Dynamical Teichmüller Spaces

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$$\mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$$, the unit circle in \( \mathbb{C} \). \\
\( \mathbb{R} \) is the real line.

\[ \pi(x) = e^{2\pi i x} \]

\[ H, \text{ an increasing function, } H(x + t) = H(x) + H(t) \]

\[ H(0) = 0 \]

\[ h, \text{ an orientation-preserving homeomorphism of } \mathbb{T} \]

\[ h(1) = 1 \]

\[ \pi \circ h = h \circ \pi, \text{ lifting} \]

We call \( h \) or \( H \) a circle homeomorphism.

Let \[ E_h(t) = \sup_{x \in \mathbb{R}} \left| \log \left| \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \right| \right|, \quad t > 0, \]

be the quasi-symmetric distortion function for \( h \).

Then \( h \) is quasi-symmetric if \( \sup_{t > 0} E_h(t) = M < \infty \).

Furthermore, \( h \) is symmetric if \( E_h(t) \to 0^+ \text{ as } t \to 0^+ \).

If \( h \) is a C\(^1\)-diffeomorphism, then \( h \) is symmetric.

and we have the modulus of continuity

\[ W_h(t) = \sup_{x, y \in \mathbb{R}} \left| \log \frac{H(x) - H(y)}{H(x+t) - H(y+t)} \right| \]

\[ 1 \leq t \]

\[ x, y \in \mathbb{R} \].
In this case \( \varphi(x) \leq C \varphi(x) \) for some constant \( C > 0 \). We say \( \phi \) is \( C^{1+\alpha} \) if \( \varphi(x) \leq C x^\alpha \) for some \( 0 < \alpha < 1 \) and some constant \( C > 0 \).

\[
\begin{align*}
Q &= \{ h \mid h \text{ is quasisymmetric} \} \\
S &= \{ h \mid h \text{ is symmetric} \} \\
C_{1+\alpha} &= \{ h \mid h \text{ is } C^{1+\alpha} \text{ for some } 0 < \alpha < 1 \}
\end{align*}
\]

- universal Teichmüller space \( TQ = \mathbb{R}/M \)
- universal symmetric Teichmüller space \( TS = S/M \)
- universal smooth Teichmüller space \( TC_{1+\alpha} = C_{1+\alpha}/M \)

where \( M = \) the space of all Möbius transformations preserving the unit disk in \( \mathbb{C} \).

\[
\begin{align*}
TQ &= \{ h \in Q \mid h(0) = 1, h(i) = 0, h(-1) = -1 \} \\
TS &= \{ h \in S \mid h(0) = 1, h(i) = i, h(-1) = -1 \} \\
TC_{1+\alpha} &= \{ h \in C_{1+\alpha} \mid h(0) = 1, h(i) = i, h(-1) = -1 \}
\end{align*}
\]
K-metric means the Kobayashi metric
T-metric means the Teichmüller metric

1) On $TQ$, $K$-metric = $T$-metric
A simple proof is to use holomorphic motions.

2) On $TS$, $K$-metric = $T$-metric
We expect a holomorphic motion proof.

3) On $T C^4 H$, $K$-metric = $T$-metric
We will talk 2) and 3) in the next talk.

Note that all of them are complex Banach manifolds through Bers embedding.

A more important Teichmüller space relating to dynamics is $AT = Q/S$, the asymptotical conformal universal Teichmüller space.

Conjecture: On $AT$, $K$-metric = $T$-metric

Note: each point in $AT$ is an equivalent class $\{ \phi(\mathbf{h}) | \mathbf{h} \in \mathbf{S} \}$
We are interested in two subspaces called dynamical Teichmüller spaces of $AT$.
consider \( f(\tau) = \tau^2: \Pi \rightarrow \Pi \), \( \varphi(x) = 2x \)

we identify \( \Pi \) with \([0,1]/\text{mod } 1\), then

\( f(x) = 2x \mod 1 \) (In general, we can consider \( f_d(x) = x^d \) for \( d \geq 2 \)).

\[ g_f(0) = 0 \text{ is a fixed point} \]

\[ g_f^{-1}(0) = \{ 0, \frac{1}{2} \} \]

\[ g_f^{-2}(0) = \{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \} \]

\[ g_f^{-3}(0) = \{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 = \frac{5}{4} \} \]

\[ 0 = 0.2 + 0, \quad 1 = 0.2 + 1, \quad 2 = 1.2 + 0, \quad 3 = 1.2 + 1. \]

In general \( g_f^{-n}(0) \) cuts \([0,1] \) into \( 2^n \) equal-sized intervals \( I_n \), \( k = 0, 1, \ldots, 2^n - 1 \)

\[ k = c_0 2^{n-1} + c_1 2^{n-2} + \ldots + c_{n-2} 2 + c_{n-1} \]

\[ \sum_{n}^{+} = \sum_{n}^{+} \omega_n = c_0, c_1, \ldots, c_{n-1} | \forall k = 0, 1 \}

\[ \prod_{\text{discrete}} \sum_{\text{topology}} \]

\[ \begin{array}{c}
\end{array} \]
\[ \Sigma^+ \rightarrow \Sigma^+ = \sum c^+ \quad \text{with} \quad \sum c^+ \rightarrow \sum^+ \quad \text{discrete topology} \]

\[ \Sigma^+ \text{ with the product topology is a compact metric space. The standard metric (Lebesgue metric)} \]
\[ c^+_2(w, w') = \sum_{k=1}^{\infty} \frac{|c_{k-1} - c'_{k-1}|}{2^k} \]

Then one can check \( I_{n, k} = I_{n, n+1} \) and \( I_{n, k} = I_{n+1, 2k} \cup I_{n+1, 2k+1} = I_{n+1} o U I_{n+1} \).

\[ \forall w = c_0 c_1 \cdots c_{n-1} \cdots, \quad w_n = c_0 c_1 \cdots c_{n-1} \]
\[ \cdots \subset I_{n+1} \subset I_{n+2} \subset \cdots \subset I_{n} \subset [0, 1] \]
\[ \Rightarrow \exists \chi w \cap \bigcap_{n=1}^{\infty} I_{n+1} \quad \text{is a single point set} \]
\[ \chi w = c_0 + \frac{1}{2} c_1 + \cdots + \frac{c_{n-1}}{2^{n-1}} + \cdots \in [0, 1] \]
\[ \pi_0: \Sigma^+ \rightarrow (0, 1], \quad \pi_0(w) = \chi w \quad \text{is a onto and 1-1 except for a countable subset} \]
\[ \sigma^+: \Sigma^+ \to \Sigma^+, \quad \sigma^+(w) = c, c, \ldots, \quad \text{if } w = c_1c_2\ldots, \]

is the \text{left shift}.

\[ \Sigma^+ \xrightarrow{\sigma^+} \Sigma^+ \]

\[ \Downarrow \pi_0 \quad \Downarrow \pi_0 \]

\[ [0,1] \xrightarrow{\varphi} [0,1] \]

On \([0,1] \), we have the \text{Lebesgue metric}.

On \(\Sigma\), \(d_L\) is the \text{Lebesgue metric}.

\[ \sigma: (\Sigma, d_L^+) \cong \sigma: (\Sigma, d_R^+) \]

\[ \Downarrow \pi_0 \quad \Downarrow \pi_0 \]

\[ \begin{array}{c}
0 \quad \frac{1}{2} \quad 1 \\
\end{array} \quad \begin{array}{c}
0 \quad H(\frac{1}{4})
\end{array} \]

\[ d^+_R(w, w') = |H(w) - H(xw)| \]

\[ \bigcirc \]
$f = h \circ g \circ h^{-1} : \Pi \to \Pi$ is a circle endomorphism of degree 2, $f(\phi) = 1$.

$F$, lift of $f$, a homomorphism of $\mathbb{R}$

$F(x+1) = F(x) + 2$

\[ \downarrow \]

\[ \Pi \]

$F \subset \bigcirc$ - 1

Consider $E_f(t) = \sup_{x \in \mathbb{R}} \sup_{n \in \mathbb{N}} \left| \log \left| \frac{F(x+t) - F(x)}{F^{-n}(x) - F^{-n}(x-t)} \right| \right|

Then we say $f$ is uniformly quasi-symmetric (UQS) if $\sup_{t > 0} E_f(t) = M < \infty$.

Furthermore, we say $f$ is uniformly symmetric (US) if $E_f(t) \to 0^+$ as $t \to 0^+$.

We say $f$ preserves the Lebesgue measure if $|f^{-1}(A)| = |A|$ for all Borel subset $A$ of $\Pi$.

\[ \text{(plm)} \]
The first dynamical Teichmüller space is

\[ \mathcal{T} \Omega \mathcal{S} = \{ \mathcal{H} \in \Omega \mid f \text{ is \upsilon \upsilon /S} \} \]

**Theorem (5):** For any \( \tau = [\mathcal{H}] = \{ \mathcal{H} \circ h \mid h \in S \} \), we have one \( \mathcal{H} = \mathcal{H} \circ h \), such that \( f = \mathcal{H} \circ f_0 \circ h \).

\[ \therefore \mathcal{T} \Omega \mathcal{S} = \{ \mathcal{H} \in \Omega \mid f \text{ is \upsilon \upsilon and \upsilon \text{plm} /S} \} \]

Furthermore, we like to study our second dynamical Teichmüller space

\[ \mathcal{T} \Omega \mathcal{Q} = \{ \mathcal{H} \in \Omega \mid f \text{ is \upsilon \upsigma \upsigma /S} \} \]

\[ \subseteq \{ \mathcal{H} \in \Omega \mid f \text{ is \upsigma \upsigma /S} \} \]

\[ f \text{ plm } \Leftrightarrow \mathcal{H}(I_{\omega_n}) = |\mathcal{H}(I_{\omega_0})| + |\mathcal{H}(I_{1\omega_n})| \]

\[ \mathcal{H}(I_{\omega_n}) = \sum_{\omega_{n+1} \in \Omega(\omega_n)} |\mathcal{H}(I_{\omega_n})| \]
Theorem (Adamski-Hu-J.-Wang)

Suppose both \( f_1 = h_1 \circ f \circ h_1^{-1} \) and \( f_2 = h_2 \circ f \circ h_2^{-1} \) have bounded geometry and plm. Suppose \( h \in S \) such that \( f_1 = h \circ f \circ h^{-1} \). Then \( h = 1_d \).

\[ \implies T(U) = \{ h \in \mathbb{Q} | f \text{ is us and plm} \} \]

\[ T(UQ) = \{ h \in \mathbb{Q} | f \text{ is uqs and plm} \} \]

Since \( f \) is uqs (or us) \( \Rightarrow f \) has bounded geometry

It makes the following conjecture is easier to study:

Conjecture: On \( T(U) \) or \( T(UQ) \), \( k \)-metric \( = T \)-metric.

In general \( f = h \circ f \circ h^{-1} \), \( h \in T(U) \) or \( T(UQ) \), is not differentiable, even may be totally singular. However, it has its dual derivative as we explain below.

\[ 9 \]
Dual symbol representation:

\[ \Sigma = \{ v_n = d_{n-1} \ldots d_1 d_0 \mid \gamma = \sum_{i=0}^{n-1} \alpha_i (d_i) \} \]

with product topology

\[ v_n = d_{n-1} \ldots d_1 d_0 = c_0 c_1 \ldots c_{n-2} c_{n-1} = \omega_n. \]

\[ \Sigma_\omega \rightarrow \Sigma = \{ v = \ldots d_{n-1} \ldots d_1 d_0 \mid \gamma = \sum_{i=0}^{n-1} \alpha_i (d_i) \} \]

In other words, \( I\omega_n, I\omega_n = V_n \) are \( n \)-closed in \( \Sigma^+ \).

\[ I\omega_n = Ic\omega_n, I\omega_n = Ic\omega_n \] are \( n \)-closed in \( \Sigma^- \).

\[ \sigma^-(v) = \ldots d_{n-1} \ldots d_1, \] if \( v = \ldots d_{n-1} \ldots d_1 d_0 \) is called the right shift.

If \( \alpha \in T \mathcal{U} \), then \( \alpha \) induces a metric on \( \Sigma^- \):

\[ d^-(v, v') = |\alpha(I\omega_n)| (|\alpha(I\omega_n) + |\alpha(I\omega_n)|)) \]

if \( v_n = v_n' \) but \( v_n \neq v_n' \).
Theorem (Hu-J-Wang), \( \forall \theta \in T \Omega \),

\[ \sigma : (\Sigma^-, d^-) \to \mathbb{R} \] is a differentiable a.e and the derivative

\[ \frac{d^- \sigma^-(v)}{d^- v} \]

is a \( C^1 \)-function. on \((\Sigma^-, d^-)\)

We proved the theorem by using Martingale theory.

Theorem (5). \( \forall \theta \in T \Omega \),

\[ \sigma : (\Sigma^-, d^-) \to \mathbb{R} \] is a \( C^1 \)-map, i.e.,

its derivative

\[ \frac{d^- \sigma^-(v)}{d^- v} \]

is a continuous function on \( \Sigma^- \).

The proof uses the following lemma.
Lemma: Suppose $h : [0, 1] \to [0, 1]$, $h(0) = 0$, $h(1) = 1$, is a $M$-quasiregular homeomorphism. Then $|h(x) - x| \leq M - 1$, $\forall x \in [0, 1]$. Moreover, $M - 1$ is the best estimation.

From lemma, we see

\[ \tilde{g} = \rho \circ g \circ \delta^{-1} : [0, 1] \to [0, 1] \text{ is a } e \in \mathbb{R} - \mathbb{Z} \]

\[ \Rightarrow \frac{|h(I_{\sigma}(v_n))|}{|h(I_{\sigma^{-1}}(v_n))|} - \frac{|h(I_{\sigma^{-1}}(v_n))|}{|h(I_{\sigma}(v_n))|} = |g(x) - x| \leq e \frac{\epsilon}{h(I_{\sigma}(v_n)) - 1} \to 0 \text{ as } n \to \infty \]
\[ \Rightarrow \sum_{n=1}^{\infty} \frac{|h(I_{\sigma}(v_n))|}{|h(I_{\nu})|} \text{ is a Cauchy sequence} \]

\[ \frac{d_{h} \sigma^{-}(v)}{d_{h} \nu} = \lim_{n \to \infty} \frac{|h(I_{\sigma^{-}}(v_n))|}{|h(I_{\nu})|} \text{ exists} \]

and is a continuous function on \( \Sigma^- \).

(7) We have two characterizations for all\[ \frac{d_{h} \sigma^{-}(v)}{d_{h} \nu} \text{ for } h \in TUS \]

(**) We still want to know some characterizations of\[ \frac{d_{h} \sigma^{-}(v)}{d_{h} \nu} \text{ for } h \in TUA \].
Another smooth dynamical Teichmüller space
\[ TC^{\#\#}E = \{ \phi \in \mathcal{Q} | f = h \circ \phi \circ h^{-1} \text{ is } C^{\#\#} \text{ for some } 0 < \alpha / \gamma < 1 \} / S \]

my solution to Katok conjecture in 1-dim
\[ \{ \phi \in \mathcal{Q} | f = h \circ \phi \circ h^{-1} \text{ is } C^{\#\#} \text{ expanding } \} / S \]

Ruelle-Perron-Frobenius Thm
\[ \{ \phi \in \mathcal{Q} | f \text{ is } C^{\#\#} \text{ expander and } \text{plun } \} \]

For \( \phi \in TC^{\#\#}E \), \( f \) has both the derivative and the dual derivative and both are Hölder continuous.

Theorem (5): \( TUS = TE^{\#\#}E \),

i.e., \( TC^{\#\#}E \) is not a complete space under T-metric, and the completion of \( TC^{\#\#}E \) is \( TUS \) under T-metric.