Extendibility of Holomorphic Motions

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This talk is based on the work I collaborated with my former students Michael Beck and Zhe Wang and my colleagues Fred Gardiner and Sudeb Mitra and Hiroshige Shiga.
One of the most important problems in complex dynamical systems is the **Hyperbolicity Conjecture** that says all hyperbolic rational maps (or polynomials) of degree $d \geq 2$ form an open and dense subset in the space of all rational maps (or polynomials) of degree $d$. For example, for the space of quadratic polynomials, $q_c(z) = z^2 + c$, $c \in \mathbb{C}$, this conjecture can be visualized by the Mandelbrot set,

$$M = \{ c \in \mathbb{C} \mid \{ q_c^n(0) \}_{n=0}^{\infty} \text{ is bounded} \}$$
MLC Conjecture: The Mandelbrot is locally connected.
The Hyperbolicity Conjecture and the MLC Conjecture are still largely open. However, Mañe, Sad, Sullivan (1983, On the dynamics of rational maps. Ann. Ec. Norm. Sup., 96:193-217) showed that all structurally stable rational maps (or polynomials) of degree $d \geq 2$ form an open and dense subset in the space of all rational maps (or polynomials) of degree $d$. In their paper, an important concept, holomorphic motions, was introduced.
A General Definition of a Holomorphic Motion

Suppose $V$ is a connected complex manifold with a basepoint $t_0$ and $E$ is a subset of the Riemann sphere $\hat{\mathbb{C}}$. A map

$$h(t, z) : V \times E \to \hat{\mathbb{C}}$$

is called a holomorphic motion of $E$ over $V$ if

i) $h(t_0, z) = z$ for all $z \in E$;

ii) for any fixed $t \in V$, $h_t(\cdot) = h(t, \cdot) : E \to \hat{\mathbb{C}}$ is injective;

iii) for any fixed $z \in E$, $h^z(\cdot) = h(\cdot, z) : V \to \hat{\mathbb{C}}$ is holomorphic.
\(\lambda\)-Lemma

- \(h(t, z)\) of \(E\) over \(V\) can be extended to a holomorphic motion \(\overline{h}(t, z)\) of \(\overline{E}\) over \(V\).

- \(\overline{h}(t, z) : V \times \overline{E} \to \widehat{\mathbb{C}}\) is a continuous map.

- \(\overline{h}_t(\cdot) = \overline{h}(t, \cdot) : \overline{E} \to \widehat{\mathbb{C}}\) has some “quasiconformal property” for any given \(t \in V\).
Without loss of generality, we always assume that $E$ is closed and contains $\{0, 1, \infty\}$ and that $h(t, 0) = 0$, $h(t, 1) = 1$, and $h(t, \infty) = \infty$ for all $t \in V$. 
Given a holomorphic motion $h$ of $E$ over $V$, can we extend it to a holomorphic motion $H$ of the Riemann sphere $\hat{\mathbb{C}}$ over $V$?

If the answer is yes, then we call $h$ fully extendable.

Any holomorphic motion $h$ of $E$ over $\Delta$ is fully extendable (Slodkowski’s Theorem).

After Slodkowski’s proof, several people tried to give other proofs in order to have a clear and deep understandings of the extension problem with Teichmüller theory, dynamical systems, several complex variables, and partial differential equations.

There is a counter-example of a holomorphic motion $h$ of $E$ over a simply connected higher-dimensional complex manifold $V$ such that $h$ is NOT fully extendable, see, for example, J-Mitra (2006, Some applications of universal holomorphic motions. Kodai Math J., Vol. 30, No. 1, 85-96).
Example 1

Douady constructed a counter-example of a holomorphic motion \( h \) of a four-point subset \( E = \{0, 1, \infty, t_0\} \) over the thrice-punctured sphere \( \mathbb{C}_{0,1} \) with a basepoint \( t_0 \),

\[
h_1(t, z) = \begin{cases} 
    z & \text{if } z = 0, 1, \infty \text{ and } t \in \mathbb{C}_{0,1}; \\
    t & \text{if } z = t_0 \text{ and } t \in \mathbb{C}_{0,1}.
\end{cases}
\]

Earle modified this example to get a counter-example of \( h_2 \) of a four-point subset \( E = \{0, 1, \infty, t_0\} \) over an annulus \( A \) with a basepoint \( t_0 \), see Earle (1997, Some maximal holomorphic motions, Contemp. Math., 211, 183-192). A maximal property is used to show that \( h_1 \) (or \( h_2 \)) is NOT fully extendable.
Example 2

A different type counter-example of a holomorphic motion $h$ of a five-point set $E = \{0, 1, \infty, a = -t_0 + 2i, b = t_0 + 2i\}$ over the punctured unit disk $\Delta^*$ with a basepoint $t_0$,

$$
\begin{align*}
    h_3(t, z) &= \begin{cases} 
        z & \text{if } z = 0, 1, \infty \text{ and } t \in \Delta^*; \\
        -t + 2i & \text{if } z = a \text{ and } t \in \Delta^*; \\
        t + 2i & \text{if } z = b \text{ and } t \in \Delta^*. 
    \end{cases}
\end{align*}
$$

In Chirka’s paper (2004), a problem has been raised about whether
the zero winding number condition is a necessary and sufficient
condition for the extension problem: Suppose \( h \) is a holomorphic
motion of \( E \) over a hyperbolic Riemann surface \( X \) with a basepoint
and suppose \( \alpha(\theta) : [0, 1] \rightarrow X \) is a simple closed curve and
\( z_1 \neq z_2 \in E \) is a pair of points, then

\[
\eta(\alpha, z_1, z_2) = \frac{1}{2\pi} \oint_{\alpha} d\arg \delta(\cdot, z_1, z_2)
\]

is the winding number of the closed curve
\( \delta(\alpha, z_1, z_2) = h(\alpha, z_1) - h(\alpha, z_2) \).

We say that \( h \) satisfies the zero winding number condition if
\( \eta(\alpha, z_1, z_2) = 0 \) for all \( \alpha \) and all pairs \( z_1 \neq z_2 \in E \).
Example 3

An example of a holomorphic motion of a four-point subset $E = \{0, 1, \infty, t_0\}$ over an annulus $X = \{t_0/R < |z| < t_0R\}$, where $R > 1$ and $t_0 > 0$ and both $R - 1$ and $z_0 = 1/t_0$ are small numbers, with a basepoint $t_0$,

$$h_4(t, z) = \begin{cases} 
  z & \text{if } z = 0, 1, \infty \text{ and } t \in X; \\
  t_0 \left( z_0 t \frac{z_0 t - a_n}{1 - a_n z_0 t} \right)^n & \text{if } z = t_0 \text{ and } t \in X.
\end{cases} \quad (1)$$

Then $h_4$ is a holomorphic motion of $E$ over $X$ satisfying the zero winding number condition and fully extendable.
Example 4

In my paper (Winding numbers and full extendibility in holomorphic motions. Conformal Geometry and Dynamics, AMS, May 26, 2020, Vol. 24, 109-117), we have the following counter-example. We have an annulus $A$ such that $-2, 0, 1/2, 1/3, i, -i \not\in A$. Let $E = \{0, 1, 2, 4, \infty\} \subset \hat{\mathbb{C}}$ (a five-point set). Define

$$\phi(t, z) = \begin{cases} 
  z & \text{if } z = 0, 1, \infty \text{ and } t \in A; \\
  -\frac{1}{t} + 3 & \text{if } z = 2 \text{ and } t \in A; \\
  t + 3 & \text{if } z = 4 \text{ and } t \in A. 
\end{cases} \quad (2)$$

The map $h(t, z) : A \times E \to \hat{\mathbb{C}}$ is a holomorphic motion and satisfies the zero winding number condition but can not be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over $A$. 
The zero winding number condition is NOT a sufficient condition for the extension problem. Then it is a good problem to find a necessary and sufficient topological condition for the extension problem.
Let

$$M(\mathbb{C}) = \{ \mu \in L^\infty(\mathbb{C}) \mid \| \mu \|_\infty < 1 \}$$

For any \( \mu \in M(\mathbb{C}) \), the Beltrami equation \( w_z = \mu w_z \) always has a solution \( w \) which is a \( K \)-quasiconformal homeomorphism of \( \hat{\mathbb{C}} \) for \( K = (1 + \| \mu \|_\infty)/(1 - \| \mu \|_\infty) \). Moreover, if we consider the normalized solution \( w^\mu \) fixing 0, 1, \( \infty \), then \( w^\mu \) is unique and depends on \( \mu \) holomorphically.
Let $E$ be a closed subset of $\hat{\mathbb{C}}$. Suppose $0, 1, \infty \in E$. We say $\mu, \nu \in M(\mathbb{C})$ are $E$-equivalent if $(w^{\nu})^{-1} \circ w^{\mu}$ is isotopic to the identity rel $E$. The space of all $E$-equivalence classes

$$T(E) = \{[\mu]_E | \mu \in M(\mathbb{C})\}$$

is called the Teichmüller space of the closed subset $E$. It is a simply connected complex Banach manifold such that the projection

$$P_E(\mu) = [\mu]_E : M(\mathbb{C}) \to T(E)$$

Universal Holomorphic Motion

Let

\[ \Psi_E(t, z) = w^\mu(z) : T(E) \times E \to \widehat{\mathbb{C}}, \quad t = P_E(\mu), \ \mu \in M(\mathbb{C}). \]

It defines a holomorphic motion of \( E \) over \( T(E) \) with the basepoint \([0]_E\). The reason people call it \textit{universal} is that when \( V \) is a simply connected complex Banach manifold with a basepoint, any holomorphic motion \( h(t, z) : V \times E \to \widehat{\mathbb{C}} \) can be treated as the pullback \( f^*(\Psi_E)(t, z) \) of \( \Psi_E \) for a unique basepoint preserving holomorphic map \( f : V \to T(E) \), that is,

\[ h(t, z) = f^*(\Psi_E)(t, z) := \Psi(f(t), z). \]

Holomorphic Map Problem

In general, for a connected complex Banach manifold $V$ with a basepoint and for a given holomorphic motion $h$ of $E$ over $V$, when can we find a basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*(\Psi_E) = h$?
Lifting Problem

For a connected complex Banach manifold $V$ with a basepoint and a given basepoint preserving holomorphic map $f : V \to T(E)$, when can we find a basepoint preserving holomorphic map $\tilde{f} : V \to M(\mathbb{C})$ such that $P_E \circ \tilde{f} = f$?
An affirmative answer to the Holomorphic Map Problem and an affirmative answer to the Lifting Problem give an affirmative answer to the Extension Problem.
The answer to the lifting problem for any basepoint preserving holomorphic map $f : \Delta \to T(E)$ is affirmative, as proved in J-Mitra-Wang (2009, Liftings of holomorphic maps into Teichmüller spaces, Kodai Mathematical Journal, 32 (2009), No. 3, 544-560). This implies Slodkowski’s theorem. Furthermore, Slodkowski’s theorem and the universal holomorphic motion property implies the lifting theorem.
For a non-simply connected hyperbolic Riemann surface with a basepoint, the study of the Holomorphic Map Problem and the study of the Lifting Problem are different. We need to introduce some topological and geometric concepts.

Suppose $X$ is a hyperbolic Riemann surface with a basepoint $t_0$ and $\{0, 1, \infty\} \subset E$ is a subset of the Riemann sphere $\mathbb{C}$. Suppose $h$ is a (normalized) holomorphic motion of $E$ over $X$. 
Let $\pi_1(X, t_0)$ denote the fundamental group of $X$. Then for any $z \neq 0, 1, \infty \in E$, $h^z(\cdot) = h(\cdot, z) : X \to \mathbb{C}_{0,1}$ is a holomorphic map. It induces a homomorphism $\rho_{h,z} : \pi_1(X, t_0) \to \pi_1(\mathbb{C}_{0,1})$.

We say that $h$ satisfies the trivial trace-monodomy condition if $\rho_{h,z}$ is trivial for all $z \neq 0, 1, \infty \in E$. 
Suppose $\pi : \Delta \to X$, $\pi(0) = t_0$, is the holomorphic universal cover with the group of deck transformations $\Gamma$. Consider the pullback holomorphic motion $H = \pi^*(h)$ of $E$ over $\Delta$. Since $\Delta$ is simply connected, there exists a basepoint preserving holomorphic map $f : \Delta \to T(E)$ such that $H = f^*(\Psi_E)$. For any $c \in \pi_1(X, t_0)$, let $\beta$ be the representation of $c$ in $\Gamma$. Then the map $w^\mu$ for any $P_E(\mu) = (f \circ \beta)(0)$ fixes each point in $E$. Thus it is a quasiconformal self-map of the hyperbolic Riemann surface $X_E := \hat{\mathbb{C}} \setminus E$. Therefore, it represents a mapping class $[w^\mu]$ of $X_E$. 
When $E'$ contains $n$ points, we use $\text{Mod}(0, n)$ to denote the mapping class group of the $n$-times punctured sphere $X_{E'}$. Then we have a homomorphism $\rho_{E'} : \pi_1(X, t_0) \to \text{Mod}(0, n)$ given by $\rho_{E'}(c) = [w^\mu] \in \text{Mod}(0, n)$. We say that $\rho_{E'}$ is trivial if $\rho_{E'}(c) = [\text{Id}]$ for all $c \in \pi_1(X, t_0)$.

We say $h$ satisfies the trivial monodromy condition if the homomorphism $\rho_{E'}$ is trivial for any finite subset $\{0, 1, \infty\} \subset E' \subset E$. 
Trivial and Non-Trivial Monodromy

A punctured Riemann sphere

Trivial Monodromy

Non-Trivial Monodromy

Figure: The closed curve $\gamma$. Yunping Jiang
Suppose $V$ is a connected complex Banach manifold with a base point and $E$ is a closed subset of $\hat{\mathbb{C}}$. Suppose $h$ is a holomorphic motion of $E$ over $V$. If the restriction $h|_{V \times E'}$ is fully extendable for any finite subset $\{0, 1, \infty\} \subset E' \subset E$, then $h$ itself is fully extendable, see Beck-J-Mitra-Shiga (2012, Extending holomorphic motions and monodromy. Annales Academiæ Scientiarum Fennicæ Mathematica, Vol. 37, 53-67) (Riemann surfaces case) and Beck-J-Mitra (2012, Normal families and holomorphic motions over infinite dimensional parameter spaces. Contemporary Mathematics, AMS, Vol. 573, 2012, 1-10) (higher or infinite-dimensional case).
Monodromy of Example 1

A four-punctured Riemann sphere

Example 1 has non zero winding number, non-trivial trace monodromy (as well as non-trivial monodromy), thus it is not fully extendable.

A pure braid of four threads in Example 3

Example 1 has non zero winding number, non-trivial trace monodromy (as well as non-trivial monodromy), thus it is not fully extendable.
Suppose $E$ contains only 4 points. Then any holomorphic motion $h$ of $E$ over any hyperbolic Riemann surface with a basepoint $t_0$ is fully extendable if and only if it satisfies the trivial trace-monodromy condition, see Beck-J-Mitra-Shiga (2012, Extending holomorphic motions and monodromy. Annales Academiæ Scientiarum Fennicæ Mathematica, Vol. 37, 53-67).
Example 2 has trivial trace-monodromy, non-zero winding number, and non-trivial monodromy, thus it is not fully extendable. The four-point subset theorem cannot be generalized to a holomorphic motion of more than four points over a non-simply connected Riemann surface.

A five-punched Riemann sphere

The pure braid of five threads given in Example 3
Monodromy of Example 3

Example 3 has zero winding number and trivial trace monodromy and trivial monodromy, thus it is fully extendable.

A four-punched Riemann sphere

The pure braid of four threads given in Example 3
The pure braid of five threads given in Example 4
A five-punctured Riemann sphere
Example 4 has zero winding number but non-trivial monodromy, thus it is not fully extendable
Suppose \( h \) is a holomorphic motion of \( E \) over a hyperbolic Riemann surface \( X \) with a basepoint \( t_0 \) satisfying the trivial monodromy condition. Then for any finite subset \( \{0, 1, \infty\} \subset E' \subset E \), the answer to the Holomorphic Map Problem is affirmative, that is we have a basepoint preserving holomorphic map \( f_{E'} : X \to T(E') \) such that \( f_{E'}^*(\Psi_{E'}) = h|X \times E' \), as proved in J-Mitra (2018, Monodromy, liftings of holomorphic maps, and extensions of holomorphic motions. Conformal Geometry and Dynamics, Volume 22 (2018), 333–344).
Suppose $h$ is a holomorphic motion of $E$ over a hyperbolic Riemann surface $X$ with a basepoint $t_0$ satisfying the trivial monodromy condition. Then for any finite subset $\{0, 1, \infty\} \subset E' \subset E$, we have a basepoint preserving holomorphic map $f_{E'} : X \to T(E')$ such that $f_{E'}^*(\Psi_{E'}) = h|X \times E'$. Moreover, the answer to the Lifting Problem for $f_{E'} : X \to T(E')$ is affirmative, that is, we have a holomorphic map $\tilde{f}_{E'} : X \to M(\mathbb{C})$ such that $f_{E'} = P_{E'} \circ \tilde{f}_{E'}$, as proved in J-Mitra (2018, Monodromy, liftings of holomorphic maps, and extensions of holomorphic motions. Conformal Geometry and Dynamics, Volume 22 (2018), 333–344).
Suppose \( h \) is a holomorphic motion of \( E \) over a hyperbolic Riemann surface \( X \) with a basepoint \( t_0 \). Then the trivial monodromy condition is a necessary and sufficient condition that \( h \) extends to a holomorphic motion of \( \hat{\mathbb{C}} \) over \( X \). In other words, it is indeed a necessary and sufficient topological condition for the Extension Problem, see J-Mitra (2018, Monodromy, liftings of holomorphic maps, and extensions of holomorphic motions. Conformal Geometry and Dynamics, Volume 22 (2018), 333–344).
Take any finite subset \( \{0, 1, \infty\} \subset E' \subseteq E \).

Suppose \( \pi : \Delta \to X \) is the holomorphic universal cover such that \( \pi(0) = t_0 \). Let \( \Gamma \) be the corresponding group of deck transformations of \( X \). Consider \( h' = h|_{X \times E'} \). Then \( H' = \pi^*(h') \) is a holomorphic motion of \( E' \) over \( \Delta \). Since \( \Delta \) is simply connected, we can apply the universal property of \( \Psi_{E'} : T(E') \times E' \to \hat{\mathbb{C}} \) to get a holomorphic map \( f_0 : \Delta \to T(E') \) such that \( H' = f_0^*(\Psi_{E'}) \). The trivial monodromy condition is equivalent to say that \( f_0 \circ \gamma = f_0 \) for any \( \gamma \in \Gamma \). Thus we have an induced holomorphic map \( f = f_0/\Gamma : X \to T(E') \).
Proof of the Lifting Theorem, I

Suppose $E_n = \{0, 1, \infty, p_1, \cdots, p_n\}$ is a set of $n + 3$ points and $E_{n+1} = E_n \cup \{p_{n+1}\}$ for $p_{n+1} \notin E_n$. The key point in the whole proof is to find a holomorphic map $\tilde{f}_n : X \to T(E_{n+1})$ such that $P_{E_n, E_{n+1}} \circ \tilde{f}_n = f_n$, where $f_n$ is the the holomorphic map constructed in the previous slide for $E_n = E'$ and $P_{E_n, E_{n+1}} : T(E_{n+1}) \to T(E_n)$ is the forgetful map.

\[
\begin{array}{ccc}
X & \xrightarrow{f_n} & T(E_n) \\
\downarrow \quad \tilde{f}_n & \quad & \downarrow P_{E_n, E_{n+1}} \\
T(E_{n+1}) & & \\
\end{array}
\]
Let

\[ Y_n = \{ c_n = (z_1, \cdots, z_n) \mid z_i \neq 0, 1 \in \mathbb{C}, z_i \neq z_j, 0 \leq i < j \leq n \} \]

and, similarly, \( Y_{n+1} = \{ c_{n+1} \} \). Then we have a natural projection \( P(c_{n+1}) = c_n \). The maps

\[
\pi_n([\mu]_{E_n}) = (w^\mu(p_1), \cdots, w^\mu(p_n)) : T(E_n) \to Y_n
\]

and, similarly, \( \pi_{n+1}([\mu]_{E_{n+1}}) : T(E_{n+1}) \to Y_{n+1} \) are holomorphic universal covers, see Nag (1981, The Torelli spaces of punctured tori and spheres. Duke Math. J. 48, 359-388).
The map
\[ \hat{f}_n(t) = \pi_n \circ f_n(t) = (w^\mu(p_1), \ldots, w^\mu(p_n)) : X \to Y_n, \quad P_{E_n}(\mu) = f_n(t) \]
is holomorphic. Now finding a lifting map \( \tilde{f}_n \) for \( f_n \) is equivalent to finding \( \tilde{f}_n : X \to Y_{n+1} \) such that \( P \circ \tilde{f}_n = \hat{f}_n \) and it is equivalent finding the last component \( g_{n+1}(t) : X \to \mathbb{C} \) of \( \tilde{f}_n(t) \).
For $\Delta$, in J-Mitra-Wang (2009, Liftings of holomorphic maps into Teichmüller spaces, Kodai Mathematical Journal, 32 (2009), No. 3, 544-560), we have constructed a bounded operator

$$\mathcal{P} = \mathcal{P}(w^\mu(p_1), \cdots, w^\mu(p_n)) : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$$

such that it has a unique fixed point, that is,

$$g_{n+1} = \mathcal{P}(g_{n+1}).$$

Thus, for a general hyperbolic surface $X$, we can find $g_{n+1,0} : \Delta \to Y_{n+1}$ for $f_{n,0} = \hat{f}_n \circ \pi : \Delta \to Y_n$. 
Now we just need to check if $g_{n+1,0}$ is invariant under $\Gamma$. For any $\gamma \in \Gamma$, we have also

$$g_{n+1,0} \circ \gamma = (\mathcal{P} \circ \gamma)(g_{n+1,0} \circ \gamma)$$

where $\mathcal{P} \circ \gamma = \mathcal{P}(w^\mu \circ \gamma(p_1), \ldots, w^\mu \circ \gamma(p_n))$. When $h$ satisfies the trivial monodromy condition, we have $\mathcal{P} \circ \gamma = \mathcal{P}$. Thus we have that

$$g_{n+1,0} \circ \gamma = \mathcal{P}(g_{n+1,0} \circ \gamma).$$

Since the fixed point of $\mathcal{P}$ is unique, $g_{n+1,0} = g_{n+1,0} \circ \gamma$. Thus, we find the last component

$$g_{n+1} = g_{n+1,0}/\Gamma : X \to \mathbb{C}$$

for $\tilde{f}_{\mu}^n$. 

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In Gardiner-J-Wang (2015, Guiding isotopies and holomorphic motions. Annales Academiæ Scientiarum Fennicæ Mathematica Volumen 40, 485-501), a geometric concept called the guiding isotopy condition was introduced: A holomorphic motion $h$ of $E$ over a hyperbolic Riemann surface $X$ with a basepoint is said to satisfy the guiding isotopy condition if for any finite subset $\{0, 1, \infty\} \subset E' \subset E$, $h' = h|_{X \times E'}$ can be extended to a quasiconformal motion $H'$ of the Riemann sphere $\mathbb{C}$ over $X$. 
From Mitra (2007, Extensions of holomorphic motions. Israel Journal of Mathematics, 159, 277-288), we knew that for a holomorphic motion $\phi$ of $E$ over a connected complex Banach manifold $V$, the following statements are equivalent: (1) it can be extended to a continuous motion of $\hat{\mathbb{C}}$ over $V$; (2) it can be extended to a quasiconformal motion of $\hat{\mathbb{C}}$ over $V$; and (3) there exists a basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*(\Psi_E) = \phi$.

Thus in the definition of the guiding isotopy condition, we can replace quasiconformal motion by continuous motion.

The guiding isotopy condition implies the trivial monodromy condition.
Thus we completed this research program by the following theorem:

The zero winding number condition, the trivial trace-monodromy condition, the trivial monodromy condition, and the guiding isotopy condition are all necessary for the affirmative answer to the Extension Problem. However, only the trivial monodromy condition and the guiding isotopy condition are, indeed, sufficient.
Thanks for Your Listening!