Teichmüller's metric and Kobayashi's metric on the smooth Teichmüller space.

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A talk given in the summer seminar on quasiconformal mappings and Teichmüller spaces in Nanjing University of Science and Technology

August 10, 2021
9:30 am - 10:30 am
Suppose \( h: [0,1] \to [0,1] \) is a homeomorphism with \( h(0) = 0, \ h(1) = 1 \). Consider the quasi-symmetric distortion,

\[
\varepsilon(t) = \sup_{x \in [0,1]} \left( \frac{\log \left| \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \right|}{t+1} \right)
\]

0. If \( M = \sup_{t>0} \varepsilon(t) < \infty \), then we call \( h \) a \( M \)-quasi-symmetric homeomorphism of \( [0,1] \).

1. If \( \varepsilon(t) \to 0^+ \) as \( t \to 0^+ \), then we call \( h \) a symmetric homeomorphism of \( [0,1] \).

... a \( C^1 \)-diffeomorphism \( h \) is symmetric.

Let \( w(t) \) be the modulus of continuity of \( \log h'(x) \), that is, \( w(t) > 0 \), increasing \( w(t) \to 0 \) as \( t \to 0 \) and \( w(0) = 0 \), such that

\[
\sup_{x, y \in [0,1], |x-y| \leq t} \left| \log h'(x) - \log h'(y) \right| \leq C w(t)
\]

We say \( h \in C^1 w \).

Furthermore, \( \varepsilon(t) \leq C w(t) \).

\( \square \)
Let $I_{0,0} = [0,1]$

$$I_{n,k} = \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right], \quad 0 \leq k \leq 2^n-1$$

Then $I_{n-1,k} = I_{n,2k} \cup I_{n,2k+1}, \quad 0 \leq k \leq 2^{n-1}$

\[ n=1 \]

\[ n=2 \]

\[ \text{two new points} \]

\[ 0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1 \]

$$\Rightarrow \frac{1}{1+e^{(2^n)}} \left| \mathcal{H}(I_{n-1,k}) \right| \leq \left| \mathcal{H}(I_{n,j}) \right| \leq \frac{1}{1+e^{(2^n)}} \left| \mathcal{H}(I_{n-1,k}) \right|$$

\[ j = 2k \text{ or } 2k+1. \]

\[(\dagger) \quad \prod_{i=m+1}^{n} \frac{2}{1+e^{\epsilon(2^n)}} \left| \mathcal{H}(I_{m,k}) \right| \leq \left| \mathcal{H}(I_{m,k}) \right| \leq \prod_{i=m+1}^{n} \frac{2}{1+e^{-\epsilon(2^n)}} \left| \mathcal{H}(I_{m,k}) \right| \]

for any $I_{n,k} \subset I_{m,k}$

Consider the infinite product

$$S = \prod_{n=1}^{\infty} \frac{1+e^{\epsilon(2^n)}}{2}$$

\[ 2 \]
If $S$ is convergent, then

$$\left\{ \log \frac{|h(I_w)|}{|I_w|} \right\}_{n=1} \to$$

is a Cauchy sequence, $w_n \to k$, i.e.

$$R = c_0 2^{n-1} + c_1 2^{n-2} + \cdots + c_{n-1} 2^{n-1}, \quad w = c_0, c_1, \ldots, c_{n-1},$$

$$\Rightarrow \quad q(\omega) = \lim_{n \to \infty} \log \frac{|h(I_{w_n})|}{|I_{w_n}|} : \Sigma^+ \to \mathbb{R}^+$$

defines a continuous function, moreover,

$$q(w_{n 00\ldots}) = q(w_{n 11\ldots})$$

$\Rightarrow q$ pushdown to $[0,1]$ defines a continuous function on $[0,1]$.

$S$ is convergent $\iff \tilde{E}(t) = \int_0^t \frac{E(t)}{t} \, dt < \infty$

$\tilde{E}(t)$ is called Dini.

Proposition 1. If $E(t)$ is Dini, then $h \in C^+ \tilde{E}(t)$,

i.e. $h$ is a diffeomorphism whose derivative has the modulus of continuity $\tilde{E}(t)$. 

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In particular, if \( \mathcal{E}(t) = t^x, 0 < x < 1 \), then \( \widetilde{\mathcal{E}}(t) = ct^x \Rightarrow \widetilde{\mathcal{E}} \circ \mathcal{E} \) is a \( C^{+x} \) diffeomorphism.

In general, \( \mathcal{E}(t) \neq \widetilde{\mathcal{E}}(t) \). Actually, \( \mathcal{E}(t) \approx \widetilde{\mathcal{E}}(t) \)
\( \Rightarrow \mathcal{E}(t) = t^x \).

In general, a quasisymmetric homeomorphism (or symmetric homeomorphism) is not differentiable or even totally singular.

**Proposition 2 (The best estimated quasisymmetric distortion)**

Suppose \( h : [0,1] \to [0,1] \), \( h(0) = 0 \), \( h(1) = 1 \), \( B \) a \( M \)-quasisymmetric homeomorphism. Then

\[
|h(x) - x| \leq M - 1, \quad \forall x \in [0,1]
\]

The bound \( M - 1 \) is the best sharpest estimation.

(see the paper for a detailed proof)
Now let us back to circle homeomorphism \( h \) (we will think \( h \) either on \( \mathbb{T} \), on \( \mathbb{R} \) or \([0,1]/01\))

**A) Beurling-Ahlfors's extension.**

Here we think \( h \) on \( \mathbb{R} \).

\[
H(x, y) = u + i v
\]

\[
u = \frac{1}{2} \int_{0}^{1} (h(x + ty) + h(x - ty)) \, dt = \frac{1}{2y} \int_{x-y}^{x+y} h(t) \, dt
\]

\[
v = \frac{1}{y} \left( \int_{x}^{x+y} h(t) \, dt - \int_{x-y}^{x} h(t) \, dt \right)
\]

\[
\Rightarrow \quad H(x, y) = H(x), \quad H|\mathbb{R} = h
\]

\[
\left| H(x, y) - H(x) \right| \leq \epsilon(y) (e^{-1} - 1) \leq C \epsilon(y).
\]

\( \forall z = x + y \, i \)
Proposition 3. Suppose \( h \in C^1 \). Then

\[
| \kappa_r(z) | \leq C W(1-r) \log \frac{1}{r} \quad \text{near } \Gamma.
\]

(Here we think \( h \) is on \( \Gamma \)).

Let \( BM(\Delta) = \) the unit ball of \( L^\infty(\Delta) \)

\[
d_{BM}(\mu, \omega) = \max \{ \| \mu - \omega \|_{L^\infty}, \| \mu - \omega^* \|_{L^\infty} \} = \frac{1}{2} \log \left( \frac{1 + \| \mu - \omega \|_{L^\infty}}{1 - \| \mu - \omega \|_{L^\infty}} \right).
\]

\[
d_{BM}(\mu, \omega) = \max \{ \| \mu - \omega \|_{L^\infty}, \| \mu - \omega^* \|_{L^\infty} \} = \frac{1}{2} \log \left( \frac{1 + \| \mu - \omega \|_{L^\infty}}{1 - \| \mu - \omega \|_{L^\infty}} \right).
\]

B) Projection \( P \):

\[
\forall \mu \in BM(\Delta), \text{ define }
\]

\[
\widetilde{\mu}(z) = \begin{cases} \mu(z), & z \in \Delta, \\ \mu(z^*), & z \in \Delta^c = \hat{\Delta} \setminus \Delta. \end{cases}
\]

\( z^* \) is the reflection of \( z \) wrt \( \Gamma \).

Let \( \widetilde{W} \) be the quasiconformal mapping whose Beltrami coefficient is \( \widetilde{\mu} \). Then \( \widetilde{W} \mid \Gamma \) is a circle quasisymmetric homeomorphism.

\[
P(\mu) = [\widetilde{W} \mid \Gamma] \in TA : BM(\Delta) \to TA
\]

It is onto and \( P(BM(\Delta)) = TA \).
let $A_r = \{ z \in \mathbb{C} : |z| < r \}$, $0 < r < 1$.

For a modulus of continuity $\omega(t)$, let $BM^\omega(\Delta) = \{ \mu \in BM(\Delta) \mid \| \mu \|_{\Delta} \leq C \omega(1-r) \}$. 

**Proposition 4.** $P : BM^\omega(\Delta) \to TE^{+\omega}$

$P$ is not onto and $P(BM^\omega(01)) \supset TE^{+\omega}$.

For a general $\mu \in BM^\omega(\Delta)$, we have only $w_\mu | \Pi \subset C_{+\omega}$.

For $\hat{w}(t) = t^{1-\beta} + \tilde{\omega}(t^\beta)$

$$
\int_0^t \frac{\omega(t^\beta)}{t} \, dt , \quad 0 < \beta < 1.
$$

$\Rightarrow$ We may not have $=$.

Our estimation re-verify that $P(BM^0(\Delta)) = TS$

where $BM^0(\Delta) = \{ \mu \in BM(\Delta) \mid \| \mu \|_{\Delta} \to 0, r \to 1 \}$

(7)
c) Bers embedding.

\[ \forall \mu \in \overline{BM(\omega)}, \text{ let} \]
\[ \hat{\mu}(z) = \sum_{z \in \Delta} \mu(z), \quad z \in \Delta \]

\[ \hat{\mu}(z) = \sum_{z \in \Delta} \mu(z), \quad z \in \Delta \omega = \mathbb{C} \setminus \Delta \]

Let \( W^u \) be the quasi-conformal mapping whose Beltrami coefficient is \( \mu \)
\[ \Rightarrow W^u/\Delta \omega \text{ is holomorphic} \]

Let \( S(w^u) = \left( N'(w^u) - \frac{1}{2} (N(w^u))^2 \right) dz^2 \)

be the Schwarzian derivative, where
\[ N(w^u) = \frac{\partial w^u}{\partial z}, \quad = \log(Dw^u) \quad \text{non-linearity} \]
\[ Dw^u = \log |w^u| \quad \text{derivative} \]

Let \( \mathcal{QD} = \) the space of all holomorphic quadratic differentials \( q = q dz^2 \) on \( \Delta \omega \)
with
\[ \| q \| = \sup_{z \in \Delta \omega} |q(z)| p_{\omega}(z) \]

where
\[ p_{\omega}(z) dz^2 = \frac{|dz|}{|z|^2 - 1} \]

is the hyperbolic metric on \( \Delta \omega \).
Then \( i: z = [u] \in TQ \to Sw^u \in QD \)

is the Bers embedding such that

\[
i(TQ) \subseteq B_6 \subseteq QD
\]

open

\(B_6\) is the ball of radius 6 centered at \(0\).

Let \( QD^w = \{ f \in QD | |f(z) - f'(z)\| \leq C w(1 - \frac{1}{x}) \} \)

for a modulus of continuity \( w(x) \).

Then we have for \( z = [u] \in TE^{+w} \),

\[
|Sw^u(z) - f^u_w(z)| \leq C \left( 1 - \frac{1}{2z} \right)^{2(1-\beta)} + w((1-\frac{1}{2z})^2)
\]

where \( \frac{1}{2} < \beta < 1 \)

\(\Rightarrow\) \( \tilde{\omega} \subset QD \)

\[
\tilde{\omega} = t\cdot(1-\beta) + \omega(t^\beta)
\]
Consider $\beta$) and $\gamma$) if we want $w(t), \tilde{w}(t), \hat{w}(t)$ in the same class of moduli of continuity, then $w(t) = t^z$.

$\Rightarrow \tilde{w}(t) = t^{2z}$, $\hat{w}(t) = t^{2z}$, \(z\) are all Hölder.

Consider

\[ TE^{++} = U \quad TE^{++} \]

\[ BM^{+}(\Delta) = U \quad BM^{+}(\Delta) \]

\[ QD^{+} = U \quad QD^{+}(\Delta) \]

Then we have

\[ P : BM^{+}(\Delta) \to TE^{++} \to QH^{+} \]

and $P(BM^{+}(\Delta)) = TE^{++}$

Thus we have the Bers complex manifold structure on $TE^{++}$ such that $P$ is a holomorphic split submersion.

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$$d_{BM}(u, w) = \text{cosh}^{-1}\left(\frac{|w-u|}{1-\overline{w}u}\right)$$

induces the Teichmüller metric on $T\mathcal{C}$

$$d_T(z, z') = \sup\{d_{BM}(u, w) \mid P(u) = z, P(w) = z'\}$$

$d_T^H$ be the restriction of $d_T$ on $T\mathcal{C}^{\mathbb{H}_1}$

Since $T\mathcal{C}^{\mathbb{H}_1}$ is a complex Banach manifold, it has a natural Kobayashi metric $d_{K, H}$ which is the largest pseudo metric on $T\mathcal{C}^{\mathbb{H}_1}$ such that

$$d_{K, H}(f(z), f(w)) \leq f_{\Delta}(z, w), \quad z, w \in \Delta$$

for any holomorphic map $f: \Delta \to T\mathcal{C}^{\mathbb{H}_1}$.

where $f_{\Delta}(z, w) = \text{cosh}^{-1}(z, w) = \frac{1}{2} \log \left(1 + \frac{|z-w|^2}{1-\overline{z}w}\right)$

be the hyperbolic metric on $\Delta$.
Then we have

\[ \text{Lemma 1. } \forall z, z' \in TC^{1+H}, \]
\[ d_{K,H}(z, z') \geq d_{T,H}(z, z') \]

Proof. Since \( TC^{1+H} \subset TA \)
\[ \Rightarrow d_{K,H}(z, z') \geq d_{K}(z, z') = d_{T}(z, z') = d_{T,H}(z, z') \]

Lemma 2. \( \forall z, z' \in TC^{1+H}, \)
\[ d_{K,H}(z, z') \leq d_{T,H}(z, z') \]

Lemma 1 and Lemma 2 \( \Rightarrow \)

Theorem: On \( TC^{1+H}, \) \( d_{K,H} = d_{T,H}. \)

We need to prove Lemma 2.

\( \Box \)
moves \( \eta \) to \( \mathcal{P}(0) = \{0\} \) and preserves both Teichmüller’s metric and Kobayashi’s metric. Thus to prove (5.7) for any points \( \tau, \eta \in \mathcal{T}C^{1+H} \), we only need to prove

\[
d_{K,H}(\{0\}, \tau) \leq d_{T,H}(\{0\}, \tau).
\]

(5.8)

Before to prove this inequality, we review some properties in Teichmüller theory without proofs. The reader who is interested in them may refer to [12, 19, 26].

5.1 Extremal Point

Suppose \( \phi \) is a holomorphic function on \( \Delta \). Let

\[
\|\phi\| = \int_{\Delta} |\phi(z)| \, dxdy, \quad z = x + iy.
\]

Given a point \( \tau = [\mu] \in T Q \), let

\[
k_0 = \inf_{\mu \in \tau} \|\mu\|_{\infty}.
\]

From the normal family theory in quasiconformal theory, we have a \( \mu_0 \in \tau \) such that \( \|\mu_0\|_{\infty} = k_0 \). We call \( \mu_0 \) an extremal point in \( \tau \).

A sequence \( \{\varphi_n\} \) of holomorphic functions is called a Hamilton sequence for \( \mu_0 \) if

\[
\|\varphi_n\| = 1 \quad \text{and} \quad \lim_{n \to \infty} \sup \int_{\Delta} \mu_0 \varphi_n \, dxdy = \|\mu_0\|_{\infty}.
\]

**Theorem 5.4** (Hamilton–Krushkal Theorem)  Given any point \( \tau = [\mu] \in T Q \), if \( \mu_0 \in \tau \) is an extremal point, then \( \mu_0 \) has a Hamilton sequence \( \{\varphi_n\} \).

5.2 Frame Point

Given a point \( \tau = [\mu] \in T Q \), an element \( \mu_1 \in \tau \) is called a frame point if there is a compact set \( D \subset \Delta \) such that

\[
\|\mu_1|\Delta \setminus D\|_{\infty} < k_0.
\]

Lemma 2.9 says that if \( \tau \neq \{0\} \in \mathcal{T}C^{1+H} \), then it always has a frame point.

**Theorem 5.5** (Strebel’s Frame Mapping Theorem)  For any \( \tau \neq \{0\} \in T Q \), if it has a frame point, then it has a unique extremal point \( \mu_0 \) in the Teichmüller form,

\[
\mu_0 = k_0 \frac{[\varphi_0]}{\varphi_0},
\]

for a holomorphic function \( \varphi_0 \) with \( \|\varphi_0\| = 1 \). Moreover, for any \( \nu \in \tau \),

\[
K_0 = \frac{1 + k_0}{1 - k_0} \leq \int_{\Delta} \frac{|1 + \nu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\nu|^2} |\varphi_0| \, dxdy.
\]

5.3 Holomorphic Functions

Suppose \( \{\varphi_n\} \) is a sequence of holomorphic functions with \( \|\varphi_n\| = 1 \). Suppose \( D \subset \Delta \) is a compact subset. We claim that \( \{\varphi_n\} \) is uniformly bounded on \( D \). We prove the claim by contradiction. Suppose not, then there exists a sequence of points \( \{z_n\} \subset D \) and a subsequence of \( \{\varphi_n\} \), still denoted by \( \{\varphi_n\} \), such that \( |\varphi_n(z_n)| \geq n \). Since \( D \) is compact, \( \{z_n\} \) has an accumulation point \( z_0 \in D \). Then there exists a subsequence of \( \{z_n\} \), still denoted by \( \{z_n\} \), such that \( z_n \) converges to \( z_0 \). Choose a small \( r > 0 \) such that the closed disk \( D_r(z_0) = \{ |z - z_0| \leq r \} \subset \Delta \). Then \( z_n \in D_{r/4}(z_0) \) when \( n \) is large enough, say \( n > N \).
For any \( n > N \), one can apply the Cauchy integral formula for \( \varphi_n(z_n) \) to obtain

\[
 n \leq |\varphi_n(z_n)| \leq \frac{1}{2\pi} \int_{|z-z_0|=r} \frac{|\varphi_n(z)|}{|z-z_n|} r' d\theta
\]

for each \( \frac{r}{2} \leq r' \leq r \). And then

\[
 n \leq \frac{1}{2\pi} \int_{|z-z_0|=r} |\varphi_n(z)| \frac{4}{r} r' d\theta = \frac{2}{\pi} \int_{|z-z_0|=r} |\varphi_n(z)| d\theta.
\]

Multiplying the previous inequality by \( r' \) and integrating both sides in radial direction from \( \frac{r}{2} \) to \( r \), we obtain

\[
\frac{3}{8}nr^2 = n \int_{\frac{r}{2}}^r r' dr' \leq \frac{2}{\pi} \int_{\frac{r}{2}}^r \int_{|z-z_0|=r'} |\varphi_n(z)| d\theta dr' \leq \frac{2}{\pi} \|\varphi_n\| = \frac{2}{\pi}.
\]

Hence \( \frac{3}{8}nr^2 \leq \frac{2}{\pi} \) for any \( n > N \). This is a contradiction when \( n \) is large enough. We proved the claim.

Applying the Cauchy integral formula for derivatives \( \{\phi'_n\} \), one can see it is also uniformly bounded on \( D \) and thus \( \{\phi_n\} \) is a uniformly bounded equi-continuous family. The Ascoli–Arzela Theorem implies \( \{\phi_n\} \) has a convergent subsequence, still denoted as \( \{\phi_n\} \), on \( D \). Taking an increasing sequence of compact sets \( \{D_m\} \) such that \( \Delta = \bigcup_m D_m \), we get a convergent subsequence of \( \{\phi_n\} \), still denoted as \( \{\phi_n\} \), on \( \Delta \). Suppose \( \phi_0 \) is its limiting function. By Fatou’s Lemma, \( \|\phi_0\| \leq 1 \).

### 5.4 The Proof of Lemma 5.3

For any \( \tau \in TC^{1+H} \), take \( \mu \in \tau \) in Lemma 2.9. Let \( k = \|\mu\|_\infty \). Let

\[
\Delta_n = \|z \in \Delta \mid |z| < r_n = 1 - \frac{1}{n}\}
\]

and \( A_n = \Delta \setminus \Delta_n \).

Let \( l_n = \|\mu|A_n\|_\infty \). Lemma 2.9 implies that \( l_n < k_0 \) for \( n \) large enough, say \( n > N \). So \( \mu \) is a frame point in \( \tau \). This implies that \( \tau \) has a unique extremal point \( \mu_0 \) in the Teichmüller form \( \mu_0 = k_0|\phi_0|/\phi_0 \) for some holomorphic function \( \phi_0 \) with \( \|\phi_0\| = 1 \). Moreover, \( 0 < k_0 < k \).

Let \( f_n(z) = w_\mu(r_n z) \). It maps \( \Delta \) to a quasi-disk \( D_n = f_n(\Delta) \). Let \( g_n : D_n \to \Delta \) be the Riemann mapping. Then \( h_n = g_n \circ f_n \) is a quasiconformal self-homeomorphism of \( \Delta \) and \( \tau_n = |h_n|T \) is in \( TQ \). From Lemma 2.9, for \( N \) large enough, every point \( \tau_n \) has a frame point for \( n > N \). Thus for every \( n > N \), \( \tau_n \) has a unique extremal point \( \mu_{n,0} \) in the Teichmüller form,

\[
\mu_{n,0} = k_{n,0}\frac{|\phi_{n,0}|}{\phi_{n,0}}
\]

with a holomorphic function \( \phi_{n,0} \) with \( \|\phi_{n,0}\| = 1 \). By our definition, one can see that \( k_{n,0} \geq k_0 \) for all \( n > N \).

Now we define \( F_n(z) = g_n^{-1} \circ w_{\mu_{n,0}}(z/r_n) \) for \( z \in \Delta_n \) and \( F_n(z) = w_\mu(z) \) for \( z \in A_n \). It agrees on the circle \( \partial \Delta_n \). Thus it is a quasiconformal self-homeomorphism of \( \Delta \). The Beltrami coefficient \( \nu_n \) of \( F_n \) is \( \mu_{n,0}(z/r_n) \) on \( \Delta_n \) and \( \mu \) on \( A_n \). Thus \( \nu_n \in \tau \in TC^{1+\alpha} \). And \( \|\nu_n\|_\infty > k_0 \).

We have a holomorphic map

\[
p(c) = \left[ c \frac{\nu_n}{\|\nu_n\|_\infty} \right] : \Delta \to TC^{1+\alpha}
\]
such that $p(0) = 0$ and $p(||v_n||_\infty) = \tau$. This implies that
\[
d_{K,\alpha}([0], \tau) \leq d_1([0], \tau) \leq \frac{1}{2} \log \frac{1 + ||v_n||_\infty}{1 - ||v_n||_\infty}.
\]
Our final step is to prove $||v_n||_\infty \to k_0$ as $n \to \infty$.

From Subsection 5.3, there exists a subsequence of $\{\varphi_{n,0}\}$, still denoted by $\{\varphi_{n,0}\}$, converging uniformly to a holomorphic function $\hat{\varphi}$ on any compact subset $D \subset \Delta$. Furthermore, $||\hat{\varphi}|| \leq 1$. We claim that $||\hat{\varphi}|| > 0$. We prove the claim by contradiction.

Suppose $||\hat{\varphi}|| = 0$. Then $\{\varphi_{n,0}\}$ has a subsequence, we still denote by $\{\varphi_{n,0}\}$, converging uniformly to zero on any compact subset $D \subset \Delta$. For any $\epsilon > 0$, we first choose a compact subset $D \subset \Delta$ such that
\[
||\mu_0(\Delta \setminus D)||_\infty < \epsilon.
\]
There exists $N_1 > N$ such that
\[
\int_D |\varphi_{n,0}(z)|dxdy \leq \epsilon
\]
and such that $D \subset \Delta_n$ for all $n > N_1$.

From Subsection 5.2,
\[
K_{n,0} = \frac{1 + k_{n,0}}{1 - k_{n,0}} \leq \int_\Delta \frac{|1 + \mu \frac{\varphi_{n,0}}{\varphi_{n,0}}|^2}{1 - |\mu|^2} |\varphi_{n,0}|dxdy.
\]
This says
\[
K_{n,0} \leq \int_{\Delta \setminus D} \frac{|1 + \mu \frac{\varphi_{n,0}}{\varphi_{n,0}}|^2}{1 - |\mu|^2} |\varphi_{n,0}|dxdy + \int_D \frac{|1 + \mu \frac{\varphi_{n,0}}{\varphi_{n,0}}|^2}{1 - |\mu|^2} |\varphi_{n,0}|dxdy.
\]
Then, for $K = (1 + k)/(1 - k)$,
\[
K_{n,0} \leq \int_{\Delta \setminus D} \frac{1 + \epsilon}{1 - \epsilon} |\varphi_{n,0}|dxdy + K \int_D |\varphi_{n,0}|dxdy,
\]
and hence
\[
K_{n,0} \leq \frac{1 + \epsilon}{1 - \epsilon} \int_\Delta |\varphi_{n,0}|dxdy + \left( K - \frac{1 + \epsilon}{1 - \epsilon} \right) \int_D |\varphi_{n,0}|dxdy.
\]
Therefore
\[
1 < k_0 < k_{n,0} \leq \frac{1 + \epsilon}{1 - \epsilon} + \left( K - \frac{1 + \epsilon}{1 - \epsilon} \right) \cdot \epsilon.
\]
This is a contradiction when $\epsilon$ is sufficient small. Therefore $||\hat{\varphi}|| > 0$.

Now let $\hat{\mu} = \frac{\hat{\varphi}}{\hat{\varphi}}$, where $\hat{k} = \lim_{n \to \infty} k_{n,0}$ (by taking a limit of a convergent subsequence if it is necessary). Then $\mu_{n,0} \to \hat{\mu}$ a.e. on $\Delta$. By the convergence theorem (see [20, Theorem 4.6]) of families of quasiconformal maps, we obtain
\[
\lim_{n \to \infty} w_{\mu_{n,0}}|T = w_{\mu_{n,0}}|T = w_\hat{\mu}|T = w_\hat{\mu}|T.
\]
By the uniqueness of the extremal point in $\tau$, $\hat{k} = k_0$. Thus $k_{n,0} \to k_0$ as $n \to \infty$ for a subsequence of $n$’s. We completed the proof of Lemma 5.3. Both Lemmas 5.2 and 5.3 give a proof of Theorem 5.1.