Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center
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Dynamics of Circle endomorphisms and Teichmüller theory

Circle endomorphisms

Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$. Suppose

$$f : T \to T$$

is an orientation-preserving covering map of degree $d \geq 2$. We call it a circle endomorphism. Suppose

$$h : T \to T$$

is an orientation-preserving homeomorphism. We call it in this paper a circle homeomorphism.

For a circle endomorphism f, it has a fixed point. We will assume that $f(1) = 1$.

The universal cover of T is the real line $\mathbb{R}$ with a covering map

$$\pi(x) = e^{2\pi i x} : \mathbb{R} \to T.$$

Then every circle endomorphism f can be lifted to an orientation-preserving homeomorphism

$$F : \mathbb{R} \to \mathbb{R}, \quad F(x+1) = F(x) + d, \quad \forall x \in \mathbb{R}.$$

We will assume that $F(0) = 0$. Then there is a one-to-one correspondence between f and F. Therefore, we also call such an F a circle endomorphism.

Every orientation-preserving circle homeomorphism h can be lifted to an orientation-preserving homeomorphism

$$H : \mathbb{R} \to \mathbb{R}, \quad H(x+1) = H(x) + 1, \quad \forall x \in \mathbb{R}.$$

We will assume throughout this paper that $0 \leq H(0) < 1$. Then there is a one-to-one correspondence between h and H. Therefore, we also call such an H a circle homeomorphism.

A circle endomorphism f is C^k for $k \geq 1$ if the k^{th}-derivative $F^{(k)}$ exists and is continuous. And, furthermore, it is called $C^{k+\alpha}$ for some $0 < \alpha \leq 1$ if $F^{(k)}$ is α-Hölder continuous, that is,

$$\sup_{x \neq y \in \mathbb{R}} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x-y|^\alpha} = \sup_{x \neq y \in [0,1]} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x-y|^\alpha} < \infty.$$

A C^1 circle endomorphism f is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C\lambda^n, \quad n = 1, 2, \cdots .$$

Definition 1. A circle homeomorphism h is called quasisymmetric if there is a constant $M \geq 1$ such that

$$M^{-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq M, \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

Furthermore, it is called symmetric if there is a bounded function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that

$$\frac{1}{1+\varepsilon(t)} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

Example 1. *A C^1-diffeomorphism of T is symmetric.*

However, the class of symmetric homeomorphisms is larger than the class of C^1-diffeomorphisms. For example, a symmetric homeomorphism may not necessarily be absolutely continuous.

Definition 2. A circle endomorphism f is called uniformly quasisymmetric if there is a constant $M > 0$ such that

$$M^{-1} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq M$$

for all $x \in \mathbb{R}$ and $t > 0$ and any $n > 0$.

Definition 3. A circle endomorphism f is called uniformly symmetric if there is a bounded function $\varepsilon(t) > 0$ for $t > 0$ such that $\varepsilon(t) \to 0^+$ as $t \to 0^+$ and such that

$$\frac{1}{1+\varepsilon(t)} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1 + \varepsilon(t), \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

Example 2. *A $C^{1+\alpha}$, for some $0 < \alpha \leq 1$, circle expanding endomorphism f is uniformly symmetric. Furthermore, $\varepsilon(t) \leq Dt^\alpha$ for some constant $D > 0$ and $0 \leq t \leq 1$.*

Proof. Since $F(x+1)=F(x)+d$, then $F'(x+1)=F'(x)$ is a periodic function. Since F is $C^{1+\alpha}$, we have a constant $C_1>0$ such that

$$|F'(x)-F'(y)|\leq C_1|x-y|^{\alpha}, \quad \forall x,y\in\mathbb{R}.$$

Since F is expanding, we have a constant $C_2>0$ and $\lambda>1$ such that

$$(F^n)'(x)\geq C_2\lambda^n, \quad \forall x\in\mathbb{R}, \;\; n>0.$$

For any $x,y\in\mathbb{R}$ and $n>0$, let $x_k=F^{-k}(x)$ and $y_k=F^{-k}(y)$, $0\leq k\leq n$. Then

$$\Big|\log\frac{(F^{-n})'(x)}{(F^{-n})'(y)}\Big| = \Big|\log\frac{(F^{n})'(y_n)}{(F^{n})'(x_n)}\Big| \leq \sum_{k=1}^{n}|\log F'(x_k)-\log F'(y_k)|$$

$$\leq \frac{1}{C_2\lambda}\sum_{k=1}^{n}|F'(x_k)-F'(y_k)| \leq \frac{C_1}{C_2\lambda}\sum_{k=1}^{n}|x_k-y_y|^{\alpha} \leq \frac{C_1}{C_2^{1+\alpha}\lambda}\sum_{k=1}^{n}\lambda^{-\alpha k}|x-y|^{\alpha}.$$

Let

$$C=\frac{C_1\lambda^{\alpha}}{C_2^{1+\alpha}(\lambda^{\alpha}-1)\lambda}.$$

Then we have the following Hölder distortion property:

$$(1) \qquad e^{-C|x-y|^{\alpha}}\leq\frac{(F^{-n})'(x)}{(F^{-n})'(y)}\leq e^{C|x-y|^{\alpha}}, \quad \forall x,y\in\mathbb{R}, \;\; \forall n>0.$$

Furthermore, let

$$\varepsilon(t)=\begin{cases} e^{Ct^{\alpha}}-1, & 0<t\leq 1,\\ e^{C}-1, & t>1.\end{cases}$$

Then $\varepsilon(t)>0$ is a bounded function such that $\varepsilon(t)\to 0$ as $t\to 0^+$ and such that

$$\frac{1}{1+\varepsilon(t)}\leq\frac{(F^{-n})'(\xi)}{(F^{-n})'(\eta)}=\frac{|F^{-n}(x+t)-F^{-n}(x)|}{|F^{-n}(x)-F^{-n}(x-t)|}\leq 1+\varepsilon(t), \;\; \forall x\in\mathbb{R}, \; \forall t>0,$$

where ξ and η are two numbers in $[0,1]$. Thus F is uniformly symmetric. Furthermore, one can see that $\varepsilon(t)\leq Dt^{\alpha}$ for some constant $D>0$ and $0\leq t\leq 1$. We have proved the example. □

Remark 1. *The uniformly symmetric condition is a weaker condition than the* $C^{1+\alpha}$ *expanding condition for any* $0<\alpha\leq 1$*. For example, a uniformly symmetric circle endomorphism could be totally singular, that is, it could map a set with positive Lebesgue measure to a set with zero Lebesgue measure.*

Another example of a uniformly symmetric circle endomorphism is a C^1 Dini expanding circle endomorphism as follows. Suppose f is a C^1 circle endomorphism. The function

$$\omega(t)=\sup_{|x-y|\leq t}|F'(x)-F'(y)|,\quad t>0,$$

is called the modulus of continuity of F'. Then f is called C^1 Dini if

$$\int_0^1\frac{\omega(t)}{t}dt<\infty.$$

Suppose f is a C^1 Dini expanding circle endomorphism. Let $C>0$ and $\lambda>1$ be two constants such that

$$(F^n)'(x)\geq C\lambda^n,\quad x\in\mathbb{R},\ \ n\geq 1.$$

Define

$$\tilde{\omega}(t)=\sum_{n=1}^{\infty}\omega(C^{-1}\lambda^{-n}t).$$

Then

$$\tilde{\omega}(t)\leq\int_0^{\infty}\omega(C^{-1}\lambda^{-x}t)dx=\frac{1}{\log\lambda}\int_0^{C^{-1}\lambda^{-1}t}\frac{\omega(y)}{y}dy<\infty$$

for all $0\leq t\leq 1$ and $\tilde{\omega}(t)\to 0$ as $t\to 0$.

Example and Exercise 1. *A C^1 Dini circle expanding endomorphism f is uniformly symmetric. Furthermore, $\varepsilon(t)\leq D\tilde{\omega}(t)$ for some constant $D>0$ and $0\leq t\leq 1$. You can refer to my paper (download site:http://qcpages.qc.cuny.edu/ yjiang/HomePageYJ/Download/ 2008TeichAndGibbsTh.pdf) "Teichmller structures and dual geometric Gibbs type measure theory for continuous potentials".*

Symbolic space and topological representation

Suppose f is a circle endomorphism with $f(1)=1$. Consider the preimage $f^{-1}(1)$. Then $f^{-1}(1)$ cuts T into d closed intervals J_0, J_1, $\cdots$, J_{d-1}, ordered by the counter-clockwise order of T. Suppose J_0 has an endpoint 1. Then J_{d-1} also has an endpoint 1. Let

$$\varpi_0=\{J_0,J_1,\cdots,J_{d-1}\}.$$

Then it is a Markov partition, that is,

i. $T=\cup_{k=0}^{d-1}J_k$,

ii. the restriction of f to the interior of J_i is injective for every $0 \leq i \leq d-1$,

iii. $f(J_i) = T$ for every $0 \leq i \leq d-1$.

Let I_0, I_1, $\cdots$, I_{d-1} be the lifts of J_0, J_1, $\cdots$, J_{d-1} in $[0,1]$. Then we have that

i) $[0,1] = \cup_{k=0}^{d-1} I_k$,

ii) $F(I_i) = [i, i+1]$ for every $0 \leq i \leq d-1$.

Let

$$\eta_0 = \{I_0, I_1, \cdots, I_{d-1}\}.$$

Then it is a partition of $[0,1]$.

Consider the pull-back partition $\varpi_n = f^{-n}\varpi_0$ of ϖ_0 by f^n. It contains d^n intervals and is also a Markov partition of T. Intervals J in ϖ_n can be labeled as follows. Let $w_n = i_0 i_1 \cdots i_{n-1}$ be a word of length n of $0's$, $1's$, $\cdots$, and $(d-1)'s$. Then $J_{w_n} \in \varpi_n$ if $f^k(J_{w_n}) \subset J_{i_k}$ for $0 \leq k \leq n-1$. Then

$$\varpi_n = \{J_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots, d-1\}.$$

Let η_n be the corresponding lift partition of ϖ_n in $[0,1]$ with the same labelings. Then

$$\eta_n = \{I_{w_n} \mid w_n = i_0 i_1 \cdots i_{n-1}, i_k \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots, d-1\}.$$

Consider the space

$$\Sigma = \prod_{n=0}^{\infty} \{0, 1, \cdots, d-1\}$$

$$= \{w = i_0 i_1 \cdots i_k \cdots i_{n-1} \cdots \mid i_k \in \{0, 1, \cdots, d-1\},\ k = 0, 1, \cdots\}$$

with the product topology. It is a compact topological space. A cylinder for a fixed word $w_n = i_0 i_1 \cdots i_{n-1}$ of length n is

$$[w_n] = \{w' = i_0 i_1 \cdots i_{n-1} i'_n i'_{n+1} \cdots \mid i'_{n+k} \in \{0, 1, \cdots, d-1\}, k = 0, 1, \cdots\}$$

All left cylinders form a topological basis of Σ. We call it the *left topology.* The space Σ with this left topology is called the *symbolic space.*

For any $w = i_0 i_1 \cdots i_{n-1} i_n \cdots$, let

$$\sigma(w) = i_1 \cdots i_{n-1} i_n \cdots$$

be the shift map. Then (Σ, σ) is called a symbolic dynamical system.

For a point $w = i_0 \cdots i_{n-1} i_n \cdots \in \Sigma$, let $w_n = i_0 \cdots i_{n-1}$. Then

$$\cdots \subset J_{w_n} \subset J_{w_{n-1}} \subset \cdots J_{w_1} \subset T.$$

Since each J_{w_n} is compact,

$$J_w = \cap_{n=1}^{\infty} J_{w_n} \neq \emptyset.$$

If every $J_w = \{x_w\}$ contains only one point, then we define the projection π_f from Σ onto T as

$$\pi_f(w) = x_w.$$

The projection π_f is 1-1 except for a countable set

$$B = \{w = i_0 i_1 \cdots i_{n-1} 1000 \cdots, i_0 i_1 \cdots i_{n-1} 0(d-1)(d-1)(d-1) \cdots\}.$$

From our construction, one can check that

$$\pi_f \circ \sigma(w) = f \circ \pi_f(w), \quad w \in \Sigma.$$

For any interval $I = [a, b]$ in $[0, 1]$, we use $|I| = b - a$ to mean its Lebesgue length. Let

$$\iota_{n,f} = \max_{w_n} |I_{w_n}|,$$

where w_n runs over all words of $\{0, 1, \cdots, d-1\}$ of length n.

Two circle endomorphisms f and g are topologically conjugate if there is an orientation-preserving circle homeomorphism h of T such that

$$f \circ h = h \circ g.$$

The following result is first proved by Shub for C^2 expanding circle endomorphisms 1960's by using the contracting mapping theorem.

Theorem 1. *Let f and g be two circle endomorphisms such that both $\iota_{n,f}$ and $\iota_{n,g}$ tend to zero as $n \to \infty$. Then f and g are topologically conjugate if and only if their topological degrees are the same.*

Proof. Topological conjugacy preserves the topological degree. Thus if f and g are topologically conjugate, then their topological degrees are the same.

Now suppose f and g have the same topological degree. Then they have the same symbolic space. Since both sets $J_{w,f} = \{x_w\}$ and $J_{w,g} = \{y_w\}$ contain only a single point for each w, we can define

$$h(x_w) = y_w.$$

One can check that h is an orientation-preserving homeomorphism with the inverse

$$h^{-1}(y_w) = x_w.$$

□

Therefore, for a fixed degree $d > 1$, there is only one topological model (Σ, σ) for dynamics of all circle endomorphisms of degree d with $\iota_n \to 0$ as $n \to \infty$.

Bounded geometry and uniformly quasisymmetry and quasisymmetric conjugacy

Definition 4. The sequence $\{\varpi_n\}_{n=0}^{\infty}$ of nested partitions of T is said to have bounded nearby geometry if there is a constant $C > 0$ such that for any $n \geq 0$ and any two intervals $I, I' \in \eta_n$ with a same endpoint or one has an endpoint 0 and the other has an endpoint 1 (in which case we say they have a common endpoint by modulo 1),

$$C^{-1} \leq \frac{|I'|}{|I|} \leq C.$$

The sequence $\{\varpi_n\}_{n=0}^{\infty}$ of nested partitions of T is said to have bounded geometry if there is a constant $C > 0$ such that

$$\frac{|L|}{|I|} \geq C, \quad \forall\, L \subset I, \;\; L \in \eta_{n+1}, \; I \in \eta_n, \quad \forall\, n \geq 0.$$

The bounded nearby geometry implies the bounded geometry since each interval $I \in \eta_n$ is divided into d subintervals in η_{n+1}. But it is not true for the other direction.

Theorem 2. *If f is a uniformly quasisymmetric circle endomorphisms, then the sequence $\{\varpi_n\}_{n=0}^{\infty}$ of nested partitions of T has bounded nearby geometry and thus bounded geometry.*

Proof. Let F with $F(0) = 0$ be the lift of f. Define

$$G_k(x) = F^{-1}(x+k) : [0,1] \to [0,1], \quad \text{for} \quad k = 0, 1, \cdots, n-1.$$

For any word $w_n = i_0 i_1 \cdots i_{n-1}$, define

$$G_{w_n} = G_{i_0} \circ G_{i_1} \circ \cdots \circ G_{i_{n-1}}.$$

Then

$$I_{w_n} = G_{w_n}([0,1]) = F^{-n}([m, m+1]),$$

where $m = i_{n-1} + i_{n-2} d + \cdots + i_0 d^{n-1}$. Suppose I'_{w_n} is an interval in η_n having a common endpoint with I_{w_n} modulo 1. Then

$$I'_{w_n} = F^{-n}([m+1, m+2]) \quad \text{or} \quad F^{-n}([m-1, m]).$$

Thus

$$\frac{1}{1+\varepsilon(1)} \leq \frac{|I_{w_n}|}{|I'_{w_n}|} \leq 1+\varepsilon(1).$$

Let $C = 1+\varepsilon(1)$. Then we have that

$$C^{-1} \leq \frac{|I|}{|I'|} \leq C$$

for any intervals $I, I' \in \eta_n$ with a common endpoint modulo 1, $n = 0, 1, \cdots$. This means that $\{\varpi_n\}_{n=0}^{\infty}$ has the bounded nearby geometry. We have proved the theorem. $\square$

Remark 2. *The converse is also true in the above theorem. Refer to the proof of a corollary in the next lecture.*