Math 143 Final Exam Review Sheet

12/3/2017

Here is an itemized list of the material that the (cumulative) final exam is based on. Make sure you study them carefully. For each topic, you can review the examples that I gave in lecture, the worked-out examples in the book, past quizzes, and homework problems and solutions available on WebAssign.

- Useful integration formulas to remember:

$$\begin{aligned}
\int x^n\,dx &= \frac{1}{n+1}\,x^{n+1} + C \quad (n \neq -1) \\
\int \frac{dx}{x} &= \ln|x| + C \\
\int \sin x\,dx &= -\cos x + C \\
\int \cos x\,dx &= \sin x + C \\
\int \sec^2 x\,dx &= \tan x + C \\
\int e^x\,dx &= e^x + C \\
\int \ln x\,dx &= x\ln x - x + C \\
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \sin^{-1}\left(\frac{x}{a}\right) + C \\
\int \frac{dx}{a^2 + x^2} &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C
\end{aligned}$$

- The substitution method
- Integration by parts
- Trigonometric integrals and substitutions
- Partial fractions and integration of rational functions
- L'Hospital's rule, application in finding limits of indeterminate forms
- Improper integrals
- Sequences and their limits, basic limit laws for sequences
- The geometric series:

$$\sum_{n=0}^{\infty} r^n \begin{cases} \text{converges to } \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

- The p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } 0 < p \leq 1 \end{cases}$$

  In particular, the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges while Euler's example $\sum_{n=1}^{\infty} 1/n^2$ converges
- Basic divergence test: If $\{a_n\}$ doesn't tend to zero, the series $\sum a_n$ diverges.
- The (limit) comparison test for series with positive terms
- The ratio and root tests for series with positive terms
- Alternating series test, error estimate $|s - s_n| < b_{n+1}$ for the alternating series $\sum (-1)^n b_n$
- Definition of absolute and conditional convergence, an absolutely convergent series is convergent
- Power series, finding the radius and interval of convergence using the ratio or root test
- Manipulating power series, inside the interval of convergence a power series can be differentiated and integrated term-by-term
- Taylor's formula:

$$f(x) = T_n(x) + R_n(x),$$

  where

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x.$$

  In particular, if $\lim_{n\to\infty} R_n(x) = 0$ for all x in some interval $(a-R, a+R)$, then f is equal to its Taylor series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \end{aligned}$$

  inside the interval $(a-R, a+R)$
- Taylor series of a few basic functions near $a = 0$. The most important examples are given below, but many other examples can be constructed using these power series, as we discussed in class:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots && -\infty < x < +\infty \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && -\infty < x < +\infty \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && -\infty < x < +\infty \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots && -1 < x < 1 \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - \cdots && -1 < x < 1 \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots && -1 < x < 1 \\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots && -1 < x < 1 \\
(1+x)^k &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots && -1 < x < 1
\end{aligned}
$$

- Applications of Taylor series in approximating functions or integrals, or in finding limits
- Useful limits to remember:

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$$
$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Practice Problems

1. Evaluate the following integrals:

- $\displaystyle\int \frac{\ln x}{x^3}\,dx$ [Hint: Integration by parts]
- $\displaystyle\int \sin^2(7x)\,dx$ [Hint: Double-angle formula]
- $\displaystyle\int \sqrt{9-x^2}\,dx$ [Hint: Trigonometric substitution]
- $\displaystyle\int \cos^2 x\ \sin^3 x\,dx$ [Hint: Split $\sin^3 x$ as $\sin^2 x\ \sin x$]
- $\displaystyle\int \frac{1}{x^3+4x}\,dx$ [Hint: Partial fractions]

2. Determine the convergence/divergence of the improper integral

$$\int_1^\infty xe^{-5x}\,dx.$$

3. Use integration by parts to express the definite integral $I_n = \int_0^1 x^n e^x \, dx$ in terms of $I_{n-1} = \int_0^1 x^{n-1} e^x \, dx$. Apply this reduction formula to compute I_3.

4. Classify the following series as absolutely convergent, conditionally convergent, or divergent:

- $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$
- $\sum_{n=1}^{\infty} \frac{(-2)^n \, n!}{n^n}$

5.

(i) Use the Leibniz test to show that the series

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots$$

converges.

(ii) Use your calculator (the built-in sum command for a sequence) to find the partial sum s_{100} of the above series. How far is the estimate s_{100} from the actual sum s?

6. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{3^n (x+1)^n}{n}$.

7. Use Taylor series to find $\lim_{x \to 0} \frac{1 + x^3 - e^{x^3}}{x^6}$.

8. Write the 2nd degree Taylor polynomial $T_2(x)$ for the function $f(x) = \sqrt[3]{x}$ at the point $a = 8$. Then find the approximate value of $\sqrt[3]{10}$ by computing $T_2(10)$. Estimate the error in your approximation using Taylor's formula for the remainder term $R_2(x)$.