Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
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Maps with extremely complicate dynamics but we now understood quite well(continued):

The map f in Example 1 is a linear map. In order to study the dynamics of a non-linear map, we need first to study some distortion property.

Naive distortion lemmas:

Let f be a function defined on a set U of the real line $\mathbb{R}$. It is said to be C^1 (or $C^{1+\alpha}$ for $0 < \alpha \leq 1$ or C^{1+bv}) if it can be extended to a differentiable function defined on an open set containing U and if the derivative of the extension is continuous (or is α-Hölder continuous or is of bounded variation).

Suppose f is a C^1 function on a set U of the real line $\mathbb{R}$ and $P_1 = \{x_i\}_{i=1}^n$ and $P_2 = \{y_i\}_{i=1}^n$ are two sequences of points in U. The number

$$\log\Big|\prod_{i=1}^n \frac{f'(x_i)}{f'(y_i)}\Big|$$

is called the distortion of f along P_1 and P_2.

Lemma 1. *Suppose $\kappa = \inf_{x\in U} |f'(x)| > 0$. Then the distortion of f along P_1 and P_2 can be estimated as*

$$\Big|\log\Big|\prod_{i=1}^n \frac{f'(x_i)}{f'(y_i)}\Big|\Big| \leq \frac{1}{\kappa}\sum_{i=1}^n |f'(x_i) - f'(y_i)|$$

Proof. The proof of this lemma is easy because

$$\Big|\log\Big|\prod_{i=1}^n \frac{f'(x_i)}{f'(y_i)}\Big|\Big| \leq \sum_{i=1}^n \Big|\log|f'(x_i)| - \log|f'(y_i)|\Big|$$

$$\leq \frac{1}{\kappa} \sum_{i=1}^{n} |f'(x_i) - f'(y_i)|.$$

□

The next two lemmas are easily derived from Lemma 1

Lemma 2. *($C^{1+\alpha}$-Denjoy distortion lemma) . Suppose f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ and $\kappa = \inf_{x \in U} |f'(x)| > 0$. Let $\iota = \sup_{x \neq y \in U} \big(|f'(x) - f'(y)|/|x-y|^{\alpha}\big) < \infty$. Then the distortion of f along P_1 and P_2 is bounded by $(\iota/\kappa) \sum_{i=1}^{n} |x_i - y_i|^{\alpha}$, that is,*

$$\Big|\log\Big|\prod_{i=1}^{n} \frac{f'(x_i)}{f'(y_i)}\Big|\Big| \leq \frac{\iota}{\kappa} \sum_{i=1}^{n} |x_i - y_i|^{\alpha}$$

Proof. Since $|f'(x_i) - f'(y_i)| \leq \iota |x_i - y_i|^{\alpha}$, it follows directly from Lemma 1. □

Lemma 3. *(C^{1+bv}-Denjoy distortion lemma) . Suppose f is C^{1+bv}. Then there is a constant $C > 0$ so that the distortion of f along P_1 and P_2 is bounded by C, that is,*

$$\Big|\log\Big|\prod_{i=1}^{n} \frac{f'(x_i)}{f'(y_i)}\Big|\Big| \leq C,$$

whenever the open intervals I_i, bounded by x_i and y_i, for $i = 1, \ldots, n$, are pairwise disjoint.

Proof. Let V be the total variation of f on U. Then $\sum_{i=1}^{n} |f'(x_i) - f'(y_i)|$ is bounded by V for $\{I_i\}_{i=1}^{n}$ are pairwise disjoint. We can take $C = V/\kappa$. □

Dynamics of non-linear expanding maps.

Theorem 1. *($C^{1+\alpha}$-hyperbolic Cantor set) If $f : I_0 \cup I_1 \to I$ is a $C^{1+\alpha}$ degree two expanding map for some $0 < \alpha \leq 1$. Then the non-escaping set Λ under f is a Cantor set whose Lebesgue measure is zero.*

Proof. Let f_i be the restriction of the function f to I_i, and $g_i = f_i^{-1} : I \to I_i$ be the inverse of f_i for $i = 0$ or 1. We can consider compositions $g_{w_n} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_n}$ for all strings $w_n = i_0 i_1 \ldots i_n$ of $0's$ and $1's$. These compositions are contracting; this means that there are constants $C > 0$ and $0 < \mu < 1$ so that $|g'_{w_n}(x)| < C\mu^n$ for all x in I.

Suppose $w_n = i_0 i_1 \ldots i_n$ is a string of $0's$ and $1's$. Let $I_{w_n} = g_{w_n}(I)$ be the image of I under g_{w_n}, and let $G_{w_n} = g_{w_n}(G)$ be the set of all the points escaping to G under $g_{w_n}^{-1}$. The union $\cup_{w_n} I_{w_n}$ is the set of all points not escaping to G under the iterates $f^{\circ k}$ for $k = 0, 1, \ldots, n$, where w_n runs over all the strings of $0's$ and $1's$ of length $n+1$. The set $\{I_{w_n}\}$ is a collection of pairwise disjoint closed intervals and one to one correspondence with the set $\{w_n\}$ of all the strings of $0's$ and $1's$ of length $n+1$. Hence $\Lambda = \cap_{n=0}^{\infty} \cup_{w_n} I_{w_n}$, where w_n runs over all the strings of $0's$ and $1's$ of length $n+1$.

Let us first prove that Λ is uncountable. For a string $w_n = i_0 i_1 \ldots i_n$ of $0's$ and $1's$ and a digit $i_{n+1} = 0$ or 1, $I_{w_n i_{n+1}} \subseteq I_{w_n}$ since $I_{i_{n+1}} \subseteq I$. This implies that

$$\cdots \subseteq I_{i_0 i_1 \ldots i_n} \subseteq \cdots \subseteq I_{i_0 i_1} \subseteq I_{i_0}$$

and that $I_w = \cap_{n=0}^{\infty} I_{i_0 i_1 \ldots i_n}$ is a non-empty closed subset for any infinite string $w = i_0 i_1 \ldots$ of $0's$ and $1's$. Hence the set $\{I_w\}$ is a collection of pairwise disjoint non-empty closed subsets and is in one to one correspondence with the uncountable set $\{w = i_0 i_1 \ldots\}$ of all infinite strings of $0's$ and $1's$. Hence the set $\{I_w\}$ is uncountable. So too is the set Λ because $\Lambda = \cup_w I_w$ where $w = i_0 i_1 \ldots$ runs over all infinite strings of $0's$ and $1's$.

Since g_{w_n} is contracting, the length of I_{w_n} is less than $C\mu^n$ for any string $w_n = i_0 i_1 \ldots i_n$ of $0's$ and $1's$ of length $n+1$. This implies that I_w contains a single number x_w, and the map $\pi(w) = x_w$ from $\{w\}$ to Λ is bijective. We use this to prove that Λ is totally disconnected, that is, every (connected) component Π of Λ contains only one number. Suppose there is a component Π of Λ which contains two different numbers x_w and $x_{w'}$ where $w = i_0 i_1 \ldots i_n i_{n+1} \ldots$ and $w' = i_0 i_1 \ldots i_n i'_{n+1} \cdots$ where $i_{n+1} \neq i'_{n+1}$. Both x_w and $x_{w'}$ are in I_{w_n} where $w_n = i_0 i_1 \ldots i_n$. The set I_{w_n} is the union of an open interval G_{w_n} and two closed intervals $I_{w_n i_{n+1}}$ and $I_{w_n i'_{n+1}}$ which are on different sides of G_{w_n}. The numbers x_w and $x_{w'}$ are in $I_{w_n i_{n+1}}$ and $I_{w_n i'_{n+1}}$, respectively. Take a point z in G_{w_n}. Then

$$\Pi = \Big(\Pi \cap (-\infty, z) \Big) \cup \Big(\Pi \cap (z, \infty) \Big).$$

This contradicts the statement that Π is a component of Λ and proves that Λ is totally disconnected.

Since Λ is closed, the set Λ' of limit points of Λ is contained in Λ. To prove that Λ is a perfect set, we only need to show that Λ is contained in Λ'. Let x_w be a number in Λ and $w = i_0 i_1 \ldots i_n i_{n+1} \ldots$. Let $r(i) = i + 1 \pmod 2$; $r(i)$ is 1 for $i = 0$ and 0 for $i = 1$. Take $w^{(n)} = i_0 \ldots i_{n-1} r(i_n) i_{n+1} \ldots$; $w^{(n)}$ differs from w at $(n+1)^{th}$ position. Then $x_{w^{(n)}} \neq x_w$ and both of them are in $I_{i_0 \ldots i_{n-1}}$. Since the length of

$I_{i_0 \ldots i_{n-1}}$ tends to zero as n goes to infinity, $x_{w^{(n)}}$ tends to x_w as n goes to infinity. This says that x_w is a limit point of Λ. So Λ is contained in Λ'. Hence Λ is a Cantor set.

Now let us prove that the Lebesgue measure of the Cantor set Λ is zero. Let $m(\cdot)$ mean the Lebesgue measure and let $|J|$ mean the length of an interval. An inequality which can be easily obtained is

$$m(\Lambda) \leq \sum_{w_n} |I_{w_n}| < C2^{n+1}\mu^n,$$

where w_n runs over all the strings of $0's$ and $1's$ of length $n+1$. This inequality is true because $\{I_{w_n}\}$ is a cover of Λ and the total number of the strings of $0's$ and $1's$ of length $n+1$ is 2^{n+1}. If $\mu < 1/2$, it is much easier to see $m(\Lambda) = 0$. However, to prove that the Lebesgue measure of Λ is zero for any $0 < \mu < 1$, we need help from Lemma 2.

Suppose $w_n = i_0 \ldots i_n$ is a string of $0's$ and $1's$ of length $n+1$. The map f^{n+1} from I_{w_n} to I is a monotone function and its inverse is g_{w_n}. For any two numbers x and y in I_{w_n}, let $x_i = f^{\circ(n-i+1)}(x)$ and $y_i = f^{\circ(n-i+1)}(y)$ for $i = 0, 1, \ldots, n+1$. By the mean value theorem and the chain rule, $|x_i - y_i| < C\mu^i$ and $\sum_{i=0}^{n+1} |x_i - y_i|^\alpha < C/(1-\mu^\alpha)$. According to Lemma 1.2, the distortion of f along $X = \{x_i\}$ and $Y = \{y_i\}$ is bounded by the constant $C' = (\iota/\kappa)\big(C/(1-\mu^\alpha)\big)$, that is,

$$\left|\log\left|\frac{\left(f^{\circ(n+1)}\right)'(x)}{\left(f^{\circ(n+1)}\right)'(y)}\right|\right| \leq C',$$

where ι is the Hölder constant of f' on $I_0 \cup I_1$ and $\kappa = \inf_{x \in I_0 \cup I_1} |f'(x)|$. This implies that

$$\frac{|G_{w_n}|}{|I_{w_n}|} \geq c = e^{-C'}|G|,$$

since $G = f^{\circ(n+1)}(G_{w_n})$ and $I = f^{\circ(n+1)}(I_{w_n})$. Now we have that

$$|I_{w_n0}| + |I_{w_n1}| \leq (1-c)|I_{w_n}|$$

because $I_{w_n} = I_{w_n0} \cup G_{w_n} \cup I_{w_n1}$; moreover,

$$m(\Lambda) \leq \sum_{w_{n+1}} |I_{w_{n+1}}| = \sum_{w_n} (|I_{w_n0}| + |I_{w_n1}|)$$

$$\leq (1-c)\sum_{w_n} |I_{w_n}| \leq \cdots \leq (1-c)^{n+1}$$

for all positive integers n. Hence the Lebesgue measure of Λ is zero. $\square$

Remark and Exercise 1. *From the proof, one can see that the non-escaping set Λ of a C^1 degree two expanding map is a Cantor set in the real line. There is a Cantor set with positive Lebesgue measure. An interesting problem is to construct a C^1 degree two expanding map whose non-escaping set is a Cantor set with positive Lebesgue measure. Try to study this counter-example. You can refer to Bowen's paper "A horseshoe with positive measure." Invent. Math.,* **29** *(1975), 203-204.*