# Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
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### Horseshoe Maps

Let $R$ be a rectangle in $\mathbb{R}^2$ with two horizontal sides and two vertical sides. Suppose $f : R \to f(R) \subset \mathbb{R}^2$ is a diffeomorphism such that $R \cap f(R)$ are two separate rectangles $R_0$ and $R_1$ such that their horizontal sides parallel to the horizontal sides of $R$ and and their vertical sides are parts of the vertical sides of $R$. Let $R^0$ and $R^1$ be two components of $f^{-1}(R) \cap R$. Suppose $R^0$ and $R^1$ are also two rectangles whose vertical sides are parallel to those vertical sides of $R$ and whose horizontal sides are parts of the horizontal sides of $R$. Suppose $f : R^0 \to R_0$ and $f : R^1 \to R_1$ are hyperbolic affine maps, contracting in the vertical direction and expanding in the horizontal direction. This map is called a Smale horseshoe map. Let $\Lambda$ be the maximal invariant set of $f$ in $R$. Then

$$\Lambda = \cap_{n=-\infty}^{\infty} f^n(R).$$

The intersection

$$R \cap f(R) \cap f^2(R) = (R_0 \cap R_1) \cap f^2(R)$$

consists of four rectangles $R_{ij}$ for $i, j \in \{0, 1\}$ whose horizontal sides parallel to the horizontal sides of $R$ and and their vertical sides are parts of the vertical sides of $R$. Inductively, $\cap_{k=0}^{n} f^k(R)$ consists of $2^n$ thin rectangles whose horizontal sides parallel to the horizontal sides of $R$ and and their vertical sides are parts of the vertical sides of $R$. Let us denote these rectangles as $R_{i_0 i_1 \cdots i_{n-1}}$ for $i_k \in 0, 1$, $k = 0, 1, \cdots n-1$. Here we label them as $f(R_{i_0 \cdots i_{n-1}}) = R_{i_1 \cdots i_{n-1}}$. Then we have that

$$R_{i_0 \cdots i_{n-1}} \subset R_{i_0 \cdots i_{n-2}} \subset \cdots \subset R_{i_0}.$$

This implies that

$$R_{i_0 \cdots i_{n-1} \cdots} = \cap_{k=0}^{\infty} R_{i_0 \cdots i_{n-1}}$$

is a horizontal line in $R$. Then

$$R_+ = \cap_{k=0}^{\infty} f^k(R)$$

consists of uncountably many horizontal lines.

Similarly, the intersection

$$R \cap f^{-1}(R) \cap f^{-2}(R) = (R^0 \cap R^1) \cap f^{-2}(R)$$

consists of four rectangles $R^{ij}$ for $i, j \in \{0, 1\}$ whose vertical sides parallel to the vertical sides of $R$ and and their horizontal sides are parts of the horizontal sides of $R$. Inductively, $\cap_{k=1}^{n} f^{-k}(R)$ consists of $2^n$ thin rectangles whose horizontal sides parallel to the horizontal sides of $R$ and and their vertical sides are parts of the vertical sides of $R$. Let us denote these rectangles as $R^{j_{n-1}j_{n-2}\cdots j_1}$ for $j_k \in 0, 1$, $k = 1, \cdots n-1$. Here we label them as $f^{-1}(R_{j_{n-1}\cdots j_2 j_1}) = R_{j_{n-1}\cdots j_2}$. Then we have that

$$R_{j_{n-1}j_{n-2}\cdots j_1} \subset R_{j_{n-2}} \subset \cdots \subset R_{j_1}.$$

This implies that

$$R_{\cdots j_{n-1}\cdots j_1} = \cap_{k=1}^{\infty} R_{j_{n-1}\cdots j_1}$$

is a vertical line in $R$.

$$R_- = \cap_{k=0}^{\infty} f^k(R)$$

consists of uncountably many horizontal lines.

Give any $w_+ = i_0 \cdots i_{n-1} \cdots$, $R_{w_+} \cap R_-$ is a Cantor set on the line $R_{w_+}$. Give any $w_- = \cdots j_{n-1} \cdots j_1$, $R_{w_-} \cap R_+$ is a Cantor set on the line $R_{w_-}$. Then

$$\Lambda = R_+ \cap R_-$$

can be thought as a product of two Cantor set on a horizontal line and on a vertical line.

**Theorem 1.** *Consider the dynamical system $f : \Lambda \to \Lambda$. Then the set of periodic points of $f$ are dense in $\Lambda$. And $Per_n(f)$ contains $2^n$ different points. Furthermore, $f : \Lambda \to \Lambda$ is topologically mixing, that is, for any two open sets $U$ and $V \subset \Lambda$, there exists an integer $N = N(U, V) > 0$ such that $f^n(U) \cap V$ for is nonempty for every $n > N$.*

## One-side full shift on symbolic space.

In one-dimensional expanding maps, we used symbols to label intervals and points. Here we study a symbolic dynamical system. Through the study of symbolic dynamical systems, we can have a better understanding of dynamical systems.

Let $S = \{0, 1, \cdots, d-1\}$ be a set with the discrete topology. Consider the space

$$\Sigma = \Sigma_d = \prod_{k=0}^{\infty} S = \{w = i_0 i_1 \cdots i_{k-1} \cdots \mid i_k \in S,\ k = 0, 1, \cdots\}.$$

We give $\Sigma$ the product topology as follows. Give any

$$w = i_0 i_1 \cdots i_{k-1} \cdots .$$

Define

$$[w]_n = \{w' = i_0 i_1 \cdots i_{n-1} j_n \cdots j_{k-1} \cdots \in \Sigma\}.$$

It is called an $n$-cylinder of $\Sigma$. We call a cylinder an open set. The set of all cylinders forms a topological basis. Then $\Sigma$ with this topology is a topological space.

We can also introduce a metric on $\Sigma$:

$$d(w, w') = \sum_{k=1}^{\infty} \frac{|i_{k-1} - j_{k-1}|}{d^k}$$

if $w = i_0 \cdots i_k \cdots$ and $w' = j_0 \cdots j_{k-1} \cdots$. One can check that $d(\cdot, \cdot)$ is a metric on $\Sigma$ and the topology induced from this metric on $\Sigma$ is the same as the product topology.

One side full shift is defined as

$$\sigma : \Sigma \to \Sigma; \quad \sigma(i_0 i_1 \cdots i_{k-1} \cdots) = i_1 \cdots i_{k-1} \cdots .$$

Then it is a continuous map. The dynamical system $\sigma$ is called a symbolic dynamical system.

Suppose $I = [0, 1]$ and $I_0 = [0, b_0]$, $I_1 = [a_1, b_1]$, $\cdots$, $I_{d-1} = [a_{d-1}, b_{d-1}]$ where

$$a_0 = 0 < b_0 < a_1 < b_1 < \cdots < a_{d-1} < b_{d-1}.$$

Let

$$f : U = \cup_{i=0}^{d-1} \to I$$

be a map such that $f : I_i \to I$ is a $C^1$-diffeomorphism for each $i = 0, \cdots, d-1$. We say $f$ is expanding if there are two constants $C > 0$ and $\lambda > 1$ such that

$$|(f^n)'(x)| \geq C\lambda^n$$

for all $x \in I$ and $n > 0$ such that $f^{n-1}(x)$ is defined. Let

$$\Lambda = \cap_{k=0}^{\infty} f^{-k}(I)$$

be the maximal invariant set of $f$.

**Theorem 2.** *Suppose $f : U \to I$ is a $C^1$ expanding map. Then $f : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \Sigma \to \Sigma$. That is, there is a homeomorphism $h : \Sigma \to \Lambda$ such that*

$$f \circ h = h \circ \sigma$$

*on $\Sigma$.*

*Proof.* For each $0 \leq i \leq d-1$, let $g_i = (f|I_i)^{-1} : I \to I_i$. For each $w_m = i_0 i_1 \cdots i_{m-1}$, define

$$g_{w_m} = g_{i_0} \circ \cdots \circ g_{i_{m-1}}$$

and

$$I_{w_m} = g_{w_m}(I).$$

Then for every $w = w_m \cdots \in \Sigma$,

$$\cdots \subseteq I_{w_m} \subset I_{w_{m-1}} \subset \cdots \subset I_{w_1}.$$

Since $f$ is expanding, $|I_{w_m}| \to 0$ as $m \to \infty$. So

$$I_w = \cap_{m=1}^{\infty} I_{w_m} = \{x_w\}$$

contains only one point. Define $h : \Sigma \to \Lambda$ as $h(w) = x_w$, we have that $h$ is a homeomorphism and

$$f \circ h = h \circ \sigma$$

on $\Sigma$. □

**Lemma 1.** *All periodic points of period $m > 0$ are*

$$Per_m(f) = \{x_w \mid w = (i_0 \cdots i_{m-1})^{\infty}\}.$$

**Exerice 1.** *Prove that $\sigma$ (as well as $f$) is topologically transitive but not minimal.*

**Exerice 2.** *Find all proper closed subsets $A$ contains only finitely many points in $\Sigma$ which is forward invariant, that is, $\sigma(A) \subseteq A$. Find a proper closed subset $A$ contains infinitely many points in $\Sigma$ which is forward invariant.*

**Gibbs Measures**

If a physical system of $n$ states with the energies of these states are $E_1, \cdots, E_n$. Suppose that this system is put in contact with a much larger "heat source" which is at temperature $T$. Energy is therefore allowed to pass between the original system and the heat source. Suppose the temperature $T$ of the heat source remains constant. It is a physical fact derived in statistical mechanics that the probability $p_j$ that state $j$ occurs is given by the Gibbs distribution

$$p_j = \frac{e^{-\beta E_j}}{\sum_{i=1}^{n} e^{-\beta E_i}}$$

where $\beta = \frac{1}{kT}$ and $k$ is a physical constant. This is the starting point for the thermodynamical formalism. However, thermodynamical formalism is a mathematical subject which studies Gibbs measures for more general systems.

Consider the symbolic dynamical system $\sigma : \Sigma \to \Sigma$ (or $f : \Lambda \to \Lambda$). A probability measure $\mu$ on $\Sigma$ is called an invariant measure if

$$\mu(A) = \mu(\sigma^{-1}(A))$$

for any Borel set $A$. A function $\phi : \Sigma \to \mathbb{R}$ is called positive if $\phi > 0$ and Hölder continuous if there are two constant $C > 0$ and $0 < \tau < 1$ such that

$$|\phi(w) - \phi(w')| \leq C\tau^m$$

for any $w, w' \in \Sigma$ such that their first $m$ digits are the same. For a positive and Hölder continuous function $\phi$, the function $\psi = \log \phi$ is also Hölder continuous.

**Theorem 3.** *Consider the system* $(\sigma, \Sigma, \log \phi)$ *where* $\phi$ *is a positive Hölder continuous function. There is a unique probability measure* $\mu = \mu_{\log \phi}$ *such that*

$$C^{-1} \leq \frac{\mu([w_m])}{\exp[-mP + \sum_{k=0}^{m-1} \log \phi(\sigma^k(w))]} \leq C.$$

*where* $C > 0$ *is a constant and where* $[w_m]$ *is any cylinder of* $\Sigma$ *and* $w \in [w_m]$*. Here* $P = P(\log \phi)$ *is a constant depending on* $\log \phi$ *and is called the pressure for the system.*

**Remark 1.** *If we consider* $f : \Lambda \to \Lambda$ *as a* $C^{1+\alpha}$ *expanding map, let* $h : \Sigma \to \Lambda$ *be the conjugacy from* $\Sigma \to \Lambda$*. Consider* $\phi(w) = 1/|f'(h(w))|$*. Then it is a positive Hölder continuous function on* $\Sigma$*. Consider*

$$t \log \phi(w) = -t \log |f'(h(w))|.$$

*Then the pressure* $P(-t \log |f'(h(w))|$ *is a function of* $t$*. There is a unique number* $0 < t_0 < 1$ *such that* $P(-t \log |f'(h(w)|) = 0$*. This number is just the Hausdorff dimension of* $\Lambda$*. For more details about Gibss theory, you can refer to my lecture notes (download site: http://qcpages.qc.cuny.edu/~yjiang/HomePageYJ/Download/JiangNJLectureIFM.pdf) "Nanjing Lecture Notes In Dynamical Systems. Part One: Transfer Operators in Thermodynamical Formalism."*