part 1: Symmetric circle homeomorphism and its extensions.

**Def 1** A circle homeomorphism \( h \) is called quasi-symmetric if there exists a constant \( M > 1 \) such that

\[
\frac{1}{M} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq M, \quad \forall x \in \mathbb{R}, \forall t > 0.
\]

It is called symmetric, if there exists a positive function \( \varepsilon(t) \) such that \( \varepsilon(t) \to 0 \) as \( t \to 0^+ \) and

\[
\frac{1}{1 + \varepsilon(t)} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq H \varepsilon(t), \quad \forall x \in \mathbb{R}, \forall t > 0
\]

**Note:** \( h: \mathbb{S}^1 \to \mathbb{S}^1 \) and \( H: \mathbb{R} \to \mathbb{R} \)

\( \mathbb{R} \) is a universal cover of \( \mathbb{S}^1 \): \( \pi(x) = e^{2\pi i x}: \mathbb{R} \to \mathbb{S}^1 \)

In this way, we can think \([0,1]\) as the unit circle \( \mathbb{S}^1 \).

**Symmetric triples**

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( x+t )</th>
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<tbody>
<tr>
<td>( H )</td>
<td></td>
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**Example of symmetric:**

\( H \) is differentiable.

\( Q.S: \quad \frac{1}{|x|} \leq \frac{R}{L} \leq |x| \)

\( S: \quad \frac{1}{1+\varepsilon(t)} \leq \frac{R}{L} \leq H \varepsilon(t) \)
Let's use unit disc model: \( \Delta \).

\( M(\Delta) = \) unit ball of the complex Banach space \( L^1(\Delta) \).

An element \( M(M) \) is called a Beltrami coefficient on \( \Delta \).

Let \( f \) be a homeomorphism from \( \Delta \) onto \( \Delta \) satisfying

\[
\frac{f_3}{f_2} = \mu(\theta) \quad \text{for almost all } \theta \in \Delta.
\]

Then \( f \) is quasiconformal. (\( \| \mu(\theta) \|_{L^1} < 1 \))

**Def 2:** A quasiconformal map \( f \) is said to be asymptotically conformal if for any \( \varepsilon > 0 \), there exists a compact subset \( \Omega \) in \( \Delta \), such that \( \| \mu(\theta) \|_{L^1(\Omega)} \) is less than \( \varepsilon \).

Another understanding: \( \mu(\theta) \to 0 \) as \( \varepsilon \to 0 \).

**Prop:** Suppose \( h: S' \to S' \) is the boundary map of \( f: \Delta \to \Delta \).

Then \( \partial h \) is quasisymmetric (\( \Rightarrow \)) \( f \) is quasiconformal.

(2) \( h \) is symmetric (\( \Rightarrow \)) \( f \) is asymptotically conformal.
idea of proof of prop:

If $h$ is quasi-symmetric or symmetric, we can use Beurling–Ahlfors extension of $h$ to calculate $\mathcal{M}(\theta)$. 

Note: $h: \mathbb{R} \to \mathbb{R}$

$$BA(h)(x, y) = \frac{i}{2\pi} \int_{-\infty}^{\infty} h(x+y) \, dt$$

where $U(x, y) = \frac{1}{2y} \int_{x}^{y} h(x+t) \, dt$

$$V(x, y) = \frac{1}{y} \int_{x}^{y} (h(x+t) - h(x-t)) \, dt$$

$BA(h): \mathbb{R}^2 \to \mathbb{R}^2$

3) Partial derivatives: $U_x, U_y, V_x, V_y$ ...

---

We need a lemma for this direction.

Lemma 1: If $h$ is $M$-qs and a finite qe extension of $f$ is $k$-qc, then $M \leq C(k)$ where $(k_n) \to 1$ as $k \to 1$.

If $f$ is asymptotically conformal,

$$\text{define } f_n^* = \frac{f}{k_n} \text{ on } \Delta / \mathbb{R}^n$$

$$\lim_{n \to \infty} \frac{M_{f_n^*}}{M_{f_n}} = 1$$

$$\lim_{n \to \infty} \frac{M_{f_n}}{M_{f_n^*}} = 1$$

Boundary map of $M_{f_n^*}$ is real analytic.

Note: $M_{f_n^*} = 0$ on $\Delta / \mathbb{R}^n$
Prop 2: Suppose $h$ is symmetric on an interval $[a, b]$, then

$\exists s$ such that $f$ is AC in a nbhd of $[a+s, b-s]$. 

Note:
1. Schwartzian derivative $\rightarrow 0$ as $|h| \rightarrow 1$.
2. Douady-Earle extension: $\mu_{DE}(h) \rightarrow 0$ as $|h| \rightarrow 1$ for symmetric $h$.

and it has similar result as prop 2.

Prop 3: Teichmüller’s metric coincides with Kobayashi’s metric on $\mathcal{T}_0 =$ space of all symmetric $h$.

Idea: By Strebel’s frame mapping theorem,

$\exists \mu_0 \sim \mu_f$ and $\mu_0 = k_0 \frac{h_0}{h_0}$, Teichmüller form.

Construct

$\mu_{fn} = \mu_0$

$\Gamma_{\mu_0} \in \mathcal{T}_0$
part 2: Markov partition with Bounded Geometry

\[ q(x) = \begin{cases} \frac{3}{4} & x \in \left[\frac{1}{2}, 1\right] \\ 2x - 1 & x \in \left(0, \frac{1}{2}\right] \end{cases} \]

pre-images of $1: q^{-n}(1)$

\[ I_0 \quad I_1 \]

\[ I_{00} \quad I_{01} \quad I_{10} \quad I_{11} \]

\[ I_{000} \quad I_{001} \quad I_{010} \quad I_{011} \quad I_{100} \quad I_{101} \quad I_{110} \quad I_{111} \]

\[ 0 \quad \frac{1}{8} \quad \frac{4}{8} \quad \frac{5}{8} \quad \frac{6}{8} \quad \frac{7}{8} \quad 1 \]

\[ 0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1 \]

\[ I_{\infty} \text{ has two subintervals } I_{\infty 0} \text{ and } I_{\infty 1} \]

\[ I_{\infty} \text{ has two preimages } I_{\infty 0} \text{ and } I_{\infty 1} \]
suppose \( f \) is a degree 2 circle endomorphism.

\( f^{-n}(1) \) gives us a Markov partition.

\[ \begin{array}{c}
I_0 & I_1 \\
\hline
I_{00} & I_{01} & I_{01} & I_{11} \\
\hline
\end{array} \]

1. \( f \) has bounded geometry if \( \frac{|I_{w_0}|}{|I_{w_1}|} \leq M, \frac{|I_{w_1}|}{|I_{w_0}|} \leq M \) for any \( w_n \)

2. \( f \) has bounded nearby geometry if \( \frac{|I_{w_n}|}{|I_{w_n}|} \leq M \) for any adjacent intervals \( I_{w_n}, I_{w_{n+1}} \)

**Prop 4:** \( h : [0,1] \to [0,1] \) is \( M \)-q.s.s.

\( h \) maps partition pts of \( g \) to partition pts of \( f \).

Then \( f \) has bounded nearby geometry.

Idea: on level \( n \):

\[
\begin{array}{c}
\frac{L}{2^n} & \frac{R}{2^n} \\
\hline
h & 1 \\
\frac{1}{M} \leq \frac{h(L)}{h(R)} \leq M \\
\end{array}
\]
**Def 3.** A circle endomorphism \( f \) is called uniformly quasisymmetric if there exists a constant \( M > 1 \) such that
\[
\frac{1}{M} \leq \frac{F^{-n}(x+t) - F^{-n}(x)}{F^{-n}(x) - F^{-n}(x-t)} \leq M, \quad \forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}, \quad \forall t > 0.
\]

**Prop 5.** \( f \) is UQS \( \iff \) bounded nearly geometry \( \iff \) \( h \) is QS

where \( f = h \circ g \circ h^{-1} \)

**Def 4.** We say a circle endomorphism \( f \) preserves the Lebesgue measure \( m \) if
\[
m ( f^{-1}(A) ) = m(A)
\]
holds for all Borel subsets \( A \subseteq S \).

For degree 2 \( \mathbb{Z} \) Lebesgue invariant \( f \):
\[
|I_{m_1}| = |I_{m_0}| + |I_{m_1}|
\]

\[
I_0 \quad I_1
\]

\[
I_{00} \quad I_{01} \quad I_{10} \quad I_{11}
\]

\[
|I_{m_0}| = |I_{m_{00}}| + |I_{m_{10}}|
\]

\[
I_{00} \quad I_{01} \quad I_{10} \quad I_{11}
\]

\[
I_{00} \quad I_{01} \quad I_{10} \quad I_{11}
\]
Prop 6: If $f$ is UQS and Lebesgue invariant, then the limit $\lim_{n \to \infty} \frac{I_{W_n}}{I_{w_0}}$ and $\lim_{n \to \infty} \frac{|I_{W_n}|}{|I_{w_0}|}$ exists along almost all dynamical paths.

Example 1: $g(x) = x^2$

$$|I_{W_n}| = 2, \quad \frac{|I_{W_n}|}{|I_{w_0}|} = 2$$

for any $W_n$.

Example 2: $f(x) = \begin{cases} \frac{3}{2}x + \frac{1}{2}, & x \in (0, \frac{1}{2}] \\ \frac{3}{2}x + \frac{1}{2} - 1, & x \in (\frac{1}{2}, \frac{3}{4}] \\ 3x - 2, & x \in (\frac{3}{4}, 1] \end{cases}$
**Prop 7:** Suppose \( f \) and \( g \) are both UAS and Lebesgue invariant. If \( f = h \circ g \circ h^{-1} \) and \( h \) is symmetric, then partitions of \( f \) and partitions of \( g \) have same type "father/son" limits along almost all dynamical paths.

\[
\begin{align*}
\text{f:} & \\
I_0 & \quad I_0 & \quad I_0 & \quad I_1 \\
\text{g:} & \\
\text{h maps partitions to partitions.} & \\
h(I_0) & \quad h(I_0) & \quad h(I_0) & \quad h(I_1) \\
\end{align*}
\]

\[
\text{h is symmetric} \\
\begin{align*}
\frac{\text{father}}{\text{son}} & \quad \frac{|I_{\text{son}}|}{|I_{\text{father}}} \left( \text{or } \frac{|I_{\text{son}}|}{|I_{\text{son}}} \right) = \frac{|h(I_{\text{son}})|}{|h(I_{\text{father}})} \left( \text{or } h(I_{\text{son}}) \right) \\
\end{align*}
\]

Main theorem of the new paper:
Let \( f \) and \( g \) be two circle endomorphisms of degree \( d \geq 2 \) such that each has bounded geometry, preserves the Lebesgue measure, and fixes 1. Suppose \( f = h^{-1} \circ g \circ h \), then \( h \) is symmetric \( \iff h = \text{Id} \).
special cases:

(1) \( g(\frac{1}{2}) : \]

\[
\begin{align*}
0 & \quad \frac{1}{2} & \quad 1 \\
\hline
0 & \quad \frac{1}{2} & \quad \frac{3}{4} \\
\end{align*}
\]

\[
\text{limit } \lim_{x \to 0} \frac{g(x)}{x} = 2
\]

\[
\text{But Leb-invariant } \Rightarrow \text{ diverge.}
\]

(2) \( f: \]

\[
\begin{align*}
0 & \quad \frac{1}{2} & \quad 1 \\
1 & \quad 2 & \quad 2:1 \\
1 & \quad 2 & \quad 2:1 \\
\end{align*}
\]

\[
\text{same cutting + dense implies linear.}
\]

\[
\text{h is symmetric } \Rightarrow \text{ } m_1 = m_2
\]
part 3: symmetric at a point.

Sullivan's result: suppose $f$ and $g$ are two $C^1$ Lipschitz expanding endomorphisms of the same degree. Let $h$ be the conjugacy between $f$ and $g$, that is, $f \circ h = h \circ g$. Then

$$h \in C^1 \Leftrightarrow h \text{ is differentiable at one point with non-zero derivative.}$$

Jiang's result: $f$ and $g$ are two $C^{1+x}$ expanding endomorphisms of the same degree for $0 < x < 1$. Let $h$ be the conjugacy between $f$ and $g$, that is, $f \circ h = h \circ g$. Then

$$h \in C^{1+x} \Leftrightarrow h \text{ is differentiable at one point with uniform bound.}$$

Question: is $h$ symmetric?

For conjugacy map $h$, $h$ is symmetric at a point $\Leftrightarrow h$ is symmetric.

Answer is No! ($Hw$)

$$f(1/2) = 8 \frac{\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2}}{1 - \frac{1}{2} \cdot 2}, \quad g(1/2) = 8^2$$

$$f = h \circ g \circ h^{-1}, \quad h \text{ is symmetric at } 1.$$

But $h$ is not symmetric.