Math 131 Second Midterm Solutions

Problem 1. [12 points] In each case, find the derivative $y' = \dfrac{dy}{dx}$:

(i) $y = (x^4 - x^2 + 7)^8$

By the chain rule,

$$y' = 8(x^4 - x^2 + 7)^7 \cdot (x^4 - x^2 + 7)' = 8(x^4 - x^2 + 7)^7\ (4x^3 - 2x).$$

(ii) $y = x\sqrt{x^3 + 5}$

By the product and chain rules,

$$\begin{aligned} y' &= (x)' \cdot \sqrt{x^3+5} + (\sqrt{x^3+5})' \cdot x \\ &= \sqrt{x^3+5} + \left(\frac{1}{2\sqrt{x^3+5}} \cdot (x^3+5)' \right) \cdot x \\ &= \sqrt{x^3+5} + \frac{3x^3}{2\sqrt{x^3+5}}. \end{aligned}$$

(iii) $xy^2 + 5y = x^3 + 1$

We use implicit differentiation. Taking d/dx of each side gives

$$\begin{aligned} 1 \cdot y^2 + x \cdot 2yy' + 5y' &= 3x^2 \\ y'(2xy+5) &= 3x^2 - y^2 \\ y' &= \frac{3x^2 - y^2}{2xy+5}. \end{aligned}$$

Problem 2. [8 points] A manufacturer's total revenue is $R(q) = 240q - 0.05q^2$ dollars when q units are produced. The current production level is 40 units.

(i) Use marginal analysis to estimate the additional revenue that will be generated by increasing the production level to 41 units.

The additional revenue $R(41) - R(40)$ is approximately equal to the marginal revenue $R'(40)$. We have

$$R'(q) = 240 - 0.1q \Longrightarrow R'(40) = 240 - (0.1)(40) = \$236.$$

(ii) Find the actual additional revenue in part (i).

$$\begin{aligned} R(41) - R(40) &= [(240)(41) - (0.05)(41^2)] - [(240)(40) - (0.05)(40^2)] \\ &= 9755.95 - 9520.00 \\ &= \$235.95 \end{aligned}$$

Problem 3. [8 points] When the price of a certain product is p dollars per unit, consumers demand x hundred units of the product, where

$$75x^2 + 15p^2 = 3690.$$

How fast is the demand x changing with respect to time when the price is 11 dollars and is decreasing at the rate of 75 cents per month?

We want to know x' at the moment when $p = 11$ and $p' = -0.75$. First we can use the given equation to find the supply x at this moment:

$$75x^2 + 15 \cdot 11^2 = 3690 \Longrightarrow 75x^2 = 1875 \Longrightarrow x^2 = 25 \Longrightarrow x = 5.$$

Differentiating both sides of the equation $75x^2 + 15p^2 = 3690$ with respect to time gives

$$150xx' + 30pp' = 0 \Longrightarrow x' = -\frac{pp'}{5x}.$$

Substituting numerical values, we obtain

$$x' = -\frac{(11)(-0.75)}{(5)(5)} = 0.33.$$

It follows that at the moment in question, the supply is increasing at the rate of 33 units per month.

Problem 4. [12 points] Let $f(x) = 3x^4 - 6x^2 + 3$. Find the intervals of increase/decrease and concavity, critical points and their types and inflection points of f. Use this information to sketch a graph of f. Compare your answer with the graph of f plotted on your calculator.

Being a polynomial, the domain of f is all real x. We have

$$f'(x) = 12x^3 - 12x = 12x(x^2 - 1).$$

Setting $f'(x) = 0$ gives $12x = 0$ or $x^2 - 1 = 0$, so the critical points are at $x = 0$, $x = -1$ and $x = 1$. Differentiating once more, we obtain

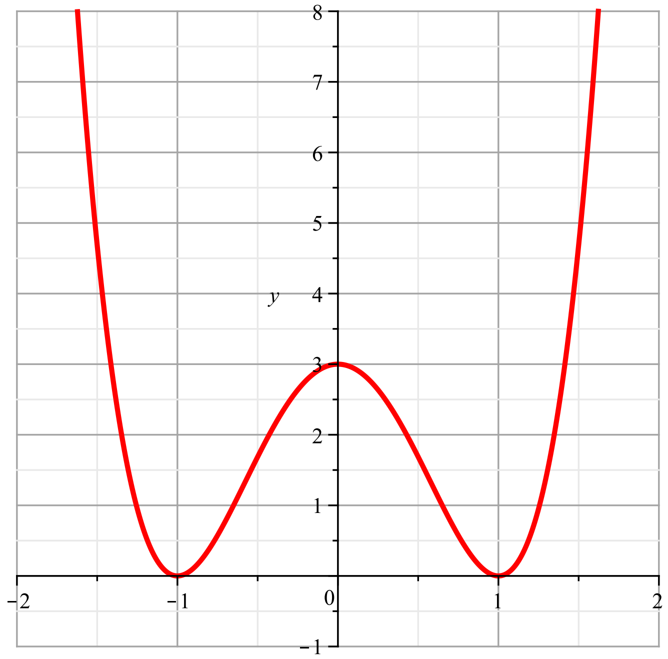
$$f''(x) = 36x^2 - 12.$$

Setting $f''(x) = 0$ gives $x^2 = 1/3$, so there are candidate inflection points at $x = -1/\sqrt{3} \approx -0.577$ and $x = 1/\sqrt{3} \approx 0.577$. We form a table, determine the sign of f' and f'' over various intervals, and draw conclusions about increase/decrease of f and the concavity of its graph. Here is the result:

x	$-\infty$		-1		$-\frac{1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$		1		$+\infty$
$f''(x)$		$+$		$+$	0	$-$		$-$	0	$+$		$+$	
$f'(x)$		$-$	0	$+$		$+$	0	$-$		$-$	0	$+$	
$f(x)$		↘	0	↗	$\frac{4}{3}$	↗	3	↘	$\frac{4}{3}$	↘	0	↗	

Thus, there are two local minima at $x = \pm 1$ and a local maximum at $x = 0$. There are two inflection points at $x = \pm 1/\sqrt{3}$. The graph of f over the interval $[-2, 2]$ is shown on the next page.

Observe that the graph is symmetric with respect to the y-axis. This was expected because f is an even function, that is, $f(-x) = f(x)$ for all x.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