in \{0,1,2\}$. Then

$$\Lambda = \{x = \sum_{n=1}^{\infty} \frac{i_n}{3^n} \in [0,1] \mid i_n \in \{0,2\}\}.$$

It is a closed subset. So it is compact.

For any

$$x = \sum_{n=1}^{\infty} \frac{i_n}{3^n}$$

and any $m > 0$, let

$$x_m = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i_m^*}{3^m} + \sum_{n=m+1}^{\infty} \frac{i_n}{3^n},$$

where $i_m^* = 2 - i_m$. Then $x_m \neq x$ and $x_m \to x$ as $m \to \infty$. This says $x \in \Lambda'$. So Λ is perfect.

Suppose $x \neq y \in \Lambda$. Then there is an integer $m > 0$ such that

$$x = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i_m}{3^m} + \sum_{n=m+1}^{\infty} \frac{i_n}{3^n}$$

and

$$y = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{i'_m}{3^m} + \sum_{n=m+1}^{\infty} \frac{i'_n}{3^n}$$

where $i_m \neq i'_m$. Let us assume that $i_m = 0$ and $i'_m = 2$ and define

$$z = \sum_{n=1}^{m-1} \frac{i_n}{3^n} + \frac{1}{3^m} + \sum_{n=m+1}^{\infty} \frac{1}{3^n}.$$

Then

$$U = (-\infty, z) \quad \text{and} \quad V = (z, +\infty)$$

are two open sets. It is clear that $U \cap V = \emptyset$ and $\Lambda \subset U \cup V$ and $x \in U$ and $y \in V$. Therefore, Λ is a Cantor set. □

Theorem 3. *Any two Cantor sets are topologically equivalent.*

Proof. Suppose (X, d) is a Cantor set. Let Λ be the 1/3-Cantor set. We only need to prove that X is homeomorphic to Λ.

Each non-empty open set U of X (respectively, Λ) can be divide into two non-empty open sets U_1 and U_2 such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Therefore, since X and Λ are compact, we can divide

$$X = X_1 \cup \cdots X_n \quad \text{and} \quad \Lambda = \Lambda_1 \cup \cdots \Lambda_n$$

into disjoint non-empty compact sets such that all of them have diameters < 1. Suppose we have divided

$$X = \cup_{i_0 \cdots i_k} X_{i_0 \cdots i_k} \quad \text{and} \quad \Lambda = \cup_{i_0 \cdots i_k} \Lambda_{i_0 \cdots i_k}$$

into disjoint non-empty compact sets such that all of them have diameters $< 1/k$. Then each pair $X_{i_0 \cdots i_k}$ and $\Lambda_{i_0 \cdots i_k}$ can be divided into same number disjoint non-empty compact sets such that all of them have diameters $< 1/(k+1)$, that is,

$$X_{i_0 \cdots i_k} = \cup_{l=1}^{n_{k+1}} X_{i_0 \cdots i_k l} \quad \text{and} \quad \Lambda_{i_0 \cdots i_k} = \cup_{l=1}^{n_{k+1}} \Lambda_{i_0 \cdots i_k l}$$

and $d(X_{i_0 \cdots i_k l}), |\Lambda_{i_0 \cdots i_k l}| < 1/(k+1)$. Thus we have a sequence of nested non-empty compact set

$$\cdots \subset X_{i_0 \cdots i_k i_{k+1}} \subset X_{i_0 \cdots i_k} \subset \cdots \subset X_{i_0} \subset X$$

and

$$\cdots \subset \Lambda_{i_0 \cdots i_k i_{k+1}} \subset \Lambda_{i_0 \cdots i_k} \subset \cdots \subset \Lambda_{i_0} \subset \Lambda.$$

Since $d(X_{i_0 \cdots i_k}), |\Lambda_{i_0 \cdots i_k}| < 1/k$,

$$\cap_{k=1}^{\infty} X_{i_0 \cdots i_k} = \{x_{i_0 \cdots i_k \cdots}\} \quad \text{and} \quad \cap_{k=1}^{\infty} \Lambda_{i_0 \cdots i_k} = \{p_{i_0 \cdots i_k \cdots}\}$$

both contain one point each. Set

$$h(p_{i_0 \cdots i_k \cdots}) = x_{i_0 \cdots i_k \cdots}.$$

Then it is a homeomorphism between X and Λ. $\square$