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A Reflection Principle for the Hyperbolic Metric and Applications to Geometric Function Theory

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To Professor Maurice Heins on his 70th birthday

We establish a reflection principle for the hyperbolic metric which has applications to geometric function theory. For instance, the reflection principle yields a number of monotonicity properties of the hyperbolic metric. The sharp form of Landau's Theorem is an immediate consequence of one of these monotonicity properties. The second main application is an interpretation of the reflection principle in terms of convexity relative to hyperbolic geometry.

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1. INTRODUCTION

This paper represents an extension of the work of Jørgensen [7]. We employ differential-geometric techniques to obtain estimates for the hyperbolic metric on a Riemann surface. These inequalities have numerous applications to geometric function theory. Jørgensen considered only regions on the Riemann sphere; his work is not sufficient for most of our uses.

Our basic tool is a slight generalization of Ahlfors' Lemma [1] which is necessary for our purposes. Pommerenke ([11], [12]) has employed a similar generalization. An immediate consequence is a refinement of a reflection principle for the hyperbolic metric that is due to Jørgensen [7]. The first byproduct of this reflection principle is a simple, geometric proof of certain monotonicity properties of the hyperbolic metric. In the special case of the twice punctured plane these monotonicity properties were established by Hempel [4], who employed more complicated analytic techniques. Our version of Ahlfors' Lemma together with one of these monotonicity properties yields a short proof of the sharp form of Landau's Theorem. The original proofs of this precise form are due to Hempel [4] and Jenkins [6], independently.

Next, we present a geometric interpretation of our reflection principle in terms of convexity relative to hyperbolic geometry. Jørgensen [7] showed that if Ω is a hyperbolic plane region and Δ is any disk contained in Ω, then Δ is convex in the hyperbolic geometry on Ω. We obtain a generalization for Riemann surfaces. As a special instance we can show that if a plane region Ω is starlike with respect to a point $a \in \mathrm{cl}(\Omega)$ and Δ is any disk with center a, then $\Omega \cap \Delta$ is hyperbolically convex. In particular, if Ω is a euclidean convex region, then $\Omega \cap \Delta$ is hyperbolically convex for any disk Δ with center in $\mathrm{cl}(\Omega)$. This result is best possible: if Ω is a half-plane and Δ does not have center in $\mathrm{cl}(\Omega)$, then $\Omega \cap \Delta$ is not hyperbolically convex.

2. CONFORMAL METRICS AND RELATED CONCEPTS

For more details on the topics of this section the reader should consult [10].

Let R be a Riemann surface. A conformal metric on R is a nonnegative invariant form $\rho(z)|dz|$. If R is actually a region in $\mathbb{C}$, then we sometimes consider just the density $\rho(z)$ rather than the metric $\rho(z)|dz|$. An important example is the hypergolic metric $\lambda_{\mathbb{D}}(z)|dz| = |dz|/(1 - |z|^2)$ on the unit disk $\mathbb{D}$.

On a Riemann surface R it generally makes no sense to speak of the value of a metric $\rho(z)|dz|$ at a point $a \in R$. However, the dichotomy of either $\rho(a) = 0$ or $\rho(a) > 0$ at the point a is independent of the choice of local coordinate at a. If $\sigma(z)|dz|$ is another metric on R which is positive at the point a, then the quotient $\rho(z)|dz|/\sigma(z)|dz|$ has a value at a which is independent of the local coordinate at a. We generally write $\rho/\sigma(a)$ to denote the value of this quotient at the point a. We also write

$\rho(z)|dz| \leqslant \sigma(z)|dz|$, or simply $\rho \leqslant \sigma$, to indicate that the quotient ρ/σ is bounded above by 1.

The (Gaussian) curvature at a of a metric $\rho(z)|dz|$ which is positive and of class C^2 in a neighborhood of the point a is defined by

$$\kappa(a, \rho(z)|dz|) = -\frac{\Delta \log \rho(z)}{\rho^2(a)}\bigg|_a .$$

The value of the right-hand side is independent of the local coordinate used at a. For example, the hyperbolic metric has constant curvature -4.

If $\rho(z)|dz|$ is a positive, continuous metric on R, then it induces a distance function on R that is given by

$$d(a, b) = \inf \int_\delta \rho(z)|dz|,$$

where the infimum is taken over all locally rectifiable paths δ on R which connect a and b. This distance function is compatible with the topology of R. A path γ connecting a and b is called a geodesic (relative to the metric $\rho(z)|dz|$) if

$$d(a, b) = \int_\gamma \rho(z)|dz|.$$

In general, a geodesic need not exist or be unique when it exists. For the hyperbolic metric $\lambda_{\mathbb{D}}(z)|dz|$ the associated distance function is

$$d_{\mathbb{D}}(z, w) = \tfrac{1}{2} \log \frac{1 + \left|\dfrac{z - w}{1 - \bar{w}z}\right|}{1 - \left|\dfrac{z - w}{1 - \bar{w}z}\right|}.$$

The unique geodesics (h-geodesics) for the hyperbolic metric on $\mathbb{D}$ are arcs of circles orthogonal to the unit circle. We often use the prefix "h" to stand for either "hyperbolic" or "hyperbolically".

The pull-back of a metric via an analytic or anti-analytic function is another useful concept. Suppose $f\colon R \to S$ is either an analytic mapping or an anti-analytic mapping of Riemann surfaces and $\sigma(z)|dz|$ is a metric on S. The pull-back of $\sigma(z)|dz|$ via f, which is denoted by $f^*(\sigma(z)|dz|)$, is the metric on R defined as follows. Initially, we assume R, S are plane regions. If f is analytic, then

$$f^*(\sigma(z)|dz|) = \sigma(f(z))\left|\frac{\partial f}{\partial z}(z)\right| |dz|$$

$$= \sigma(f(z))|f'(z)||dz|.$$

When f is anti-analytic,

$$f^*(\sigma(z)|dz|) = \sigma(f(z))\left|\frac{\partial f}{\partial \bar{z}}(z)\right| |dz|.$$

Recall that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If R, S are Riemann surfaces, then similar definitions hold, but it is necessary to work in terms of local coordinates. If $\sigma(z)|dz|$ is a positive C^2 metric on S and f is locally injective, then

$$\kappa(a, f^*(\sigma(z)|dz|)) = \kappa(f(a)), \sigma(z)|dz|).$$

In particular, Gaussian curvature is invariant under both conformal and anti-conformal mappings. If γ is any path on R, then we have the change of variable formula

$$\int_\gamma f^*(\sigma(z)|dz|) = \int_{f\circ\gamma} \sigma(z)|dz|.$$

Finally, if $f: R \longrightarrow S$ and $g: S \longrightarrow T$ are analytic or anti-analytic functions, then it is straightforward to verify that

$$(g \circ f)^* = f^* \circ g^*.$$

A metric $\rho(z)|dz|$ on R is called invariant under a conformal or anti-conformal automorphism $f: R \longrightarrow R$ if $f^*(\rho(z)|dz|) = \rho(z)|dz|$. In this case the associated distance function is also invariant, that is, $d(f(a), f(b)) = d(a, b)$ for all $a, b \in R$. For instance, the hyperbolic metric on $\mathbb{D}$ is invariant under the full group $\mathrm{Aut}(\mathbb{D})$ of conformal automorphisms of the disk.

A Riemann surface Ω is called hyperbolic if its universal covering surface is conformally equivalent to the unit disk. This is equivalent to the existence of an analytic universal covering projection $f: \mathbb{D} \to \Omega$ of the disk onto Ω. The set of all such coverings is given by $\{f \circ T\colon\ T \in \mathrm{Aut}(\mathbb{D})\}$. The only Riemann surfaces which are not hyperbolic are those conformally equivalent to $\mathbb{C} \setminus \{0\}$, $\mathbb{C}$, $\mathbb{P}$ or a torus. For a hyperbolic plane region Ω a covering is uniquely determined by specifying $f(0)$ and $\arg f'(0)$. For a hyperbolic Riemann surface Ω there is a unique, positive, real-analytic metric $\lambda_\Omega(z)|dz|$ on Ω such that $f^*(\lambda_\Omega(z)|dz|) = \lambda_{\mathbb{D}}(z)|dz|$. This metric is independent of the choice of the covering projection because $\lambda_{\mathbb{D}}(z)|dz|$ is invariant under $\mathrm{Aut}(\mathbb{D})$. If Ω is a plane region, then the hyperbolic metric is determined from

$$\lambda_\Omega(f(z))|f'(z)| = \frac{1}{1-|z|^2}.$$

In particular, if $f(0) = a$, then $\lambda_\Omega(a) = 1/|f'(0)|$.

The hyperbolic metric on Ω has constant curvature -4 and is invariant under both conformal and anti-conformal mappings. In particular, it is invariant under the group $\mathrm{Aut}(\Omega)$ of conformal automorphisms of Ω. Let d_Ω denote the distance function on Ω that is induced by $\lambda_\Omega(z)|dz|$. A geodesic relative to the hyperbolic metric will be termed an h-geodesic. For any $a, b \in \Omega$ an h-geodesic exists but it may not be unique if Ω is not simply connected. In any case, an h-geodesic γ on R is always the image of an h-geodesic $\tilde{\gamma}$ in $\mathbb{D}$ under a universal covering projection. For example, if $\mathbb{H} = \{z\colon \mathrm{Im}(z) > 0\}$ is the upper half-plane, then $\lambda_{\mathbb{H}}(z)|dz| = \frac{1}{2}\,\mathrm{Im}(z)$ and h-geodesics are circular arcs and line segments orthogonal to the real axis $\mathbb{R}$.

3. A VERSION OF THE AHLFORS' LEMMA

We require a slight extension of the usual form of the Ahlfors' Lemma. Pommerenke ([11], [12]) established a similar result but in function theoretic terms rather than the differential-geometric language that is suitable for our purposes.

THEOREM 1 *Let Ω be a hyperbolic Riemann surface. Suppose that $\rho(z)|dz|$ is an upper semicontinuous, nonnegative metric on Ω such that for any $a \in \Omega$ either $\rho/\lambda_\Omega(a) \leq 1$ or else $\rho(a) > 0$ and $\rho(z)|dz|$ has a supporting metric on a neighborhood of a. Then $\rho/\lambda_\Omega \leq 1$.*

Remarks A metric $\rho_a(z)|dz|$ is called a *supporting metric* for $\rho(z)|dz|$ at the point a if $\rho_a(z)|dz|$ is defined, positive and of class C^2 in a neighborhood U of a, has curvature at most -4 and $\rho/\rho_a \geqslant 1$ in U with equality at the point a. The usual form of the Ahlfors' Lemma [1] requires the existence of a supporting metric at every point a such that $\rho(a) > 0$, so Theorem 1 is a generalization.

Proof First, we assume the validity of the theorem for the unit disk $\mathbb{D}$. Let $f: \mathbb{D} \to \Omega$ be an analytic universal covering projection. Then $\sigma(z)|dz| = f^*(\rho(z)|dz|)$ is an upper semicontinuous, nonnegative metric on $\mathbb{D}$. Since $\lambda_{\mathbb{D}}(z)|dz| = f^*(\lambda_\Omega(z)|dz|)$, it follows that $\rho/\lambda_\Omega(a) \leqslant 1$ implies $\sigma/\lambda_{\mathbb{D}}(b) \leqslant 1$ for all $b \in f^{-1}(a)$. Similarly, if $\rho_a(z)|dz|$ is a supporting metric for $\rho(z)|dz|$ at a, then $\sigma_a(z)|dz| = f^*(\rho_a(z)|dz|)$ is a supporting metric for $\sigma(z)|dz|$ at each point $b \in f^{-1}(a)$. This is true because the invariance of curvature under a pull-back implies that $\sigma_a(z)|dz|$ has curvature at most -4 and because $\rho/\rho_a \geqslant 1$ near a with equality at a implies $\sigma/\sigma_a \geqslant 1$ in a neighborhood of each $b \in f^{-1}(a)$ with equality at b. Consequently, $\sigma(z)|dz|$ satisfies the hypotheses of the theorem on $\mathbb{D}$, so we deduce that $\sigma/\lambda_{\mathbb{D}} \leqslant 1$. This yields $\rho/\lambda_\Omega \leqslant 1$.

All that remains is to establish the theorem in the special case $\Omega = \mathbb{D}$. In fact, it is sufficient to show that for fixed $r \in (0, 1)$

$$\rho(z) \leqslant \frac{r}{r^2 - |z|^2} = \lambda_r(z)$$

holds when $|z| < r$. The general result follows by letting r increase to 1. Since $v = \log \lambda_r - \log \rho$ is lower semicontinuous and tends to $+\infty$ when $|z| \to r$, the function v attains a minimum value at a point a with $|a| < r$. If the point a is such that $\rho(a) \leqslant \lambda_{\mathbb{D}}(a) < \lambda_r(a)$, then $v(a) > 0$ and so $\rho(z) < \lambda_r(z)$ for $|z| < r$. Otherwise, $\rho(z)|dz|$ has a supporting metric at a and the proof is the same as Ahlfors' original [1].

The next result will be used to demonstrate the sharpness of some subsequent theorems that are obtained from our version of Ahlfors' Lemma.

THEOREM 2 *Suppose R is a Riemann surface and Ω, Δ are hyperbolic subsurfaces. If $a \in \Omega \cap \Delta$ and $\lambda_\Delta/\lambda_\Omega \leqslant 1$ in a neighborhood of a with equality at a, then $\Delta = \Omega$.*

Proof Take $f: \mathbb{D} \to \Omega$ and $g: \mathbb{D} \to \Delta$ to be analytic universal covering projections with $f(0) = a = g(0)$. Let g^{-1} denote the branch of the inverse of g that is defined in a neighborhood of a and satisfies

$g^{-1}(a) = 0$. Then $h = g^{-1} \circ f$ is defined in a small disk $D(\varepsilon)$ about the origin and satisfies $h(0) = 0$ and $|h'(0)| = 1$. The latter is true since $\lambda_\Delta/\lambda_\Omega(a) = 1$. From $g \circ h = f$ we have on $D(\varepsilon)$

$$\begin{aligned}\lambda_{\mathbb{D}}(z)|dz| &= f^*(\lambda_\Omega(z)|dz|) \geqslant f^*(\lambda_\Delta(z)|dz|)\\ &= h^*(g^*(\lambda_\Delta(z)|dz|)) = h^*(\lambda_{\mathbb{D}}(z)|dz|).\end{aligned}$$

If γ is the radial path from 0 to $z \in D(\varepsilon)$, then

$$\begin{aligned}d_{\mathbb{D}}(0, z) &= \int_\gamma \lambda_{\mathbb{D}}(\zeta)|d\zeta| \geqslant \int_\gamma h^*(\lambda_{\mathbb{D}}(\zeta)|d\zeta|)\\ &= \int_{h \circ \gamma} \lambda_{\mathbb{D}}(\zeta)|d\zeta| \geqslant d_{\mathbb{D}}(0, h(z)).\end{aligned}$$

This yields $|h(z)| \leqslant |z|$ for $z \in D(\varepsilon)$, so h maps $D(\varepsilon)$ into itself. Schwarz' Lemma applied to h on $D(\varepsilon)$ gives $|h'(0)| \leqslant 1$ with equality if and only if h is a rotation about the origin. Since $|h'(0)| = 1$, we conclude that $h(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$ Then $f(z) = g(e^{i\theta}z)$ for $z \in D(\varepsilon)$ and this identity must continue to hold on $\mathbb{D}$. In particular, $\Omega = f(\mathbb{D}) = g(\mathbb{D}) = \Delta$.

4. A REFLECTION PRINCIPLE FOR THE HYPERBOLIC METRIC

We begin by establishing certain notation that will be in force throughout this section. Let $\bar{R}$ be a bordered Riemann surface, R the interior, ∂R the nonempty border oriented so that R lies to the left and $\hat{R}$ the Schottky double of R across ∂R [2, p. 119]. Suppose $j\colon \hat{R} \to \hat{R}$ is the associated anticonformal involution that fixes ∂R pointwise. A subsurface Ω of $\hat{R}$ is called *symmetric about* ∂R provided $j(\Omega) = \Omega$. If Ω is hyperbolic and symmetric about ∂R, then it is straightforward to verify that the hyperbolic metric on Ω is also symmetric; that is, $j^*(\lambda_\Omega(z)|dz|) = \lambda_\Omega(z)|dz|$. In any case, $j^*(\lambda_\Omega(z)|dz|) = \lambda_{\Omega^*}(z)|dz|$ is the hyperbolic metric on $\Omega^* = j(\Omega)$, the reflection of Ω about ∂R.

THEOREM 3 *Let Ω be a hyperbolic subsurface of $\hat{R}$ such that $\Omega \cap \partial R \neq \varnothing$ and $j(\Omega\backslash R) \subset \Omega$, or equivalently, $\Omega\backslash R \subset \Omega^*$. Then $\lambda_{\Omega^*}/\lambda_\Omega(a) \leqslant 1$ for $a \in \Omega\backslash \bar{R}$ with equality if and only if Ω is symmetric about ∂R.*

Proof We have already noted that equality holds if Ω is symmetric about ∂R. Define a metric $\rho(z)|dz|$ on Ω by

$$\rho(z)|dz| = \begin{cases} \lambda_\Omega(z)|dz| & \text{on} \quad \Omega \cap \bar{R}, \\ \lambda_{\Omega^*}(z)|dz| & \text{on} \quad \Omega \setminus R. \end{cases}$$

The second portion of this definition makes sense because $\Omega \setminus R \subset \Omega^*$ by hypothesis. In order to conclude that $\rho(z)|dz|$ is a continuous metric on Ω, we must show that the two parts of the definition are the same on $\Omega \cap \partial R$. From $j \circ j(z) = z$ we obtain $(\partial j/\partial \bar{z})(j(z))(\partial \bar{j}/\partial z)(z) = 1$. This yields $|(\partial j/\partial \bar{z})(z)| = 1$ at any fixed point of j; in particular, this holds at each point of $\Omega \cap \partial R$. This demonstrates that $\rho(z)|dz|$ is well-defined and continuous on Ω. Trivially, $\rho/\lambda_\Omega \leqslant 1$ on $\Omega \cap \partial R$ and $\rho(z)|dz|$ has constant curvature -4 on $\Omega \setminus \partial R$, so $\rho(z)|dz|$ is its own supporting metric at each point of $\Omega \setminus \partial R$. Theorem 1 implies that $\rho/\lambda_\Omega \leqslant 1$, which produces the inequality of the theorem for $a \in \Omega \setminus \bar{R}$. If equality holds at a point $a \in \Omega \setminus \bar{R}$, then we have $\lambda_{\Omega^*}/\lambda_\Omega = \rho/\lambda_\Omega \leqslant 1$ in a neighborhood of a with equality at a and Theorem 2 implies $\Omega = \Omega^*$.

Remark Jørgensen [7] established this theorem in the special case in which R is an open half-plane in $\mathbb{C}$, ∂R is the circle on the Riemann sphere that bounds R, $\hat{R} = \mathbb{P}$ and $R \subset \Omega$. In this paper our applications of Theorem 3 will generally be to regions on the sphere but we will not always have $R \subset \Omega$, so Jørgensen's version of Theorem 3 does not suffice for our purposes. Also, Jørgensen used the boundary behavior of the hyperbolic metric in his proof, while this issue does not even enter into our proof.

COROLLARY 1 $(\partial/\partial n)(\lambda_{\Omega^*}/\lambda_\Omega)(b) \geqslant 0$ *for* $b \in \Omega \cap \partial R$ *with strict inequality unless* Ω *is symmetric about* ∂R, *where* $\partial/\partial n$ *denotes differentiation in the direction of the inward-pointing normal on* ∂R.

Proof Fix $b \in \Omega \cap \partial R$. Because $\lambda_{\Omega^*}/\lambda_\Omega \leqslant 1$ on $\Omega \setminus R$ with equality at b, it is elementary that $(\partial/\partial n)(\lambda_{\Omega^*}/\lambda_\Omega)(b) \geqslant 0$. Jørgensen [10, Lemma 1.2] showed that if $\lambda_{\Omega^*}/\lambda_\Omega < 1$ in a disk with equality at a boundary point, then $\partial/\partial r \log(\lambda_{\Omega^*}/\lambda_\Omega) > 0$ at this boundary point, where $\partial/\partial r$ denotes differentiation in the radial direction away from the center of the disk. By applying this result to a small disk in $\Omega \setminus \bar{R}$ that is tangent to ∂R at b, we conclude that $(\partial/\partial n)(\lambda_{\Omega^*}/\lambda_\Omega)(b) > 0$ when Ω is not symmetric about ∂R. This is essentially an instance of a strong form of the maximum principle for linear elliptic partial differential equations that is due to Hopf ([5], [13, Chapter 2]).

COROLLARY 2 *For $b \in \Omega \cap \partial R$ and $a \in \Omega \setminus R$, $d_\Omega(a, b) \geq d_\Omega(j(a), b)$ with inequality unless Ω is symmetric about ∂R.*

Proof Let γ be an h-geodesic on Ω from a to b. Take γ_1 to be the subarc of γ from a to the first point of intersection of γ with ∂R and γ_2 to be the remainder, if any, of γ. Then $\gamma = \gamma_1 + \gamma_2$ and $\gamma_1 \subset \Omega \setminus R$ so

$$\begin{aligned} d_\Omega(a, b) &= \int_\gamma \lambda_\Omega(z)|dz| = \int_{\gamma_1} \lambda_\Omega(z)|dz| + \int_{\gamma_2} \lambda_\Omega(z)|dz| \\ &\geq \int_{\gamma_1} j^*(\lambda_\Omega(z)|dz|) + \int_{\gamma_2} \lambda_\Omega(z)|dz| \\ &= \int_{j \circ \gamma_1} \lambda_\Omega(z)|dz| + \int_{\gamma_2} \lambda_\Omega(z)|dz| \geq d_\Omega(j(a), b), \end{aligned}$$

because $j \circ \gamma_1 + \gamma_2$ is a path on Ω from $j(a)$ to b. If Ω is not symmetric about ∂R, then $\lambda_{\Omega^*}/\lambda_\Omega < 1$ on $\gamma_1 \setminus \partial R$ and strict inequality holds in the above chain of inequalities.

Remark A special case of Corollary 2 for simply connected regions is due to Ullman [14] and Jørgensen [7, Theorem 3] extended it to multiply connected regions.

We shall most often apply the results of this section to the following situation. Γ will denote a circle on the Riemann sphere, R will be one of the open disks on $\mathbb{P}$ determined by Γ and $\bar{R} = R \cup \Gamma$. In this case we can regard $\mathbb{P}$ itself as the Schottky double of R across $\partial R = \Gamma$ and j is ordinary reflection in the circle Γ. If $\Omega \cap \Gamma \neq \emptyset$ and $j(\Omega \setminus R) \subset \Omega$, then

$$\lambda_\Omega(z) \geq \lambda_\Omega(j(z))\left|\frac{\partial j}{\partial \bar{z}}(z)\right| = \lambda_{\Omega^*}(z)$$

for $z \in \Omega \setminus \bar{R}$ with strict inequality unless Ω is symmetric about Γ. If Γ is a straight line, then $|\partial j/\partial \bar{z}| = 1$. Also, for $z \in \Omega \cap \Gamma$,

$$\frac{\partial \lambda_{\Omega^*}}{\partial n}(z) \geq \frac{\partial \lambda_\Omega}{\partial n}(z)$$

with strict inequality unless Ω is symmetric about Γ. When Γ is a straight line, this simplifies to

$$0 \geqslant \frac{\partial \lambda_\Omega(z)}{\partial n}$$

for $z \in \Omega \cap \Gamma$.

5. MONOTONICITY PROPERTIES OF THE HYPERBOLIC METRIC AND LANDAU'S THEOREM

Hempel [4] established several monotonicity properties for the hyperbolic metric of $\mathbb{C}\setminus\{0, 1\}$ by making use of a maximum principle for for partial differential equations together with boundary estimates for the density of the hyperbolic metric that are obtained by using the classical theory of the elliptic modular function. We present simple, geometric proofs of various monotonicity properties of the hyperbolic metric which contain those of Hempel as special instances. As one application of his results, Hempel derived a sharp form of Landau's theorem. An independent proof was given by Jenkins [6], who employed ideas from the topological theory of functions. We give a direct proof of the explicit expression for the bound in Landau's Theorem.

THEOREM 4 *Let Ω be a hyperbolic region in $\mathbb{C}$.*

(i) *If* $\{z\colon \operatorname{Im}(z) > 0\} \subset \Omega$, *then* $\partial\lambda_\Omega/\partial y < 0$ *for* $\operatorname{Im}(z) > 0$.
(ii) *If* $\partial\Omega \subset [0, \infty]$, *then* $\partial\lambda_\Omega/\partial\theta < 0$ *for* $0 < \theta < \pi$ *with the reverse inequality for* $-\pi < \theta < 0$.
(iii) *If* $\{z\colon\ 0 < |z - a| < \rho\} \subset \Omega$ *and* $z = a + re^{i\theta}$, *then* $\partial\lambda_\Omega/\partial r > -(\lambda_\Omega/r)$ *for* $r \in (0, \rho)$ *so that* $r\lambda_\Omega(a + re^{i\theta})$ *is strictly increasing on* $(0, \rho)$ *for each fixed* θ.

Proof (i) Fix $y_0 > 0$ and let $\bar{R} = \{z\colon \operatorname{Im}(z) \geqslant y_0\}$. Since $\Omega \supset \bar{R}$, Corollary 1 of Theorem 3 gives $\partial\lambda_\Omega/\partial y \leqslant 0$. Strict inequality must hold since symmetry about ∂R would give $\Omega = \mathbb{C}$ and $\mathbb{C}$ is not hyperbolic.

(ii) Because Ω is symmetric about the real axis, $\lambda_\Omega(\bar{z}) = \lambda_\Omega(z)$. Therefore, it suffices to establish $\partial\lambda_\Omega/\partial\theta < 0$ for $0 < \theta < \pi$ since then the reverse inequality for $-\pi < \theta < 0$ follows automatically. Fix $\theta_0 \in (0, \pi)$. Let Γ be the line through 0 and $e^{i\theta_0}$ and R the half-plane determined by Γ which contains -1. If $\bar{R} = R \cup \Gamma$, then $\Omega \supset \bar{R}\setminus\{0\} \supset R$ and again

Corollary 1 of Theorem 3 produces $\partial\lambda_\Omega/\partial\theta \leqslant 0$. Equality would imply Ω is symmetric about Γ and so $\Omega \supset \mathbb{C}\backslash\{0\}$ which is impossible.

(iii) There is no harm in assuming that $a = 0$. Fix $r_0 \in (0, \rho)$ and let $\Gamma = \{z: |z| = r_0\}$, $\bar{R} = \{z: 0 < |z| \leqslant r_0\}$. Then $j(z) = r_0^2/\bar{z}$ is reflection in $\Gamma = \partial R$ and clearly $j(\Omega\backslash R) \subset R \subset \Omega$. Now, Corollary 1 of Theorem 3 yields

$$\frac{\partial}{\partial r}[\lambda_\Omega(r_0^2/\bar{z})r_0^2/|z|^2] \leqslant \frac{\partial\lambda_\Omega}{\partial r}(z)$$

for $|z| = r_0$. This is equivalent to the inequality in part (iii) of the theorem. Equality would imply $\Omega \supset \mathbb{C}\backslash\{0\}$, a contradiction.

Remarks From part (ii) of Theorem 4 we conclude that $\lambda_\Omega(re^{i\theta})$ is strictly decreasing on $(0, \pi)$ and strictly increasing on $(-\pi, 0)$. This is a special case of a symmetry property of the hyperbolic metric due to Weitsman [15] In particular, this implies that on each circle about the origin the density $\lambda_{0,1}$ of the hyperbolic metric $\lambda_{0,1}(z)|dz|$ on $\mathbb{C}\backslash\{0, 1\}$ attains its minimum value on the negative real axis, a result due to Lehto, Virtanen and Vaisälä [8]. For the unit circle this was rediscovered by Jenkins [6]. Set

$$K = \frac{1}{2\lambda_{0,1}(-1)} = \frac{\Gamma(1/4)^4}{4\pi^2} = 4.3768796\ldots$$

([4], [6]); $\lambda_{0,1}(-1)$ is the minimum value of $\lambda_{0,1}$ on the unit circle and -1 is the unique point at which the minimum is attained.

THEOREM 5 *For* $z \in \mathbb{C}\backslash\{0, 1\}$

$$\lambda_{0,1}(z) \geqslant \frac{1}{2|z|(|\log|z|| + K)},$$

with strict inequality unless $z = -1$.

Proof Define a continuous metrix $\rho(z)|dz|$ on $\mathbb{C}\backslash\{0\}$ by $\rho(z) = 1/2|z|(|\log|z|| + K)$. For $|z| = 1$, $\rho(z) = 1/2K = \lambda_{0,1}(-1) \leqslant \lambda_{0,1}(z)$ with strict inequality for $z \neq -1$. Next, we show that $\rho(z)|dz|$ has curvature -4 off the unit circle. This can be accomplished by direct calculation, but here is an easier method that also yields additional information. If $r = e^K > 1$, then for $0 < |z| < 1$

$$\rho(z) = \frac{1}{2|z|(\log r - \log|z|)} = \lambda_{D'(r)}(z)$$

is the density of the hyperbolic metric on the punctured disk $D'(r) = \{z: 0 < |z| < r\}$. Thus, $\rho(z)|dz|$ obviously has curvature -4 for $0 < |z| < 1$. If $h(z) = 1/z$, then direct calculation gives $h^*(\rho(z)|dz|) = \rho(z)|dz|$. Because curvature is invariant under a conformal mapping, we may conclude that $\rho(z)|dz|$ also has curvature -4 for $|z| > 1$. Therefore, $\rho(z)|dz|$ is its own supporting metric off the unit circle. Theorem 1 gives $\rho(z) \leqslant \lambda_{0,1}(z)$ for $z \in \mathbb{C}\setminus\{0, 1\}$. We have already observed that strict inequality holds for $|z| = 1$, $z \neq -1$. It remains to show that strict inequality holds off the unit circle. If equality held at a, $0 < |a| < 1$, then $\lambda_{D'(r)}(z) = \rho(z) \leqslant \lambda_{0,1}(z)$ for z near a with equality at a and Theorem 2 would imply $\mathbb{C}\setminus\{0, 1\} = D'(r)$, a contradiction. From $h^*(\rho(z)|dz|) = \rho(z)|dz|$ and $h^*(\lambda_{0,1}(z)|dz|) = \lambda_{0,1}(z)|dz|$ (because h is a conformal automorphism of $\mathbb{C}\setminus\{0, 1\}$), we know that equality at a would also result in equality at $1/a$. Hence, strict inequality also holds for $|z| > 1$.

COROLLARY (Landau's Theorem) *If f is holomorphic in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{C}\setminus\{0, 1\}$, then for $z \in \mathbb{D}$*

$$(1 - |z|^2)|f'(z)| \leqslant 2|f(z)|(|\log|f(z)|| + K).$$

Equality holds at $a \in \mathbb{D}$ if and only if f is a holomorphic universal covering of $\mathbb{D}$ onto $\mathbb{C}\setminus\{0, 1\}$ with $f(a) = -1$.

Proof The principle of hyperbolic metric gives

$$\lambda_{0,1}(f(z))|f'(z)| \leqslant \lambda_{\mathbb{D}}(z) = \frac{1}{1 - |z|^2},$$

with equality if and only if f is a holomorphic covering of $\mathbb{D}$ onto $\mathbb{C}\setminus\{0, 1\}$. Theorem 5 gives

$$\frac{f'(z)}{2|f(z)|(|\log|f(z)|| + K)} \leqslant \lambda_{0,1}(f(z))|f'(z)|.$$

Necessary and sufficient for equality at $z = a$ where $f'(a) \neq 0$ is that $f(a) = -1$. By combining these two inequalities, we obtain the desired result.

6. HYPERBOLIC CONVEXITY

We now give an interpretation of Theorem 3 in terms of convexity relative to hyperbolic geometry. Suppose Ω is a hyperbolic Riemann surface and E a subset of Ω. E is called *hyperbolically convex*, or *h-convex* for short, if for any pair a, b of distinct points in E every h-geodesic joining a and b also lies in E. Recall that an h-geodesic need not be unique if Ω is not simply connected. Let R, $\bar{R}$, ∂R, $\hat{R}$ and j be as in Section 4.

THEOREM 6 *Let Ω be a hyperbolic subsurface of $\hat{R}$ such that $\Omega \cap \partial R \neq \emptyset$ and $j(\Omega_b R) \subset \Omega$. Then $\Omega \cap R$ is an h-convex subset of Ω.*

Proof First, suppose that Ω is symmetric about ∂R. We wish to show that any h-geodesic γ that joins $a, b \in \Omega \cap R$ must remain in $\Omega \cap R$. Let $f: \mathbb{D} \to \Omega$ be an analytic universal covering projection such that $f(0) \in \Omega \cap R$ and f maps the positive direction along the real axis at the origin into the positive direction along ∂R at $f(0)$. Then $j(f(\bar{z}))$ has the same properties, so $f(z) = j(f(\bar{z}))$. Then f maps $(-1, 1)$ onto a single contour of ∂R, f maps $\mathbb{D}^+ = \{z \in \mathbb{D} : \operatorname{Im}(z) > 0\}$ onto $\Omega \cap R$ and $\mathbb{D}^- = \{z \in \mathbb{D} : \operatorname{Im}(z) < 0\}$ onto $\Omega \cap (\hat{R} \setminus \bar{R})$. Fix $\tilde{a} \in \mathbb{D}^+$ with $f(\tilde{a}) = a$ and let $\tilde{\gamma}$ be the unique lift of γ via f with initial point $\tilde{a}$. Then $\tilde{\gamma}$ is an h-geodesic in $\mathbb{D}$ connecting $\tilde{a} \in \mathbb{D}^+$ to a point $\tilde{b}$ that lies over b. Because $f(\mathbb{D}^-) \subset \hat{R} \setminus \bar{R}$, we must have $\tilde{b} \in \mathbb{D}^+$. Since $\mathbb{D}^+$ is h-convex, it follows that $\tilde{\gamma} \subset \mathbb{D}^+$ and so $\gamma = f \circ \tilde{\gamma} \subset \Omega \cap R$.

Next, assume that Ω is not symmetric about ∂R. The initial step is to show that any h-geodesic γ connecting $a, b \in \Omega \cap R$ must lie in $\Omega \cap \bar{R}$. We are assuming that

$$d_\Omega(a, b) = \int_\gamma \lambda_\Omega(z)|dz|.$$

Because γ is a compact regular analytic arc, γ meets ∂R in only finitely many points; otherwise, γ would be contained in some contour of ∂R [9], which is nonsensical. If γ did not remain in $\Omega \cap \bar{R}$, then γ would contain a subarc δ such that the endpoints of δ lie on $\Omega \cap \partial R$ and otherwise δ is contained in $\Omega \setminus \bar{R}$. Then $j \circ \delta$ has the same endpoints as δ and since Ω is not symmetric about ∂R,

$$\int_{j \circ \delta} \lambda_\Omega(z)|dz| = \int_\delta j^*(\lambda_\Omega(z)|dz|) < \int_\delta \lambda_\Omega(z)|dz|.$$

Consequently, if we would replace the subarc δ of γ by $j \circ \delta$, then we would obtain a path on Ω joining a and b with strictly smaller hyperbolic length than γ. This is impossible since γ is an h-geodesic. Therefore, $\gamma \subset \Omega \cap \bar{R}$.

All that remains is to show that γ does not meet ∂R. Let $f: \mathbb{D} \to \Omega$ be an analytic universal covering projection with $f(0) = a$. The group G of cover transformations consists of all conformal automorphisms T of $\mathbb{D}$ such that $f \circ T = f$. Let Δ denote the Dirichlet fundamental region for G with center 0; that is, $\Delta = \{z: d_{\mathbb{D}}(0, z) < d_{\mathbb{D}}(T(0), z)$ for all $T \in G, T \neq I\}$. The covering projection f is injective on Δ. Let $\tilde{\gamma}$ be the lift of γ with initial point 0. The terminal point $\tilde{b}$ of $\tilde{\gamma}$ lies over b and $\tilde{b} \in \mathrm{cl}(\Delta)$. By moving b slightly towards a along γ, if necessary, we may assume $\tilde{b} \in \Delta$. Since γ is an h-geodesic, so is $\tilde{\gamma}$. Thus, $\tilde{\gamma}$ is a radial line segment. For c near b let δ be an h-geodesic from a to c. We know that $\delta \subset \Omega \cap \bar{R}$. Let $\tilde{\delta}$ be the lift with initial point 0. Then $\tilde{\delta}$ is a radial segment from 0 to $\tilde{c} \in \Delta$ which lies over c. Hence, there is a small closed disk about $\tilde{b}$ such that any radial segment from 0 to a point of this disk projects to an h-geodesic in $\Omega \cap \bar{R}$. Let K be the closed convex hull of the set consisting of 0 together with the closed disk about $\tilde{b}$. Then $f(K) \subset \Omega \cap \bar{R}$, so the open mapping theorem gives $f(\mathrm{int}(K)) \subset \Omega \cap R$. We conclude that $\gamma \subset \Omega \cap R$, so $\Omega \cap R$ is h-convex

Example 1 Let $\Omega = \mathbb{C}\setminus\{0, 1\}$, $R = \{z: 0 < |z| \leqslant 1\}$, $\hat{R} = \mathbb{C}\setminus\{0\}$ and $j(z) = 1/\bar{z}$. Clearly, $j(\Omega\setminus R) = \bar{R}\setminus\{1\} \subset \Omega$, so we know that $\Omega \cap R = \{z: 0 < |z| < 1\}$ is h-convex in Ω.

Example 2 Let $\Omega = \{z: \rho^{-1} < |z| < \rho\}$, $\rho > 1$, $\bar{R} = \{z: \mathrm{Im}(z) \leqslant d\}$ and j denote reflection in the horizontal line $\mathrm{Im}(z) = d$. If $d \geqslant (1/2)(\rho + \rho^{-1}) > 1$, then $\Omega \cap R$ is h-convex since $j(\Omega\setminus R) \subset \Omega$. Note that $\Omega \cap R$ is doubly connected, $\pm 1 \in \Omega \cap R$ and both arcs of the unit circle joining ± 1 are h-geodesics.

Jørgensen [7] remarked that an open disk or half-plane contained in a hyperbolic region on $\mathbb{P}$ is always h-convex. Flinn [3] showed that the only open sets E in $\mathbb{P}$ with the property that E is h-convex in every simply connected region containing E were disks and half-planes. Theorem 6 lets us conclude that certain sets besides disks and half-planes are sometimes h-convex. Examples 1 and 2 illustrate this. We now offer a simple geometric criterion for h-convexity which includes disks contained in the region as a special case. A region Ω is said to be *starlike with respect to* $c \in \mathrm{cl}(\Omega)$ provided that for every $z \in \Omega$ the half-open line segment $(c, z]$ belongs to Ω.

THEOREM 7 *Suppose Ω is a hyperbolic region in $\mathbb{C}$ and Ω is starlike with* *respect to* $c \in \mathrm{cl}(\Omega)$. *Then* $\Omega \cap \{z: |z-c| < r\}$ *is h-convex in* Ω *for any* $r > 0$.

Proof Let $\Gamma = \{z: |z-c| = r\}$, $\bar{R} = \{z: |z-c| \leqslant r\}$ and j denote reflection in Γ. It is sufficient to show that $j(\Omega \setminus R) \subset \Omega$. Consider any $z \in \Omega \setminus R$. By hypothesis the segment $(c, z]$ lies in Ω. Set $\zeta = (c, z] \cap \Gamma$. Then $(c, \zeta] \subset \Omega$ and the reflection $j(z)$ of z lies on $(c, \zeta]$ because j maps each ray emanating from c onto itself, but j interchanges the interior and exterior of Γ.

Corollary 1 *Suppose* $f: \mathbb{D} \to \Omega$ *is a conformal mapping with* $f(0) = 0$ *and* Ω *is starlike with respect to the origin. Then for any* $r > 0$ *the set* $f^{-1}(\Omega \cap \{z: |z| < r\})$ *is h-convex in* $\mathbb{D}$. *In particular,* $f^{-1}(\Omega \cap \{z: |z| < r\})$ *is starlike with respect to the origin.*

Proof The theorem insures us that $\Omega \cap \{z: |z| < r\}$ is h-convex in Ω. Because h-convexity is a conformal invariant, $f^{-1}(\Omega \cap \{z: |z| < r\})$ is h-convex in $\mathbb{D}$. Since radial line segments in $\mathbb{D}$ are h-geodesics, any set E which contains the origin and is h-convex is starlike with respect to the origin.

Corollary 2 *Let* $\Omega \neq \mathbb{C}$ *be a convex region in* $\mathbb{C}$. *Then for any* $c \in \mathrm{cl}(\Omega) \cap \mathbb{C}$ *and any* $r > 0$, $\Omega \cap \{z: |z-c| < r\}$ *is h-convex in* Ω.

Proof Since Ω is convex, it is starlike with respect to every point in $\mathrm{cl}(\Omega) \cap \mathbb{C}$.

Remark In the following sense Corollary 2 is best possible. If $\mathbb{H}$ is the upper half-plane, then the h-geodesics are circles and lines that are orthogonal to the real axis. Of course, circles orthogonal to the real axis have their center on the real axis. Simple geometric considerations show that if $\mathrm{Im}(c) < 0$, then $\mathbb{H} \cap \{z: |z-c| < r\}$ is not h-convex in $\mathbb{H}$.

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