

# Dynamical Systems and Quasiconformal Mappings:

A Course Given in Department of Mathematics at CUNY Graduate Center  
Spring Semester of 2009, Friday, 9:30am-11:30am

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## Lecture IX, March 27, 2009

**Examination Problem:** Give a detailed proof of the following theorem: Suppose  $f$  is a  $C^{1+Dini}$  circle expanding endomorphism. Then  $f$  is uniformly symmetric with  $\epsilon(t) \leq C\tilde{w}(t)$ .

### Bounded geometry and uniformly quasisymmetry and quasisymmetric conjugacy: (continued)

**Example 1.** Consider  $q_d(z) = z^d$ ,  $d \geq 2$ . Then it is a circle endomorphism of degree  $d$ . The lift of  $q$  is  $Q(x) = dx$ . Then we have

$$\eta_0 = \left\{ \left[ \frac{k}{d}, \frac{k+1}{d} \right] \right\}_{k=0}^{d-1}$$

and

$$\eta_{n-1} = \left\{ \left[ \frac{k}{d^n}, \frac{k+1}{d^n} \right] \right\}_{i=0}^{d^n-1}.$$

Every  $0 \leq k < d^{n+1}$ , can be expressed as

$$k = i_{n-1} + i_{n-2}d + \cdots + i_0d^{n-1}.$$

So for  $w_n = i_0i_1 \cdots i_{n-1}$ ,  $I_{w_n} = [i/d^n, (i+1)/d^n]$ .

**Theorem 1.** Any two uniformly quasisymmetric circle endomorphisms  $f$  and  $g$  of the same degree  $d > 1$  are topologically conjugate and the conjugacy is a quasisymmetric homeomorphism.

*Proof.* From  $f \circ h = h \circ g$  and  $g(1) = 1$ ,  $h(1)$  is a fixed point of  $f$ , that is,  $f(h(1)) = h(1)$ . Let  $k(z) = z/h(1)$  and  $\tilde{f} = k \circ f \circ k^{-1}$ . Then  $\tilde{f}(1) = 1$ . Take  $\tilde{h} = k \circ h$ . We have that  $\tilde{h}(1) = 1$  and  $\tilde{f} \circ \tilde{h} = \tilde{h} \circ g$ . So  $\tilde{h}$  is quasisymmetric if and only if  $h$  is quasisymmetric. So, without loss of generality, we assume that  $h(1) = 1$ .

Suppose

$$\eta_{n,f} = \{I_{w_n,f}\} \quad \text{and} \quad \eta_{n,g} = \{I_{w_n,g}\}, \quad n = 1, 2, \cdots$$

are two sequences of Markov partitions for  $f$  and  $g$ , respectively.

From the bounded geometry property (Theorem ??), we have a constant  $0 < \tau < 1$  such that

$$\iota_{n,f} = \max_{w_n} |I_{w_n,f}|, \quad \iota_{n,g} = \max_{w_n} |I_{w_n,g}| \leq \tau^n, \quad \forall n = 1, 2, \dots$$

Then Theorem ?? implies that  $f$  and  $g$  are topologically conjugate.

Suppose  $h$  is the topological conjugacy between  $f$  and  $g$  and  $H$  is its lift to  $\mathbb{R}$ . By adding all integers, the sequence of partitions  $\eta_{n,f}$  and  $\eta_{n,g}$  induce two sequences of partitions of  $\mathbb{R}$ , which we still denoted as  $\eta_{n,f}$  and  $\eta_{n,g}$ . Both of these sequences of partitions have bounded nearby geometry.

Let  $\Omega$  be the set of all endpoints of intervals  $I \in \eta_n$ ,  $n = 0, 1, \dots, \infty$ . Then it is dense in  $\mathbb{R}$ .

For  $x \in \Omega$ . Consider the interval  $[x-t, x]$ . There is a largest integer  $n \geq 0$  such that there is an interval  $I = [a, x] \in \eta_{n,f}$  satisfying  $[x-t, x] \subseteq I$ . Suppose  $J = [b, x] \in \eta_{n+1,f}$ . Then  $J \subseteq [x-t, x]$ . Let  $J' = [x, c] \in \eta_{n+1,f}$ . From Theorem ??, there is a constant  $C > 0$  such that

$$C^{-1} \leq \frac{|J'|}{|J|} \leq C.$$

If  $|J'| > t$ , we have  $|J'| \leq Ct$ . Let  $J'_k = [x, c_k] \in \eta_{n+k+1,f}$  for  $k > 0$ . From the bounded geometry, there is a  $0 < \tau < 1$  such that

$$|J'_k| \leq \tau^k Ct.$$

Let  $k$  be the smallest integer greater than  $-\log C / \log \tau$ . Then  $|J'_k| \leq t$ . This implies that  $J'_k \subseteq [x, x+t]$ . So we have

$$\frac{|H(J'_k)|}{|H(I)|} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq \frac{|H(J')|}{|H(J)|},$$

where  $H(I) \in \eta_{n,g}$ ,  $H(J), H(J') \in \eta_{n+1,g}$ , and  $H(J'_k) \in \eta_{n+k+1,g}$ . Now from the bounded geometry for  $g$ , we have a constant, still denote as  $C > 0$ , such that

$$C^{-1} \leq \frac{|H(J'_k)|}{|H(I)|} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq \frac{|H(J')|}{|H(J)|} \leq C.$$

If  $|J'| \leq t$ , we have  $|J'| \geq C^{-1}t$ . Let  $J'_{-k} = [x, c_{-k}] \in \eta_{n-k+1,f}$  for  $k \geq 0$ . Then from the bounded geometry, there is a constant, which we still denote as  $0 < \tau < 1$ , such that  $|J'_{-k}| \geq \tau^{-k} C^{-1}t$ . Let  $k$  be the smallest integer greater than  $-\log C / \log \tau$ . Then  $|J'_{-k}| \geq t$ . This implies that  $J'_{-k} \supseteq [x, x+t]$ . So we have

$$\frac{|H(J')|}{|H(I)|} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq \frac{|H(J'_{-k})|}{|H(J)|},$$

where  $H(I) \in \eta_{n,g}$ ,  $H(J)$ ,  $H(J') \in \eta_{n+1,g}$ , and  $H(J'_{-k}) \in \eta_{n-k+1,g}$ . Now from the bounded geometry for  $g$ , we have a constant, which we still denote as  $C > 0$ , such that

$$C^{-1} \leq \frac{|H(J')|}{|H(I)|} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq \frac{|H(J'_{-k})|}{|H(J)|} \leq C.$$

For any  $x \in \mathbb{R}$ , since  $\Omega$  is dense in  $[0, 1]$ , we have a sequence  $x_n \in \Omega$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . For any  $t > 0$ , we have that

$$C^{-1} \leq \frac{|H(x_n+t) - H(x_n)|}{|H(x_n) - H(x_n-t)|} \leq C.$$

Since  $H$  is uniformly continuous on  $\mathbb{R}$ , we get that

$$C^{-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq C.$$

We have proved the theorem.  $\square$

**Corollary 1.** *Suppose  $f$  is a circle endomorphism. If the sequence of nested Markov partitions  $\eta = \{\eta_n\}_{n=0}^{\infty}$  has the bounded geometry. Then  $f$  is uniformly quasisymmetric.*

*Proof.* Suppose  $h$  is a conjugacy from  $f$  to  $q$ , that is,  $f = h \circ q \circ h^{-1}$ . Since both sequences of nested Markov partitions for  $f$  and for  $q$  have bounded nearby and bounded geometry. Then  $h$  is quasisymmetric. Since  $f^n = h \circ q^n \circ h^{-1}$ , so  $f$  is uniformly quasisymmetric.  $\square$