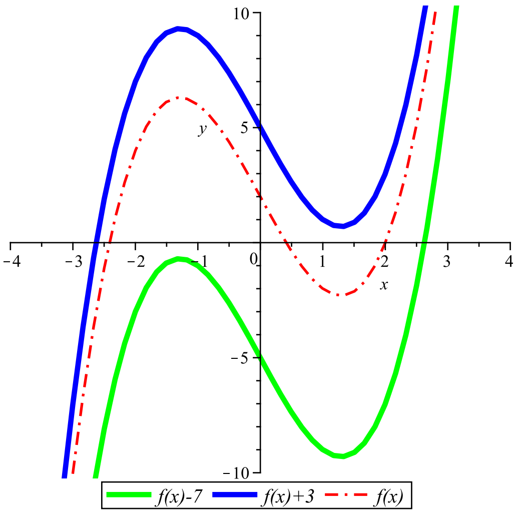
Here are the solutions to the few remaining homework problems from Spivak's chapter 4:

14. Describe the graph of g in terms of the graph of f if

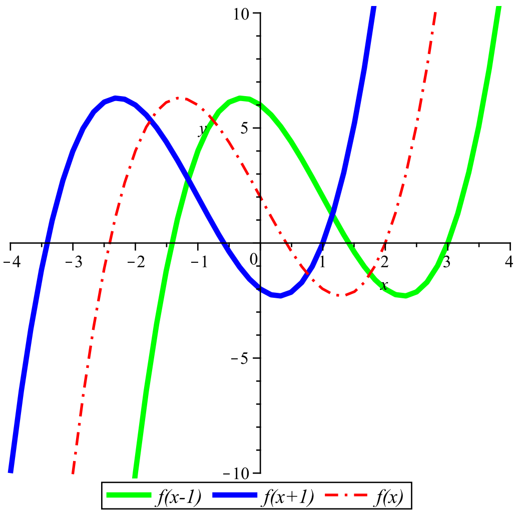
(i) $g(x) = f(x) + c$.

Shift the graph of f vertically, c units up if $c > 0$ and $-c$ units down if $c < 0$. Here and in the subsequent parts, the picture on the right is only an example chosen to illustrate the idea.



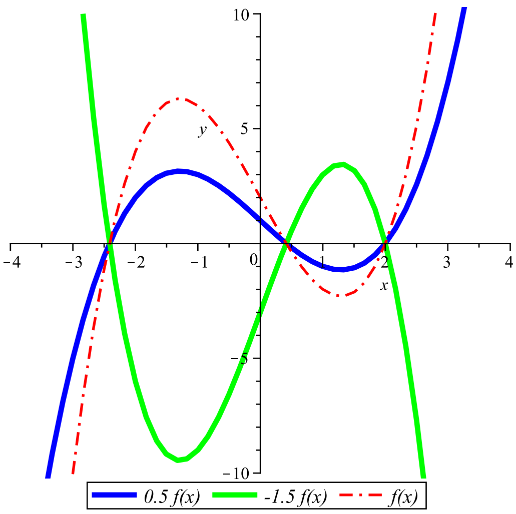
(ii) $g(x) = f(x + c)$.

Shift the graph of f horizontally, c units to the left if $c > 0$ and $-c$ units to the right if $c < 0$.



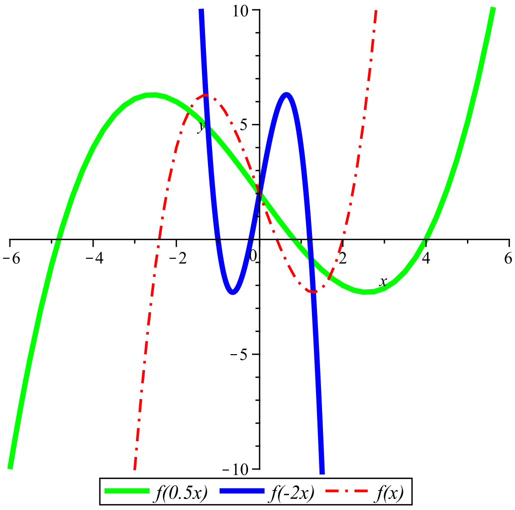
(iii) $g(x) = cf(x)$.

If $c = 0$, the graph of g is the horizontal line $y = 0$. If $c > 0$, re-scale the heights on the graph of f by a factor of c (this would be a vertical compression if $0 < c < 1$ and a vertical stretch if $c > 1$). When $c < 0$, first reflect the graph of f about the x-axis and then re-scale the heights on the reflected graph by a factor of $-c$ (which would be a vertical compression if $-1 < c < 0$ and a vertical stretch if $c < -1$).



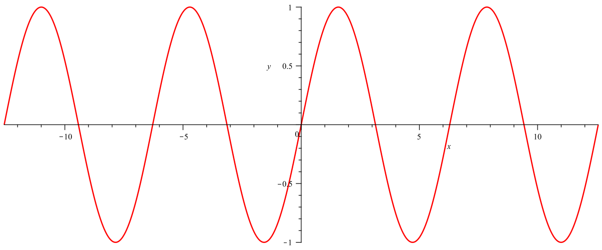
(iv) $g(x) = f(cx)$.

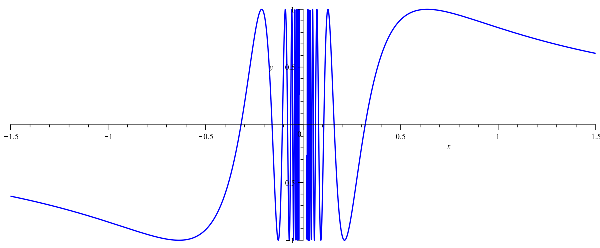
If $c = 0$, the graph of g is the horizontal line $y = f(0)$. If $c > 0$, re-scale the graph of f horizontally by a factor of $1/c$ (this would be a stretch if $0 < c < 1$ and a compression if $c > 1$). When $c < 0$, first reflect the graph of f about the y-axis and then re-scale the reflected graph horizontally by a factor of $-1/c$ (which would be a stretch if $-1 < c < 0$ and a compression if $c < -1$).



(v) $g(x) = f(1/x)$.

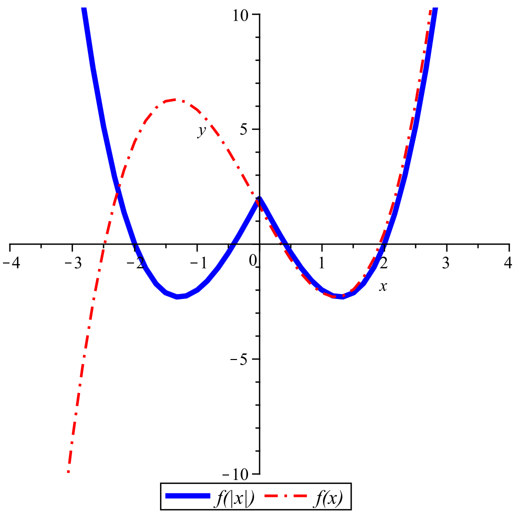
Everything that happens for large x on the graph of f happens for small x on the graph of g and vice versa, as we have seen for $f(x) = \sin(x)$, $g(x) = \sin(1/x)$.





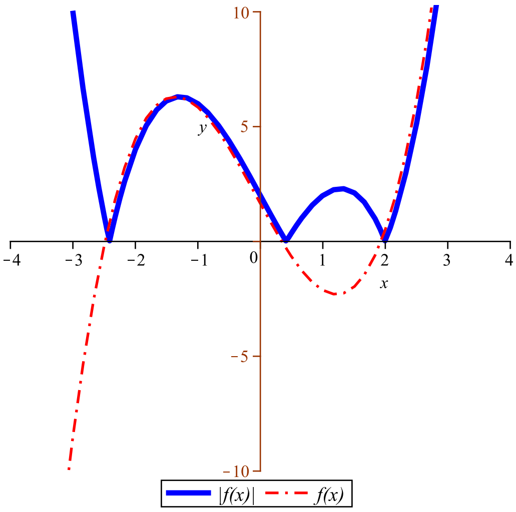
(vi) $g(x) = f(|x|)$.

Then $g(x) = f(x)$ if $x \geq 0$ and $g(x) = f(-x)$ if $x < 0$. Thus, the graph of g is obtained by keeping the part of the graph of f over $x \geq 0$ and reflecting it about the y-axis to obtain the rest.



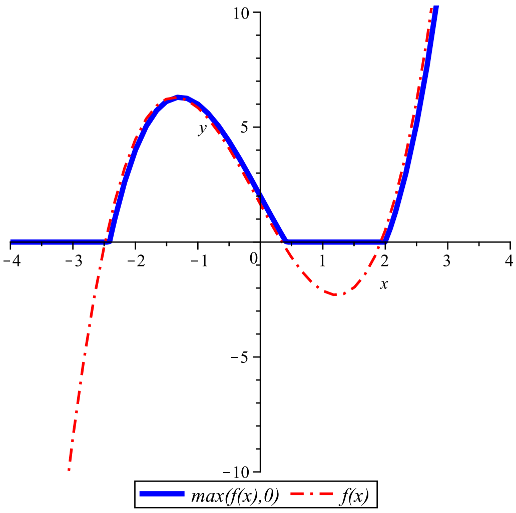
(vii) $g(x) = |f(x)|$.

Then $g(x) = f(x)$ if $f(x) \geq 0$ and $g(x) = -f(x)$ if $f(x) < 0$. Thus, the graph of g is obtained by keeping the part of the graph of f above the x-axis and reflecting what is below about the x-axis to obtain the rest.



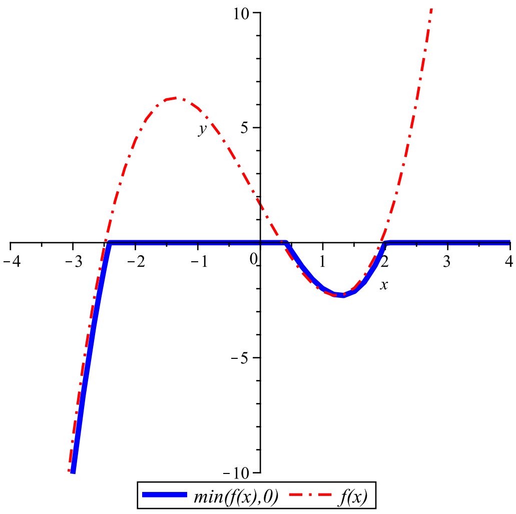
(viii) $g(x) = \max(f(x), 0)$.

Then $g(x) = f(x)$ if $f(x) \geq 0$ and $g(x) = 0$ if $f(x) < 0$. Thus, we keep the part of the graph of f above the x-axis and "cut off" what is below, replacing it by 0.



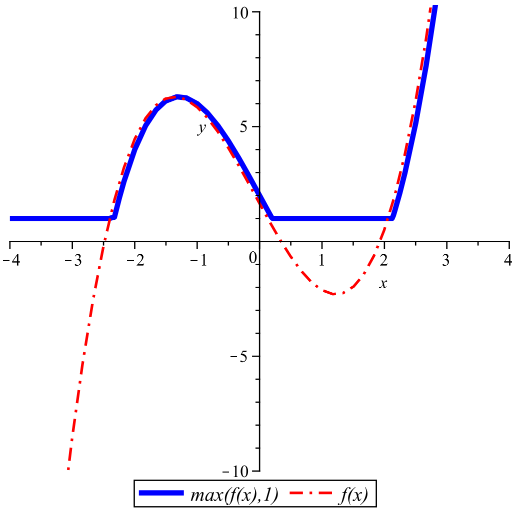
(ix) $g(x) = \min(f(x), 0)$.

Then $g(x) = f(x)$ if $f(x) \leq 0$ and $g(x) = 0$ if $f(x) > 0$. Thus, we keep the part of the graph of f below the x-axis and "cut off" what is above, replacing it by 0.



(x) $g(x) = \max(f(x), 1)$.

Then $g(x) = f(x)$ if $f(x) \geq 1$ and $g(x) = 1$ if $f(x) < 1$. Thus, we keep the part of the graph of f above the line $y = 1$ and "cut off" what is below, replacing it by 1.



17(v). Draw the graph of $f(x) = [1/x]$.

By the definition of the integer part function [ ],

$$\left[\frac{1}{x}\right] = n \text{ for some integer } n \iff n \leq \frac{1}{x} < n+1.$$

Thus, we need to figure out when $1/x$ lies between two consecutive integers n and $n+1$. For $n = 0$ we have

$$0 \leq \frac{1}{x} < 1 \iff x > 1,$$

while for $n = -1$ we have

$$-1 \leq \frac{1}{x} < 0 \iff x \leq -1.$$

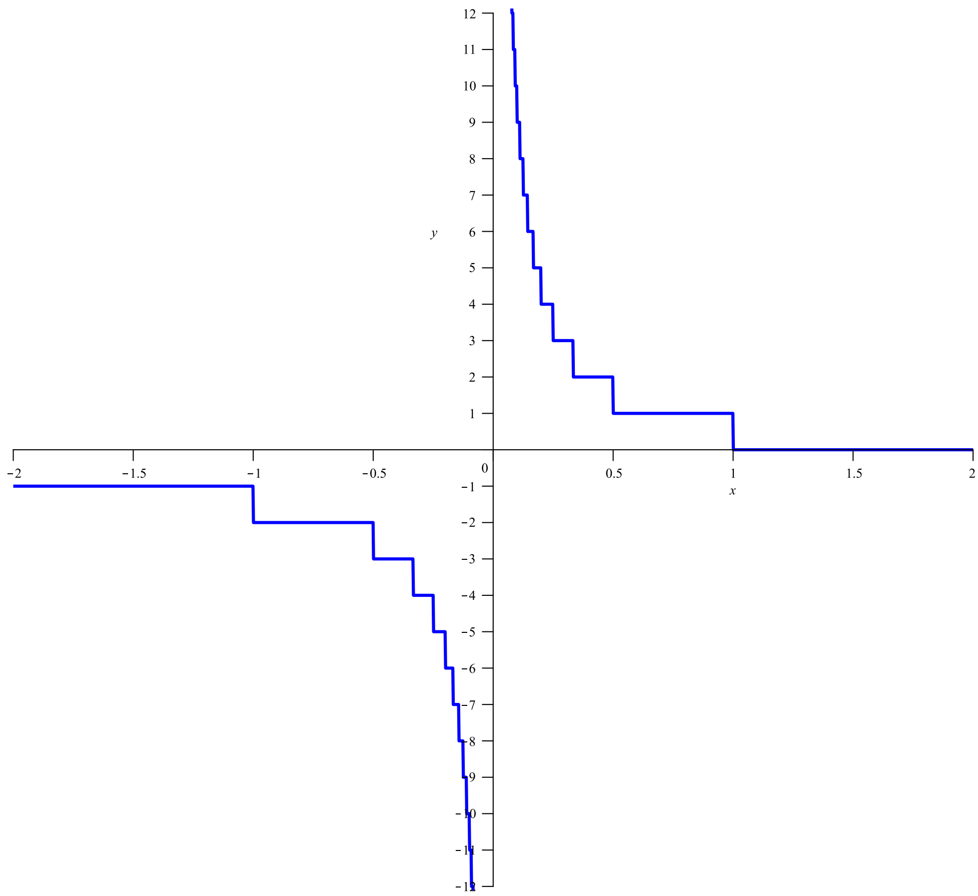
For all other values of n, we have

$$n \leq \frac{1}{x} < n+1 \iff \frac{1}{n+1} < x \leq \frac{1}{n}.$$

Summarizing, we obtain

$$f(x) = \left[\frac{1}{x}\right] = \begin{cases} 0 & \text{if } x > 1 \\ -1 & \text{if } x \leq -1 \\ n & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \text{ for some integer } n \neq 0, -1 \end{cases}$$

The resulting graph is shown below:



Appendix 3: Polar Coordinates

1. If two points have polar coordinates (r_1, θ_1) and (r_2, θ_2), show that the distance d between them is given by

$$d^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2).$$

What does this say geometrically?

Denote the cartesian coordinates of these points by (x_1, y_1) and (x_2, y_2), so

$$\begin{cases} x_1 &= r_1 \cos\theta_1 \\ y_1 &= r_1 \sin\theta_1 \end{cases} \qquad \begin{cases} x_2 &= r_2 \cos\theta_2 \\ y_2 &= r_2 \sin\theta_2 \end{cases}$$

By the distance formula, we can write

$$\begin{aligned} d^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2 \\ &= r_1^2 \cos^2\theta_1 + r_2^2 \cos^2\theta_2 - 2r_1 r_2 \cos\theta_1 \cos\theta_2 + r_1^2 \sin^2\theta_1 + r_2^2 \sin^2\theta_2 - 2r_1 r_2 \sin\theta_1 \sin\theta_2 \\ &= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) - 2r_1 r_2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \\ &= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2). \end{aligned}$$

Geometrically, this means in any triangle $\triangle ABC$,

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC| \cos(\sphericalangle BAC),$$

a fact that is often known as the **law of cosines**.

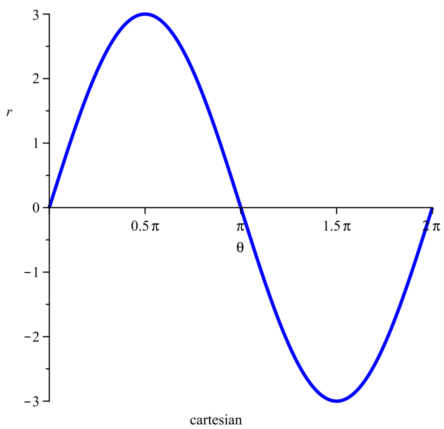
3. Sketch the graphs of the following equations:

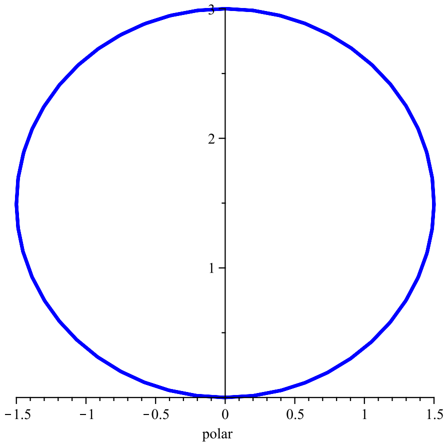
(i) $r = a \sin\theta$

If $a = 0$, this is just the origin $r = 0$. If $a \neq 0$, it will be a circle of radius $|a|/2$ centered at $(0, a/2)$. In fact,

$$r = a \sin\theta \iff r^2 = ar \sin\theta \iff x^2 + y^2 = ay \iff x^2 + (y - a/2)^2 = a^2/4.$$

As θ goes from 0 to 2π, the circle is traversed twice in the counterclockwise direction.



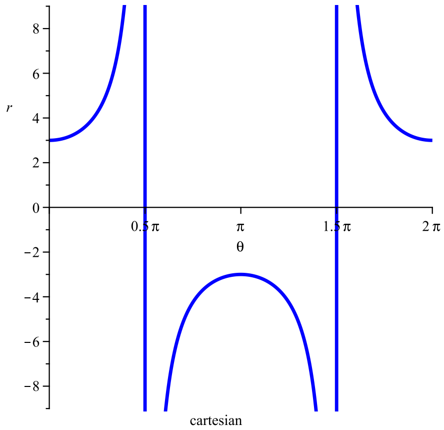


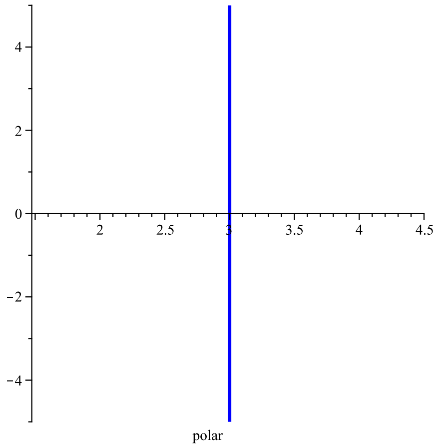
(ii) $r = a \sec\theta$

Again, if $a = 0$, this is just the origin $r = 0$. If $a \neq 0$, it will be the vertical line $x = a$. In fact,

$$r = a \sec\theta \iff r = \frac{a}{\cos\theta} \iff r\cos\theta = a \iff x = a.$$

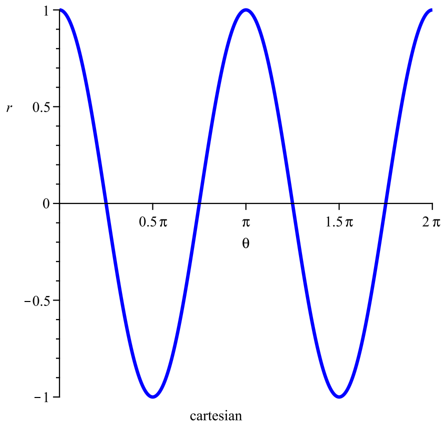
As θ goes from 0 to 2π, the line is traversed twice in the upward direction.

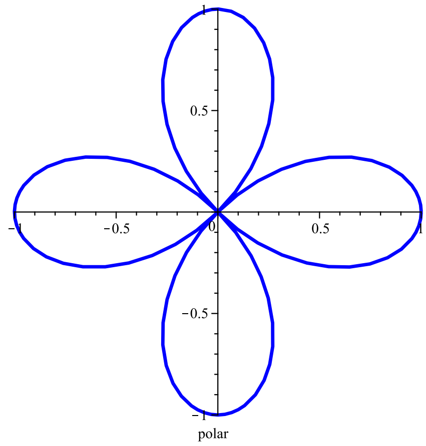




(iii) $r = \cos(2\theta)$

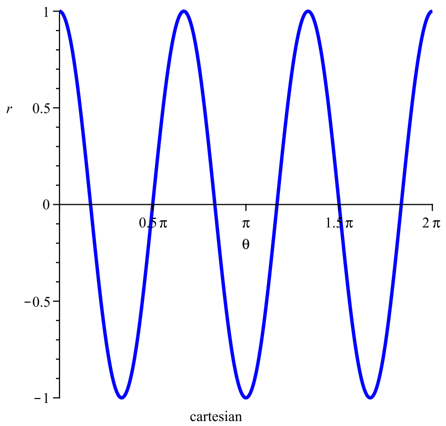
This will be a 4-leaf clover, as a simple inspection of the cartesian graph of $\cos(2\theta)$ reveals.

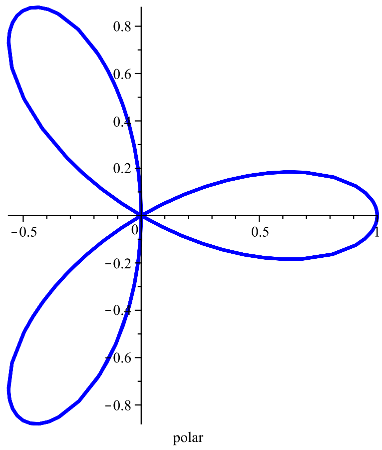




(iv) $r = \cos(3\theta)$

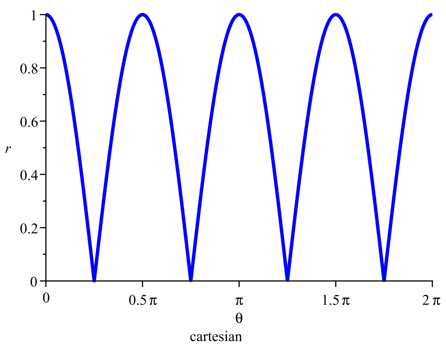
This will be a 3-leaf clover, with each petal traversed twice as θ goes from 0 to 2π.

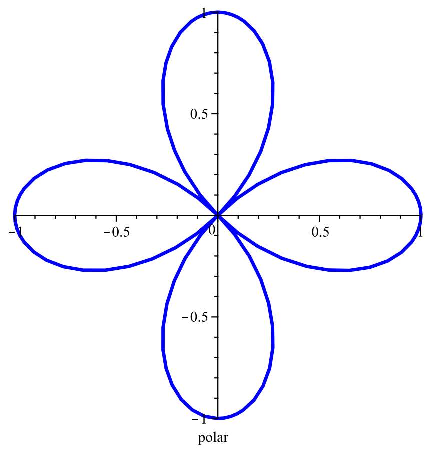




(v) $r = |\cos(2\theta)|$

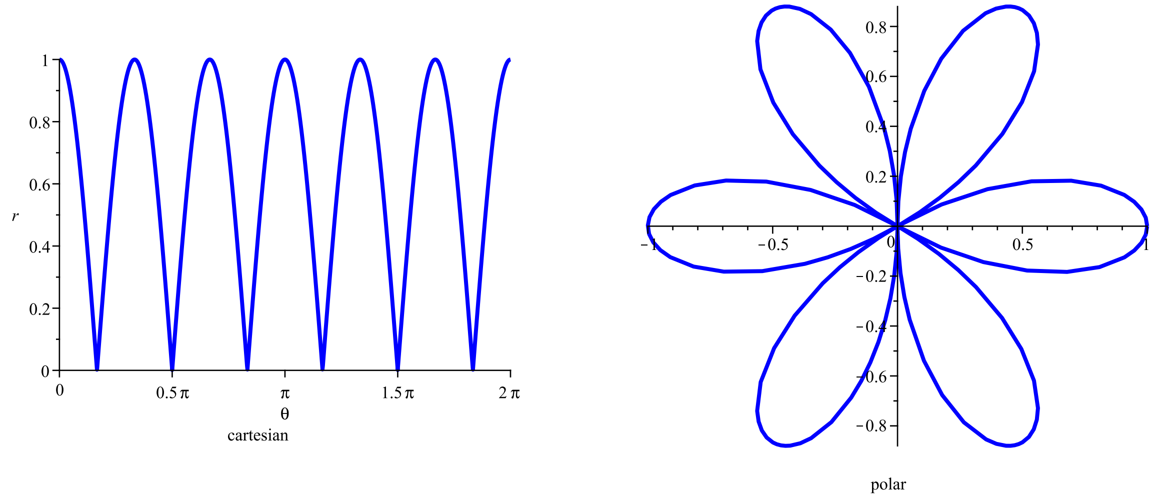
This will also be a 4-leaf clover, but the difference with (iii) is that here the petals are traversed consecutively in counterclockwise direction as θ goes from 0 to 2π.





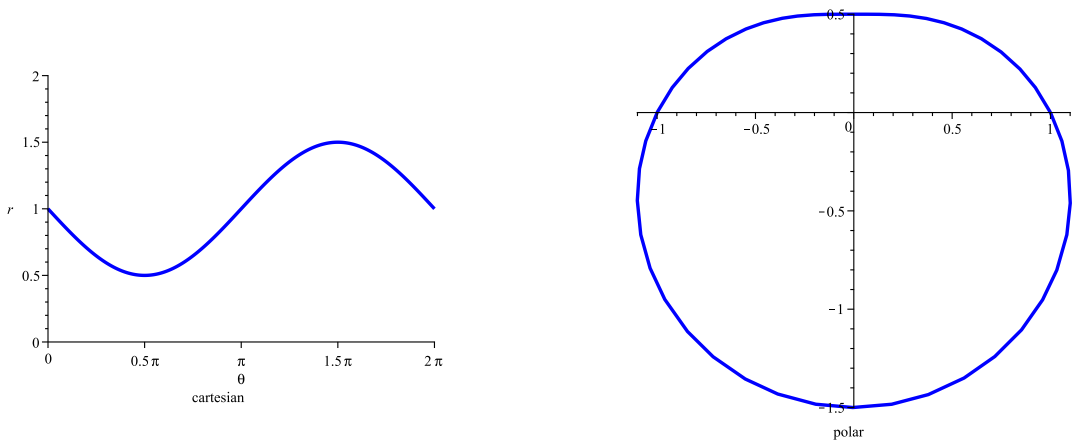
(vi) $r = |\cos(3\theta)|$

This will be a 6-leaf clover, where the petals are traversed consecutively in counterclockwise direction as θ goes from 0 to 2π.

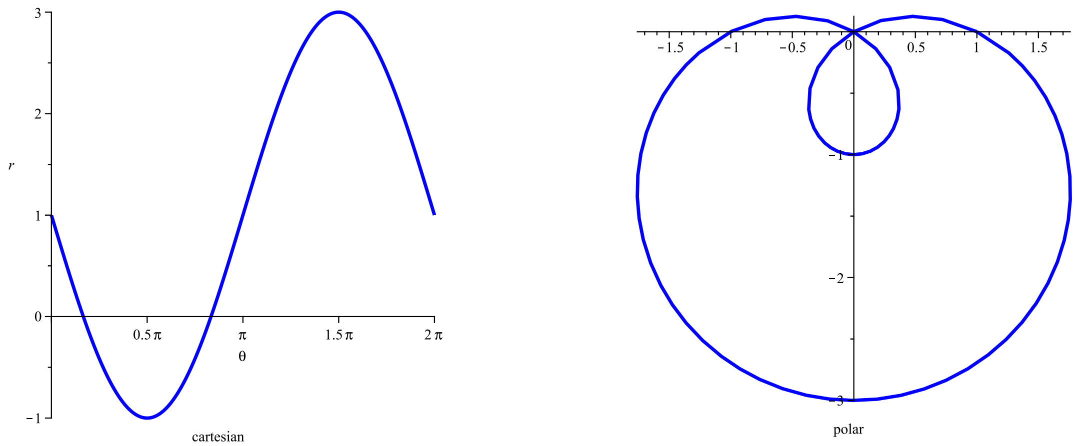


9. Sketch the graphs of the following equations:

(i) $r = 1 - \frac{1}{2}\sin\theta$



(ii) $r = 1 - 2\sin\theta$



(iii) $r = 2 + \cos\theta$