Tame Quasiconformal Motions and Teichmüller Spaces

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A talk given in
Topology/Geometry Seminar
The Mathematics Department
Rutgers, The State University of New Jersey, New Brunswick
May 7, 2019, 3:30pm-4:30pm
Suppose $D$ is a non-empty simply connected open subset of the complex plane $\mathbb{C}$, but $D \neq \mathbb{C}$. Then there is a biholomorphic map

$$f : D \rightarrow \Delta = \{z \in \mathbb{C} \mid |z| < 1\}.$$ 

Moreover, for a given $z_0 \in D$, $f$ is unique provided $f(z_0) = 0$ and $f'(z_0) > 0$.

Riemann 1851, Ph.D thesis; Caráthèodory 1912, a proof.
Suppose $S$ is a simply connected Riemann surface. Then $S$ is biholomorphic to one of the following:

\[ \Delta \ (\text{hyperbolic}); \quad \mathbb{C} \ (\text{parabolic}); \quad \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \ (\text{spherical}). \]

Klein 1883;
Poincaré 1882;
Koebe 1907 and Poincaré 1907, a proof;
Abikoff 1981 survey, AMS monthly, 88, 574-592.
Or called a variable metric Riemann mapping theorem. Consider $\mathbb{R}^2 = \{(x, y)\}$ and a Riemannian metric

$$g(x, y) = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2$$

with $E > 0$, $G > 0$, and $EG - F^2 > 0$.

Let $z = x + iy$ and $\bar{z} = x - iy$. Then

$$g(z) = \gamma(z)|dz + \mu(z)d\bar{z}|^2$$

with

$$\gamma = \frac{1}{4}(E + G - 2\sqrt{EG - F^2}), \quad \mu = \frac{E - G + 2iF}{4\gamma}.$$
Question

Does the Beltrami equation

\[ w_z = \mu w_z \]

has a solution?

Gauss 1822, isothermal coordinate on surfaces.

Morrey 1932, quasi-linear elliptic partial differential equation.

Ahlfors-Bers, 1960, a famous paper.

Bojarski 1955,

etc. .....
Measurable Riemann Mapping Theorem: Theorem

Let \[ M(\mathbb{C}) = \{ \mu \in L^\infty(\mathbb{C}) \mid \|\mu\|_\infty < 1 \} \]

For any \( \mu \in M(\mathbb{C}) \), the Beltrami equation always has a solution \( w \) which is a \( K \)-quasiconformal homeomorphism of \( \hat{\mathbb{C}} \) for \( K = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty) \). Moreover, if we consider the normalized solution \( w^\mu \) fixing 0, 1, \( \infty \), then \( w^\mu \) is unique and depends on \( \mu \) holomorphically.
Analytic definition: A $W^{1,2}_{loc}$ (first-order distribution partial derivatives in $L^2_{loc}$) map satisfies the Beltrami equation as a weak solution.

Geometric definition: A map increases the modulus of any quadrilateral at most by $K$.

Grötzsch 1928, Ahlfors 1935 formally introduced
Let $E$ be a closed subset of $\hat{\mathbb{C}}$. Suppose $0, 1, \infty \in E$. We say $\mu, \nu \in \mathcal{M}(\mathbb{C})$ are $E$-equivalent, denote as $\mu \sim_E \nu$ if $(w^\nu)^{-1} \circ w^\mu$ is homotopic to the identity rel to $E$, that is, there is a continuous map $H(t, z) : [0, 1] \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that

1) $H(0, z) = z$;
2) $H(1, z) = (w^\nu)^{-1} \circ w^\mu(z)$ for all $z \in \hat{\mathbb{C}}$; and
3) $H(t, z) = z$ for all $0 \leq t \leq 1$ and all $z \in E$. 
The space of all equivalence classes

\[ T(E) = \{ [\mu] \mid \mu \in M(\mathbb{C}) \} \]

is called the Teichmüller space of the closed subset \( E \).
For any $\mu, \nu \in M(\mathbb{C})$, let

$$d_0(\mu, \nu) = \tanh^{-1} \left\| \frac{\mu - \nu}{1 - \overline{\mu} \nu} \right\|_{\infty} = \frac{1}{2} \log \frac{1 + \left\| \frac{\mu - \nu}{1 - \overline{\mu} \nu} \right\|_{\infty}}{1 - \left\| \frac{\mu - \nu}{1 - \overline{\mu} \nu} \right\|_{\infty}}$$

The Teichmüller metric on $T(E)$ is, for any $\alpha, \beta \in T(E)$,

$$d_T(\alpha, \beta) = \inf \{ d_0(\mu, \nu) \mid \mu \in \alpha, \nu \in \beta \}$$
The complement $\hat{\mathbb{C}} \setminus E = \bigcup_i \Omega_i$. Each $\Omega_i$ is a Riemann surface. So we have the classical Teichmüller space $Teich(\Omega_i)$. We then have a product Teichmüller space

$$\prod_i Teich(\Omega_i) = \{(\tau_i) \mid \tau_i \in Teich(\Omega_i), \sup_i d_{\Omega_i, T}(0_i, \tau_i) < \infty\}$$
There is a biholomorphic map between the Teichmüller space $T(E)$ and the product Teichmüller space $\prod_i \text{Teich}(\Omega_i) \times M(E)$, that is,

$$T(E) \simeq \prod_i \text{Teich}(\Omega_i) \times M(E).$$

The map $P_E(\mu) = [\mu]$ is a holomorphic split submersion, that is, for any $\tau \in T(E)$, there is a neighborhood $U$ about $\tau$ and a holomorphic section $s_{\tau,U} : U \to M(\mathbb{C})$ such that $P_E \circ s_{\tau,U} = Id$.  

\[ 
\begin{array}{ccc}
M(\mathbb{C}) & \xrightarrow{P_E} & T(E) \\
\downarrow & & \downarrow \\
& s_{\tau,U} & \\
\end{array} 
\]
The map $P_E(\mu) = [\mu]$ has a global continuous section, that is, there is a continuous section $S : T(E) \to M(\mathbb{C})$ such that $P_E \circ S = Id$ (from Douady-Earle 1986 paper about the barycentric extension of a quasisymmetric homeomorphism of the circle).

This implies that $T(E)$ is contractible.

If $\dim T(E) \geq 2$, then $P_E$ can not have a global holomorphic section (Earle 1969).
Complex Geometry: all basepoint preserving holomorphic maps $f : V \to T(E)$ where $V$ is a connected complex Banach manifold with a basepoint $t_0$. In particular, $V = \Delta$ with metric $d\rho = |dz|/(1 - |z|^2)$.

Real Geometry: all basepoint preserving continuous maps $f : W \to T(E)$ where $W$ is a connected Hausdorff space with a based point $t_0$. In particular, $W = I = [0, 1]$ is the unit interval.
Problem (Lifting Problem)

For a basepoint preserving holomorphic map $f : V \to T(E)$, can one find a basepoint preserving holomorphic map $\tilde{f} : V \to M(\mathbb{C})$ such that $P_E \circ \tilde{f} = f$?
Lifting Theorem for $\Delta$

For $V = \Delta$ with the basepoint 0, the answer is affirmative, that is, for any basepoint preserving holomorphic map $f : \Delta \to T(E)$, one can find a basepoint preserving holomorphic map $\tilde{f} : \Delta \to M(\mathbb{C})$ such that $P_E \circ \tilde{f} = f$.

\[
\begin{aligned}
\text{M}(\mathbb{C}) & \xrightarrow{\tilde{f}} \text{M}(\mathbb{C}) \\
\Delta & \xrightarrow{f} T(E) \\
\end{aligned}
\]

One application of this lifting theorem is an easy proof of Teichmüller’s metric $=$ Kobayahi’s Metric (Royden, 1971; Gardiner, 1984).
Holomorphic Motions

Suppose $V$ is a connected complex Banach manifold with basepoint $t_0$. A map $h(t, z) : V \times E \to \hat{\mathbb{C}}$ is called a holomorphic motion if

i) $h(t_0, z) = z$ for all $z \in E$;

ii) for any fixed $t \in V$, $h(t, \cdot) : E \to \hat{\mathbb{C}}$ is injective;

iii) for any fixed $z \in E$, $h(\cdot, z) : V \to \hat{\mathbb{C}}$ is holomorphic.

We can normalized it by assuming $h(t, 0) = 0$, $h(t, 1) = 1$, and $h(t, \infty) = \infty$ for all $t \in V$.

Example: The map $\Psi_E(t, z) = w^\mu(z) : T(E) \times E \to \hat{\mathbb{C}}$ is a holomorphic motion.
The holomorphic motion $\Psi_E(t, z) = w^\mu(z) : T(E) \times E \to \hat{\mathbb{C}}$ is universal for holomorphic motions in the meaning that for any holomorphic motion $h(t, z) : V \times E \to \hat{\mathbb{C}}$, where $V$ is a simply connected Banach complex manifold with a basepoint, there is a unique basepoint preserving map $f : V \to T(E)$ such that $f^*(\Psi_E) = h$.

Extension Problem

We say $H(t, z) : V \times \hat{C} \to \hat{C}$ is a holomorphic motion extension of a holomorphic motion $h(t, z) : V \times E \to \hat{C}$ if $H|_{V \times E} = h$.

Problem (Extension Problem)

For a given holomorphic motion $h(t, z) : V \times E \to \hat{C}$, can we extend it to a holomorphic motion $H(t, z) : V \times \hat{C} \to \hat{C}$?
Due to the universal property, the extension problem is equivalent to the holomorphic lifting problem.

When $V = \Delta$, it is Slodkowski’s theorem (Slodowski 1991, Bers-Royden 1986, Sullivan-Thurston 1986, Chirka 2004, also see J-Mitra-Wang 2009 for a direct proof of the lifting problem), that is, the answer to both problems is yes.

For $V = T(E)$ with $\dim T(E) \geq 2$, the answer is no (J-Mitra 2007, Earle 1968).
**Definition**

Suppose $W$ is a connected Hausdorff space with a basepoint $t_0$. A map $\phi(t, z) : W \times E \to \mathbb{C}$ is called a quasiconformal motion if

i) $\phi(t_0, z) = z$ for all $z \in E$;

ii) for each $t \in W$, the map $\phi(t, \cdot) : E \to \mathbb{C}$ is injective;

iii) given any $t \in W$ and any $\epsilon > 0$, there is a neighborhood $U_t$ about $t$ such that for any quadruplet $a, b, c, d \in E$,

$$\rho_{0,1}(\phi_x(a, b, c, d), \phi_y(a, b, c, d)) < \epsilon, \quad \forall \ x, y \in U_t.$$ 

In the above definition, $\rho_{0,1}$ is the hyperbolic metric on the thrice punctured sphere $\mathbb{C} \setminus \{0, 1\}$ and

$$
\phi_x(a, b, c, d) = \frac{(\phi_x(a) - \phi_x(c))(\phi_x(b) - \phi_x(d))}{(\phi_x(a) - \phi_x(d))(\phi_x(b) - \phi_x(c))}
$$

is the cross-ratio of four points $\phi_x(z) = \phi(x, z)$ for $z = a, b, c, d$. 
When we study the real geometry on $T(E)$, the following problem is important.

**Problem (Universal Problem)**

Whether $\Psi_E(t,z) : T(E) \times E \to \hat{C}$ is a universal for quasiconformal motion? That is, for any quasiconformal motion, $\phi(t,z) : W \times E \to \hat{C}$, where $W$ is a simply connected Hausdorff space with a basepoint $t_0$, can we find a basepoint preserving continuous map $f : W \to T(E)$ such that $f^*(\Psi_E) = \phi$?
Problem (Extension for Quasiconformal Motions)

For a quasiconformal motion \( \phi(t, z) : W \times E \to \hat{\mathbb{C}} \), can one extend it to a quasiconformal motion \( \Phi(t, z) : W \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \)?

Note that for a quasiconformal motion \( \Phi(t, z) : W \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) and each \( t \in W \), \( \Phi_t(\cdot) : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is a quasiconformal homeomorphism and \( t \to \mu_{\Phi_t} \) is a continuous map from \( W \) to \( M(\mathbb{C}) \),
In their 1986 paper, Thurston and Sullivan asserted that the extension problem holds when $W = [0, 1]$. However, in our 2013 work, J-Mitra-Shiga-Wang, we proved the following theorem. Let $I = [0, 1]$ with the basepoint 0.

**Theorem (J-Mitra-Shiga-Wang, Tohoku, 2018)**

There exists a closed subset $E$ in $\hat{\mathbb{C}}$ with $\#(E) = \infty$, and a quasiconformal motion $\phi(t, z) : I \times E \rightarrow \hat{\mathbb{C}}$, such that $\phi$ CANNOT be extended to a quasiconformal motion $\Phi(t, z) : U \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ for any neighborhood $U$ of the basepoint 0. However, it can be extended to a continuous motion of $\hat{\mathbb{C}}$ over $I$.

In the theorem, $I$ can be replaced by any path connected Hausdorff space $W$. 
Outline of the Proof, I

We take $1 < r_1 < r_2 < \cdots < r_n < r_{n+1} < \cdots$ so that $r_{n+1}/r_n \to \infty$ as $n \to \infty$. $X = \hat{\mathbb{C}} \setminus (\{r_n\}_{n=1}^{\infty} \cup \{\infty\})$. Let $\alpha_n$ be a simple closed curve in $X$ only enclosing $r_{2n}$ and $r_{2n+1}$.

![Diagram](image-url)
Let $A_n = \{ r_{2n} \leq |z| \leq r_{2n+1} \}$ and $B_n = \{ r_{2n+1} \leq |z| \leq r_{2n+2} \}$. Take $p_n \in \mathbb{N}$ so large that

$$
\lim_{n \to \infty} \frac{l_X(\tau_{n}^{p_n}(\alpha_n))}{l_X(\alpha_n)} = \infty
$$

where $\tau_n$ is the Dehn twist in $A_n$ about its core curve and $l_X(\alpha_n)$ and $l_X(\tau_{n}^{p_n}(\alpha_n))$ mean the hyperbolic lengths of closed geodesics homotopic to $\alpha_n$ and $\tau_{n}^{p_n}(\alpha_n)$ in $X$, respectively.
Let \( E = \bigcup_{n=1}^{\infty} C_n \cup \{\infty\} \) with \( C_n = \{|z| = r_n\} \). For each \( n \in \mathbb{N} \), define \( \phi_n(t, z) = z \exp\{2\pi i n(n + 1)(t - (n + 1)^{-1})p_n\} \) for \((t, z) \in [(n + 1)^{-1}, n^{-1}] \times C_{2n}\) and \( \phi_n(t, z) = z \) elsewhere. Define

\[
\phi(t, z) = \lim_{n \to \infty} \phi_n \circ \cdots \circ \phi_1(t, z) : I \times E \to \hat{\mathbb{C}}.
\]

Then we have proven that

a) \( \phi \) can be extended to a continuous motion of \( \hat{\mathbb{C}} \) over \( I \).

b) \( \phi \) is a quasiconformal motion of \( E \) over \( I \).

c) \( \phi \) can not be extended to a quasiconformal motion of \( \hat{\mathbb{C}} \) over any \( U \).
In the proof of \( c \), we use a result of Wolpert 1979: Let \( X \) and \( Y \) be hyperbolic surfaces and \( f : X \to Y \) be a \( K \)-quasiconformal map from \( X \) to \( Y \). Then, for any non-trivial and non-peripheral closed curve \( \alpha \) on \( X \),

\[
\frac{1}{K} l_X(\alpha) \leq l_Y(f(\alpha)) \leq Kl_X(\alpha)
\]

holds, where \( l_X(\alpha) \) is the hyperbolic length of the geodesic on \( X \) homotopic to \( \alpha \) in \( X \), \( l_Y(f(\alpha)) \) is the hyperbolic length of the geodesic on \( Y \) homotopic to \( f(\alpha) \).
Corollary (J-Mitra-Shiga-Wang, Tohoku, 2018)

The motion $\Psi(t, z) = w^\mu(z) : T(E) \times E \to \hat{\mathbb{C}}$ is not universal for quasiconformal motions.
Definition (J-Mitra-Shiga-Wang, Tohoku, 2018)

Let $W$ be a connected Hausdorff space with the basepoint $t_0$. A map $\phi(t, z) : W \times E \to \hat{\mathbb{C}}$ is called a tame quasiconformal motion if

i) $\phi(t_0, z) = z$ for all $z \in E$;

ii) for each $t \in W$, the map $\phi(t, \cdot) : E \to \mathbb{C}$ is injective;

iii) given any $t \in W$, there exists a quasiconformal map $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a neighborhood $U_t$ about $t$ with the basepoint $t$ and a quasiconformal motion $\psi(t, z) : U_t \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\phi(x, z) = \psi(x, w(z))$ for all $(x, z) \in U_t \times E$. 
Theorem (J-Mitra-Shiga-Wang, Tokohu, 2018)

The motion $\Psi(t, z) = w^\mu(z) : T(E) \times E \rightarrow \hat{\mathbb{C}}$ is universal for tame quasiconformal motions. That is, for any simply connected Hausdorff space $W$ with a basepoint $t_0$ and any tame quasiconformal motion $\phi(t, z) : W \times E \rightarrow \hat{\mathbb{C}}$, there exists a unique basepoint preserving continuous map $f : W \rightarrow T(E)$ such that $f^*(\Psi_E) = \phi$. 
Corollary (J-Mitra-Shiga-Wang, Tokoku, 2018)

Let $W$ be a simply connected Hausdorff space with a basepoint $t_0$, and $\phi(t, z) : W \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion. Then, there exists a quasiconformal motion $\Phi(t, z) : W \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\Phi$ extends $\phi$. 
Let $G$ be a group of Möbius transformations, such that the closed set $E$ is invariant under $G$, that is, $g(E) = E$ for any $g \in G$. A motion $\phi : W \times E \to \mathbb{C}$ is called $G$-equivalent if for any $g \in G$ and $x \in W$, there is a Möbius transformation $\theta_x(g)$ such that $\phi(x, g(z)) = (\theta_x(g))(\phi(x, z))$.

**Corollary (G-Equivalence Extension, J-Mitra-Shiga-Wang, Tokoku, 2018)**

Let $W$ be a simply connected Hausdorff space with a basepoint, and $\phi : W \times E \to \mathbb{C}$ be a $G$-equivariant tame quasiconformal motion. Then, there exists a $G$-equivariant quasiconformal motion $\Phi(t, z) : W \times \mathbb{C} \to \mathbb{C}$ such that $\Phi$ extends $\phi$. 
Thanks!